Numerical Simulation and Optimization of Industrial Problems

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Dpto. de Matemática Aplicada • Grupo de Ingeniería Matemática

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Silicon: production and applications

- Silicon is **produced** in **electric arc furnaces** by reduction of **silicon dioxide** with **carbon**:

$$\mathrm{Si\,O_2} + \mathrm{C} = \mathrm{Si} + \mathrm{C\,O_2}$$

Silicon: production and applications

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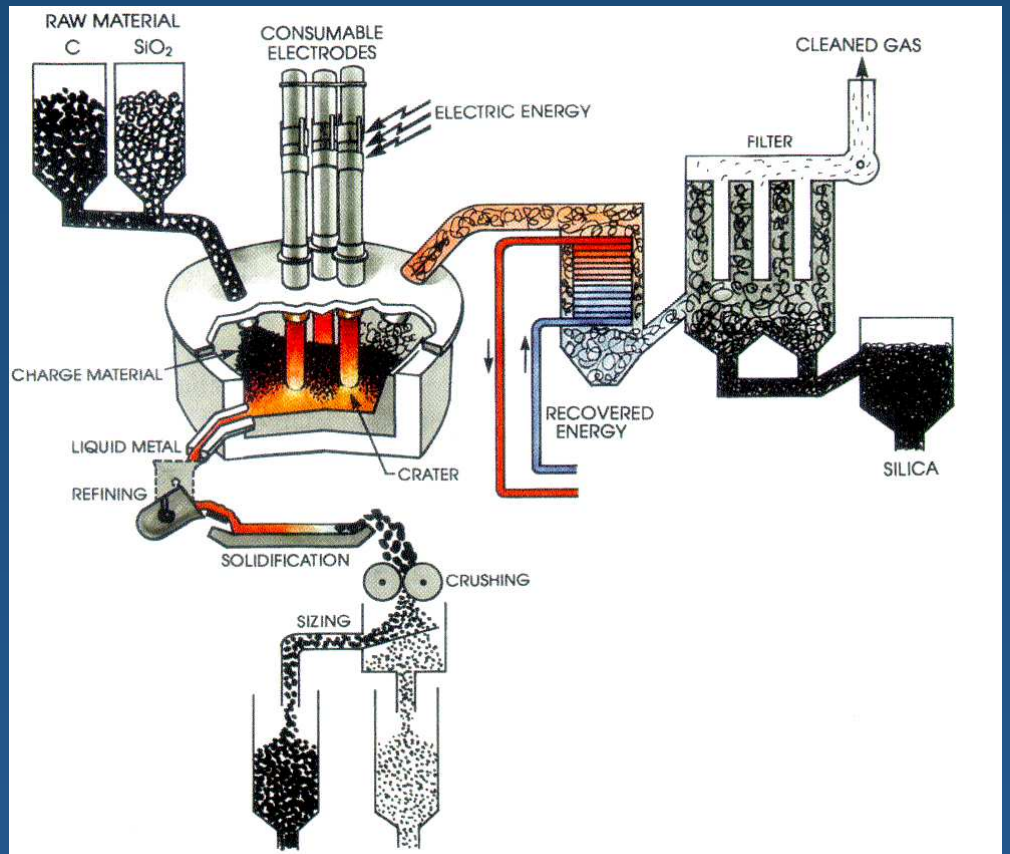


Silicon: production and applications

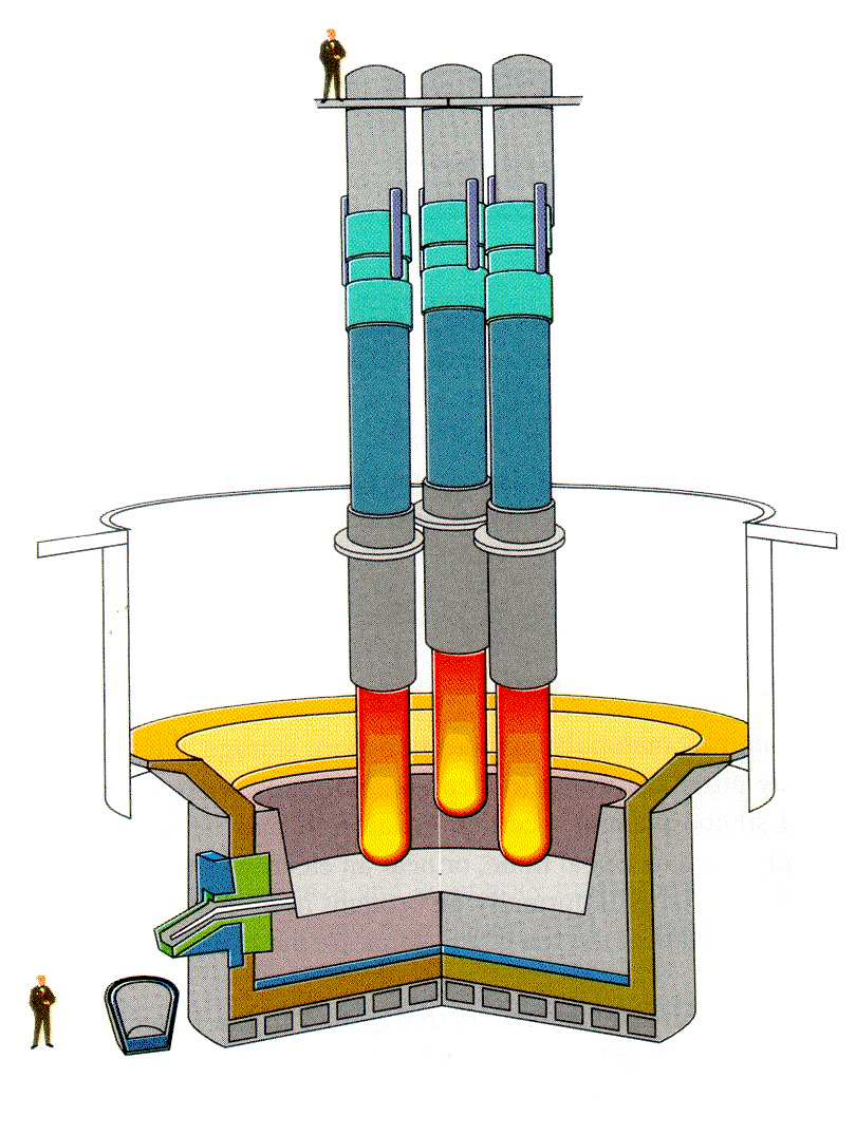
- Silicon is **produced** in **electric arc furnaces** by reduction of **silicon dioxide** with **carbon**:

$$\mathrm{Si\,O_2} + \mathrm{C} = \mathrm{Si} + \mathrm{C\,O_2}$$

Applications of silicon:

- Ferrosilicon (silicon steels, can contain more than 2% of other materials)
- Metallurgical silicon (e.g. silicon-aluminum alloys, contains about 1% of other elements)
- Chemical silicon (silicones)
- Solar silicon (solar cells)
- Electronic silicon (semiconductors, the purest silicon, "9N" = 99.9999999 of purity)

The reduction furnace



The electrodes

- Main components of the reduction furnace.
- Electric current enters each electrode through *contact clamps*.
- The current flows down through the column generating heat by Joule effect.
- Electric arc at the tip of each electrode.

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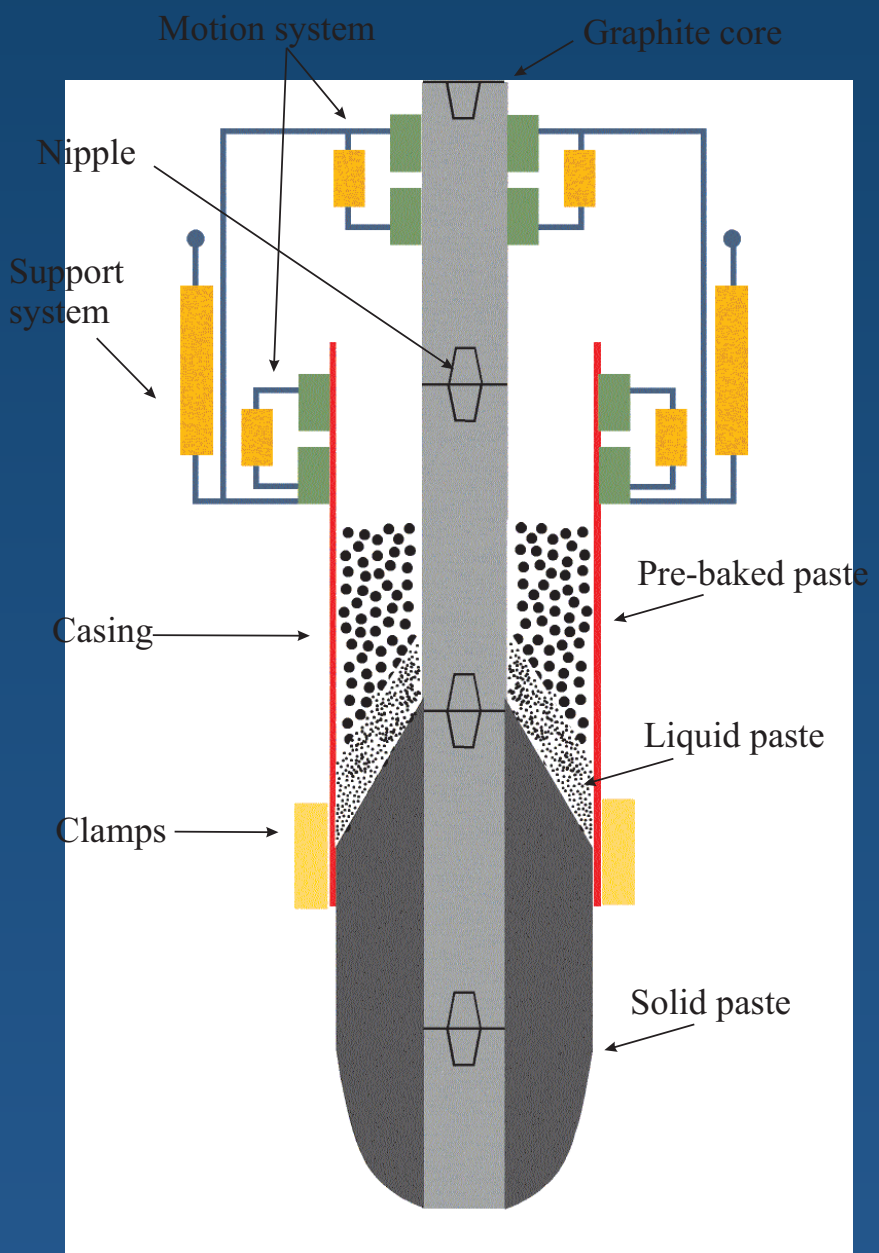
Classical electrodes:

- **Pure graphite:** steel production.
- **Prebaked:** silicon metal production.
- **Søderberg** (“self-baked” electrodes): ferrosilicon production.

The ELSA electrode

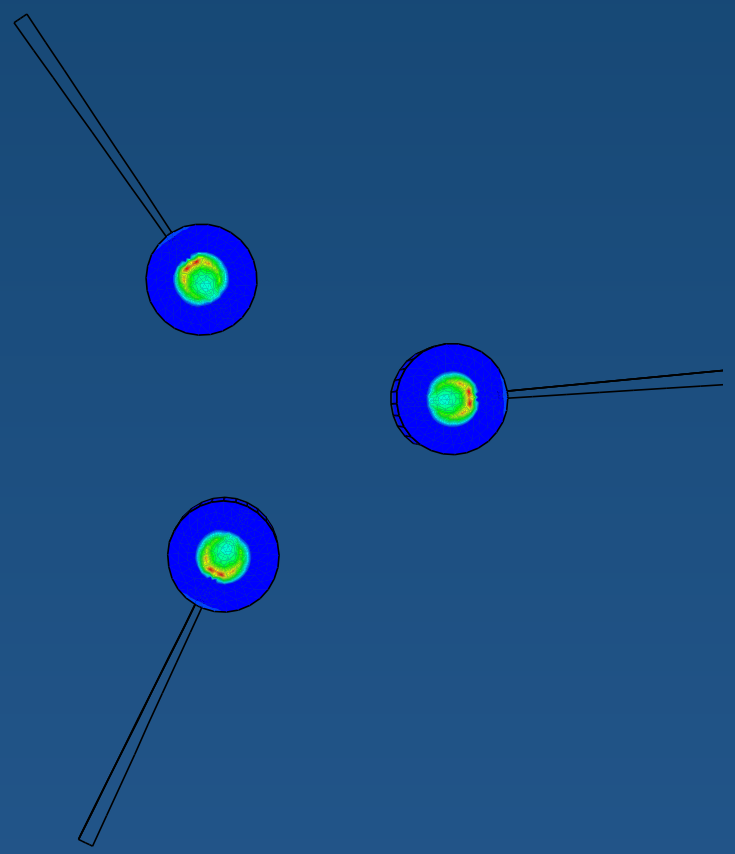


The ELSA electrode

- Compound electrode: graphite/paste.
- Suitable for metallurgical silicon production.
- Cheaper than prebaked electrodes.
- Paste baking is a crucial point: the liquid-solid interphase has to be controlled in order to prevent the liquid paste to fall down in the mixture

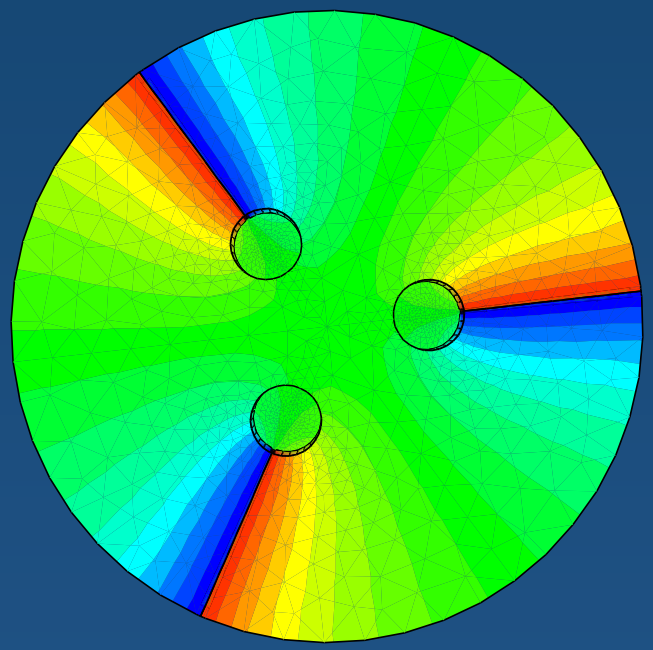
Numerical results

Modulus the current density, $|\mathbf{J}_h|$, in the conductors.



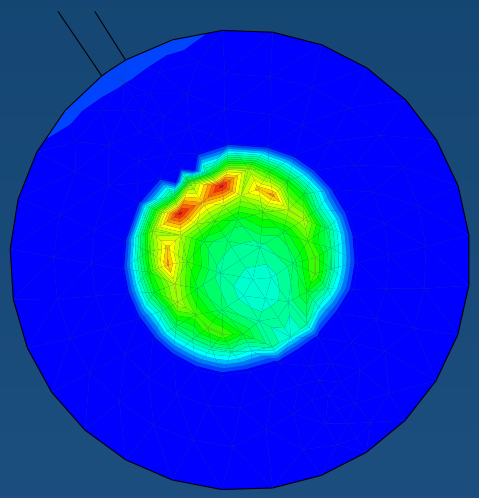
Numerical results

Magnetic potential $\widetilde{\Phi}_h$ in the dielectric.



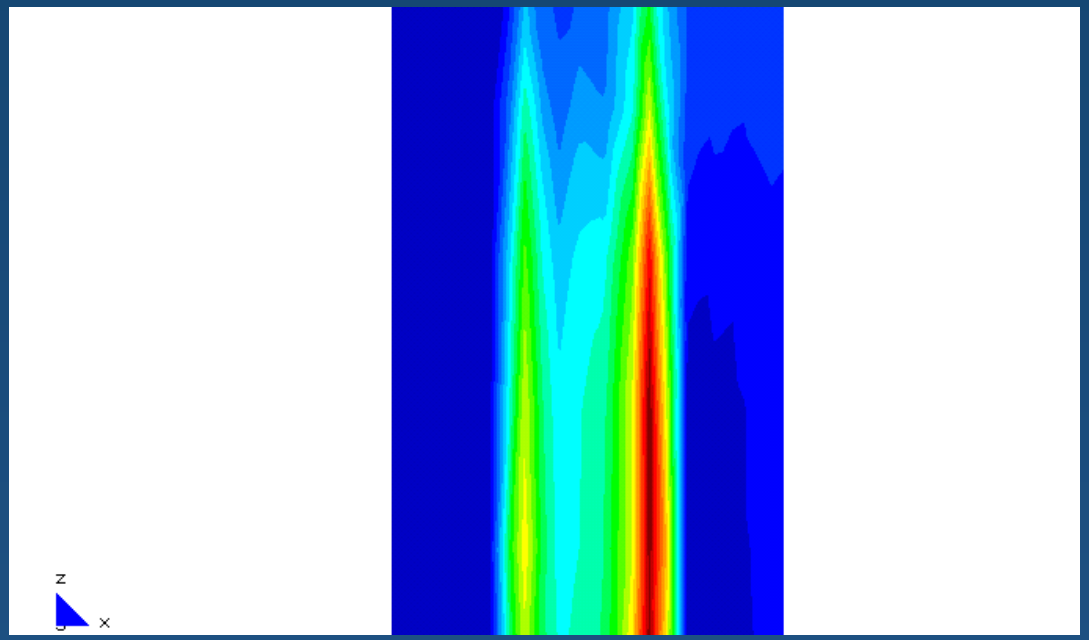
Numerical results

$|\mathbf{J}_h|$: Horizontal section of one of the electrodes.



Numerical results

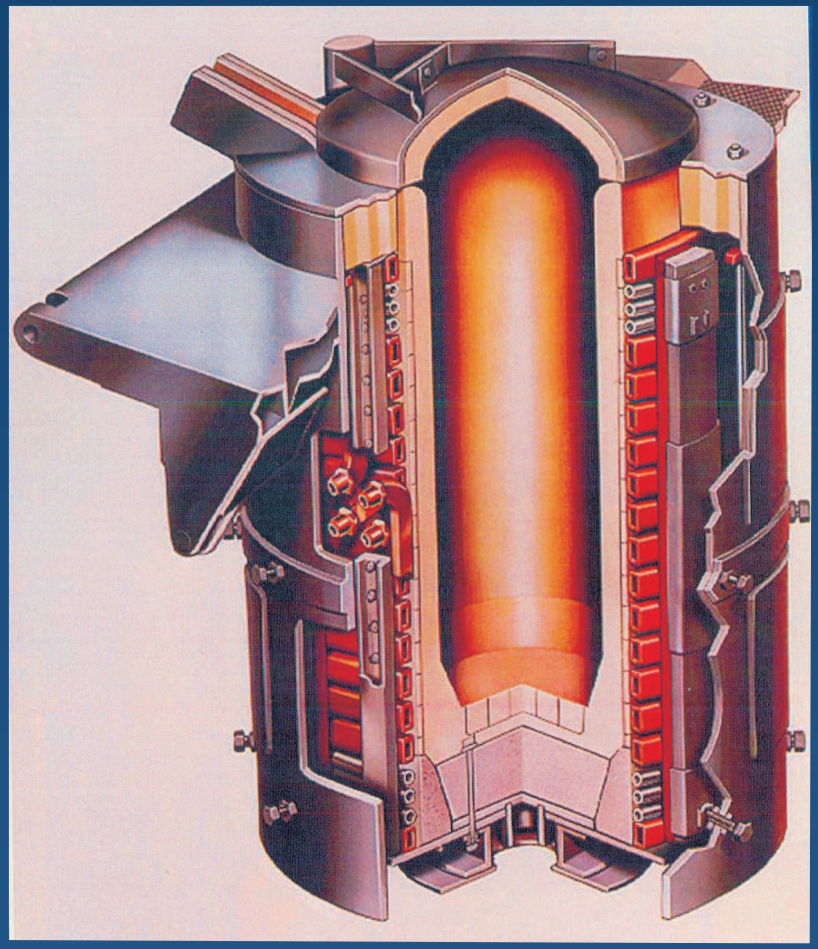
$|\mathbf{J}_h|$: Vertical section of one of the electrodes.



Numerical simulation of induction furnaces

Photographs taken from http://www.ameritherm.com

An industrial induction furnace



- Silicon for melting and purification
- Graphite crucible
- Refractory layers
- Water-cooled coil

Mathematical model

Coupled problems

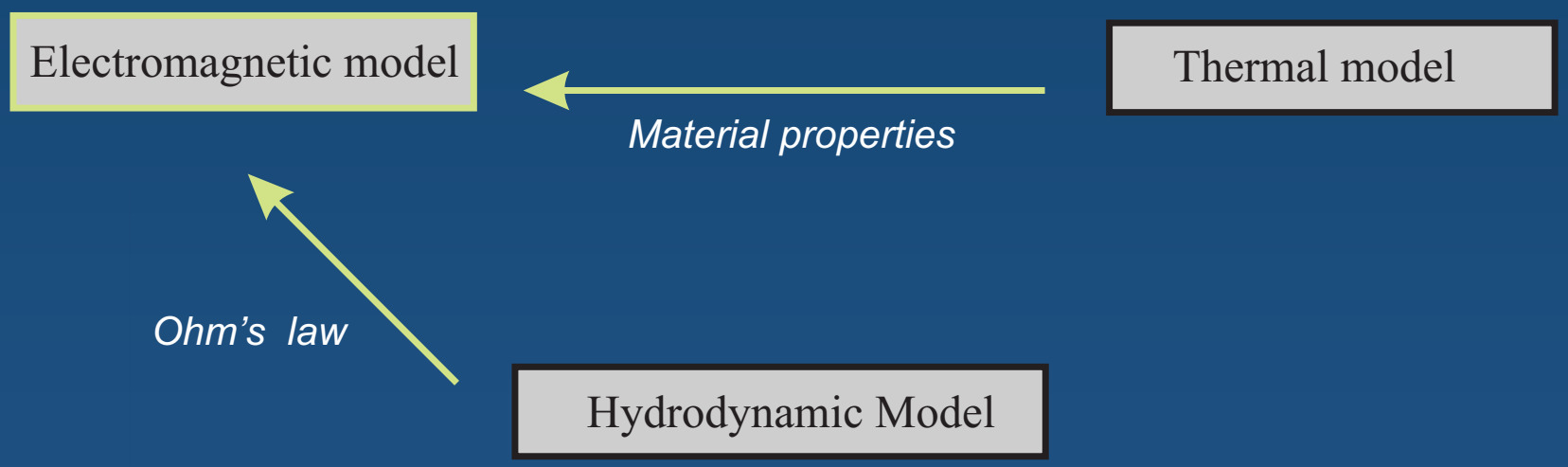
Electromagnetic model

Thermal model

Hydrodynamic Model

Mathematical model

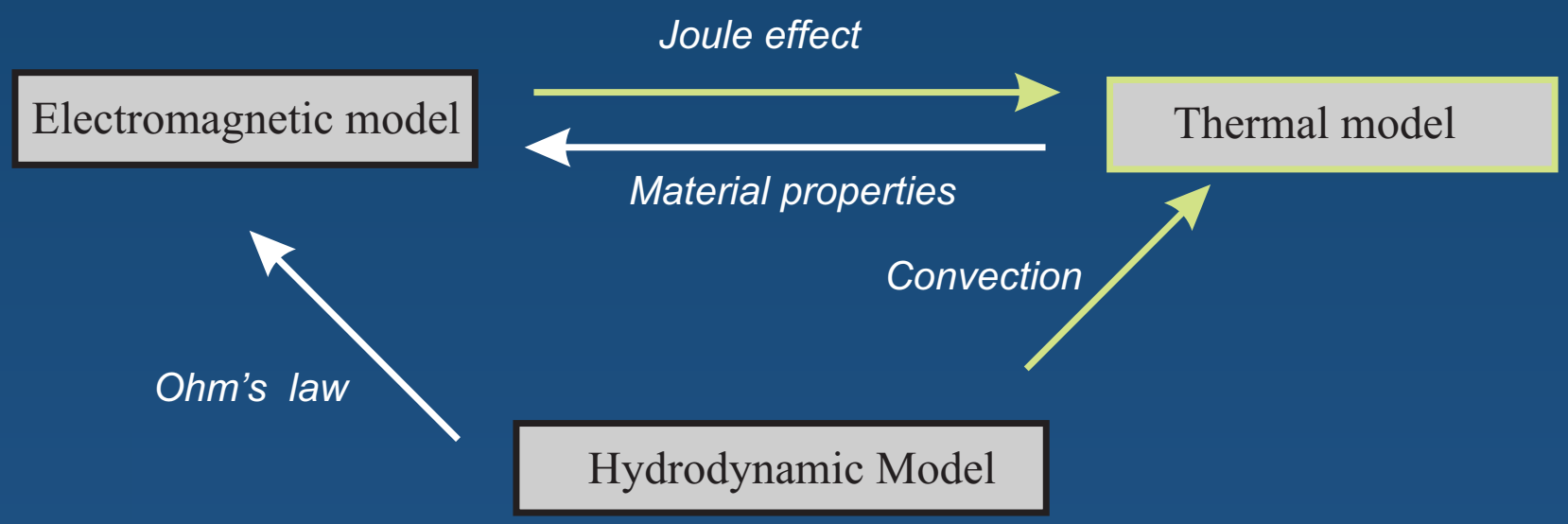
Electromagnetic model



$$
\begin{aligned}
\mathbf{curl}\,\mathbf{H} &= \mathbf{J}, \\
i\omega\mathbf{B} + \mathbf{curl}\,\mathbf{E} &= \mathbf{0}, \\
\mathrm{div}\,\mathbf{B} &= 0,
\end{aligned}
\qquad
\begin{aligned}
\mathbf{B} &= \mu(T)\mathbf{H}, \\
\mathbf{J} &= \sigma(T)(\mathbf{E} + \mathbf{v} \times \mathbf{B}).
\end{aligned}
$$

Mathematical model

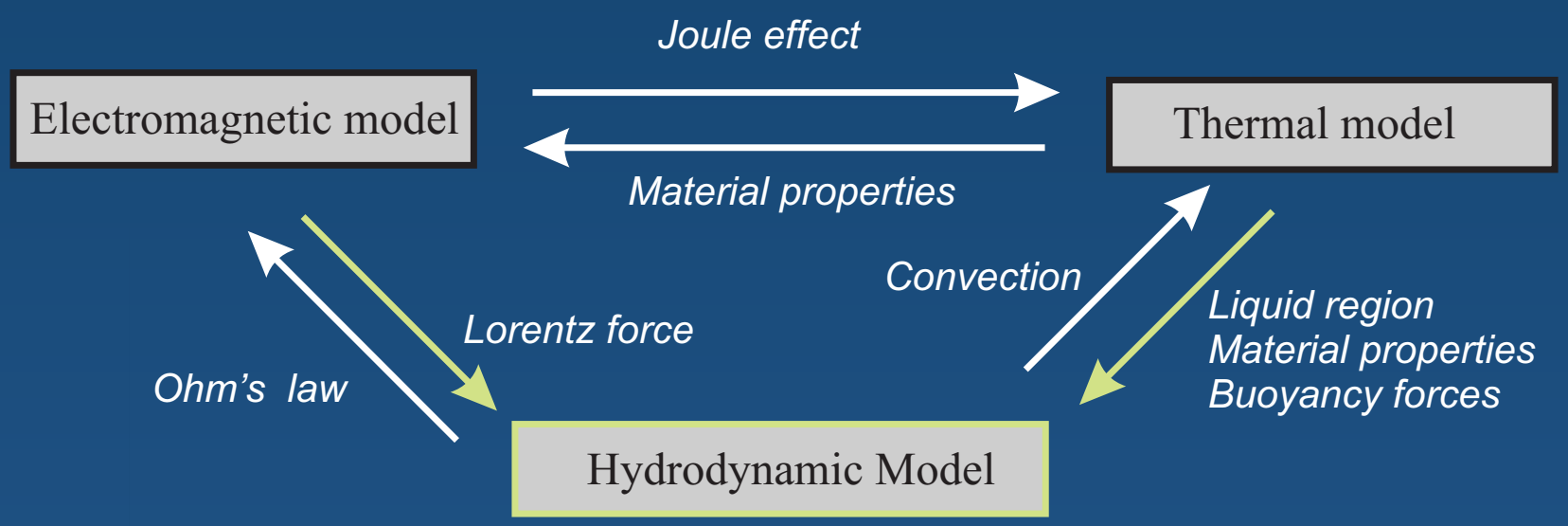
Thermal model



$$\frac{\partial e}{\partial t} + \mathbf{v} \cdot \mathbf{grad}\, e - \operatorname{div}(k\, \mathbf{grad}\, T) = \frac{|\mathbf{J}|^2}{2\sigma}$$

Mathematical model

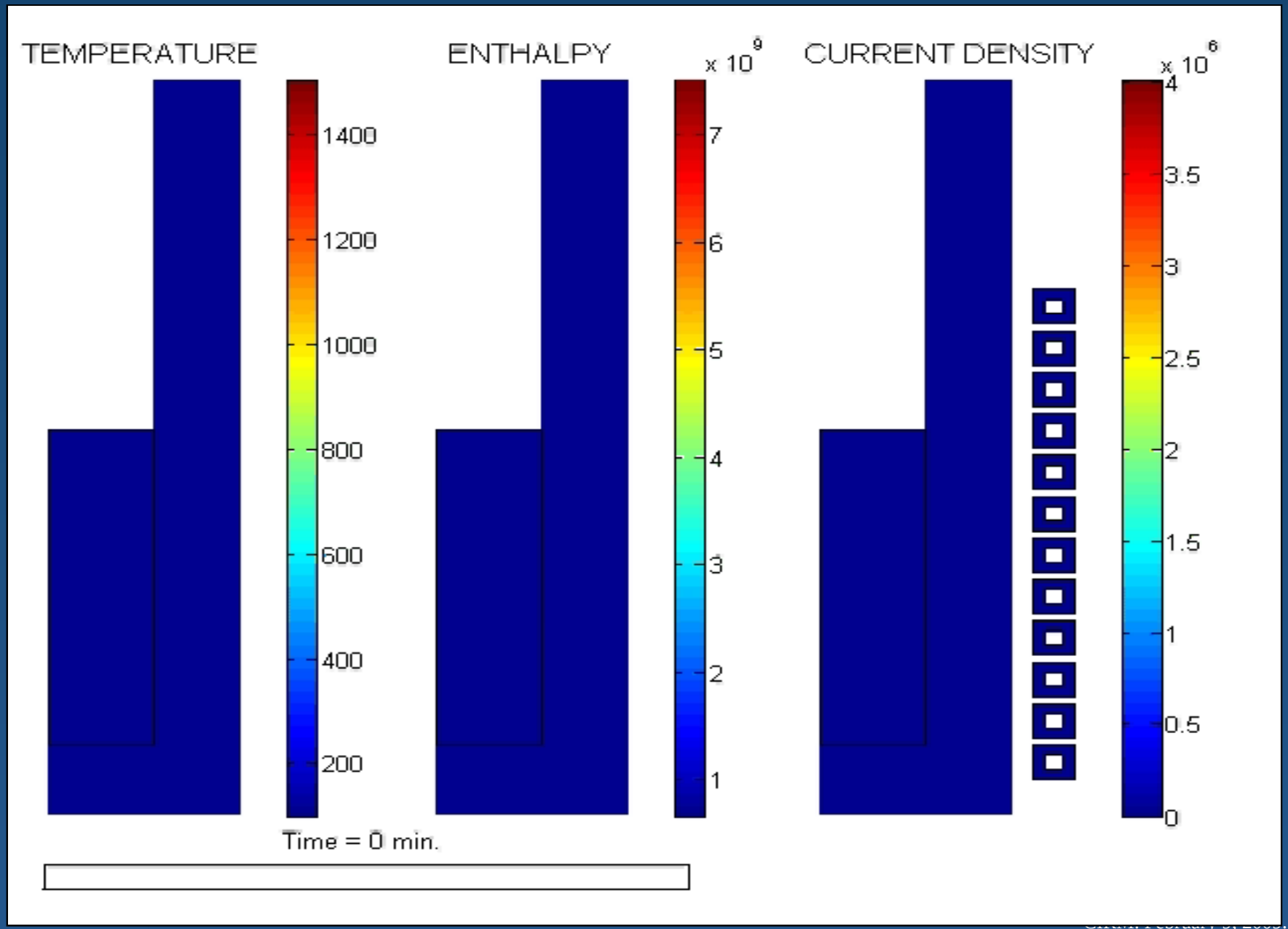
Hydrodynamic model



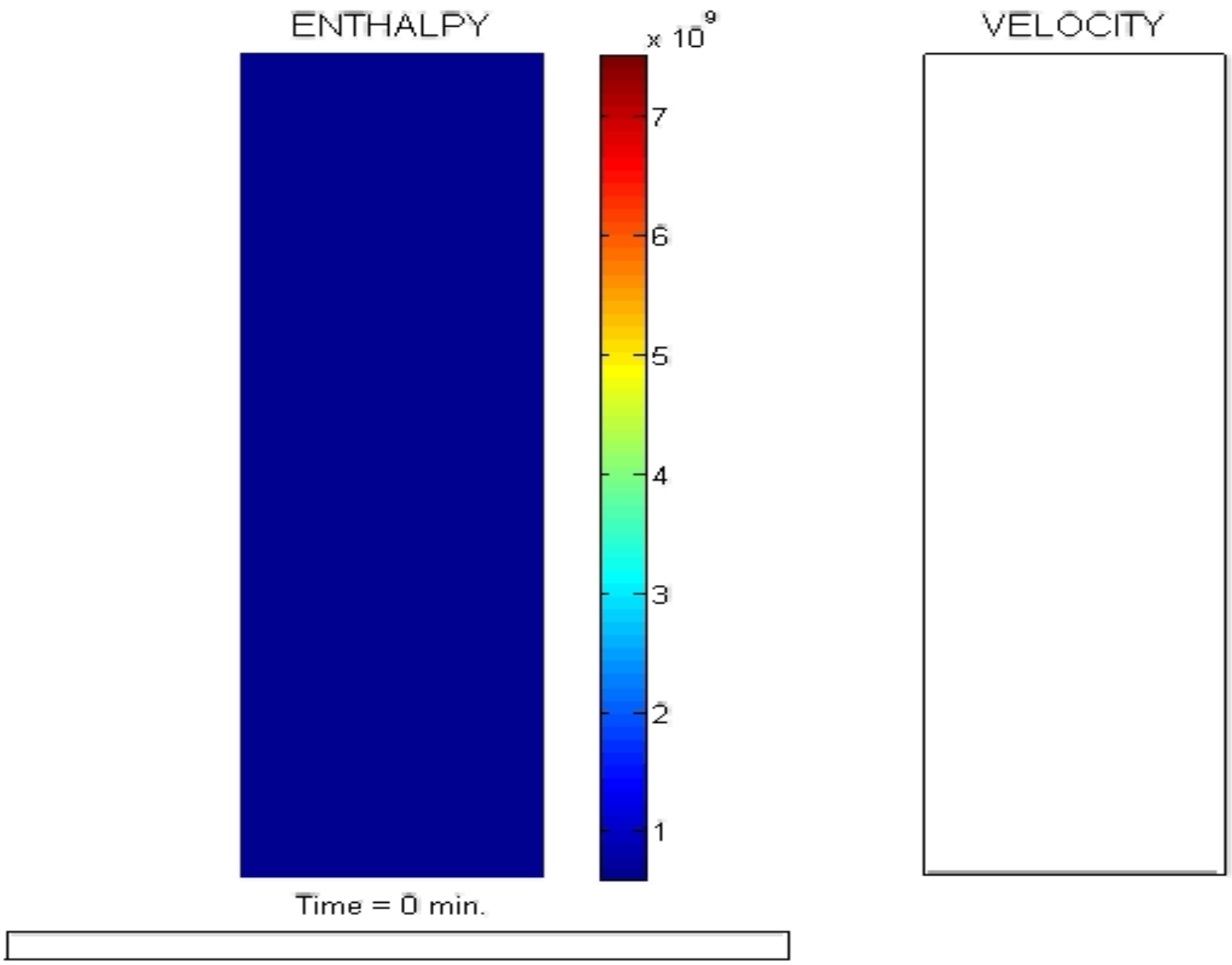
$$\rho(T)\dot{\mathbf{v}} - \operatorname{div}\left(\eta(T)(\operatorname{grad}\mathbf{v} + (\operatorname{grad}\mathbf{v})^t)\right) + \mathbf{grad}\, p = \mathbf{f}(T, \mathbf{H})$$

$$\operatorname{div}\mathbf{v} = 0$$

Temperature, enthalpy and current density evolution



Enthalpy and velocity field in silicon



Maxwell equations in free space

1. Charges and currents

Electromagnetic field theory is a discipline concerned with the study of charges, at rest and in motion, producing currents and electric and magnetic fields.

Charges are represented by density scalar fields. These densities can be volume, surface or line densities. From a mathematical point of view they will be distributions supported on a volume, a surface or a line, respectively. We will also consider point charges that will be represented by Dirac measures.

Suppose there are electric charges of density $\rho(x,t)$ in a region and the charge at point x moves with velocity $\mathbf{v}(x,t)$ at time t. Then we define the current density field $\mathbf{J}$ by

$$\mathbf{J}(x,t) = \rho(x,t)\mathbf{v}(x,t). \tag{1}$$

Let us consider a surface $\mathcal{S}$. The surface integral

$$\mathcal{I}_{\mathcal{S}} = \int_{\mathcal{S}} \mathbf{J} \cdot \mathbf{n} \mathrm{d}A \tag{2}$$

represents the charge flux through the surface $\mathcal{S}$ which is also called the **intensity** through $\mathcal{S}$.

In the MKS system charge is measured in Coulomb (C) and current density in $C/sm^2 = A/m^2$, where Coulomb over second is Ampère (A).

Hence, the MKS unit for the current flux or intensity is Ampère.

2. Electric and magnetic fields

Electric and magnetic fields are fundamentally fields of force that originate from electric charges.

Electric charges at rest relative to an observation point give rise to an **electric** field there, which is static (time independent).

The relative motion of charges provides an additional force called **magnetic** force. If charges are moving at constant velocities relative to the observation point, this added magnetic field is static (time independent).

Accelerated motions produce both time-varying electric and magnetic fields.

The symbol for the electric field intensity is the vector $\mathbf{E}$. Its unit is force per unit charge, in MKS system Newton over Coulomb (N/C).

In its turn, the magnetic field is represented by vector $\mathbf{B}$ called the **magnetic flux density**.

Its unit is Weber per square meter (Wb/m^2) or Tesla (T).

The presence of $\mathbf{E}$ and $\mathbf{B}$ at a point in space may be detected physically by means of a charge Q moving at velocity $\mathbf{v}$.

The force acting on that charge is given by the **Lorentz force law**,

$$\mathbf{F} = \mathrm{Q}(\mathbf{E} + \mathbf{v} \times \mathbf{B}). \tag{3}$$

In the case of a charge distribution with (volumetric) charge density ρ_v, the density of force of the electromagnetic field is

$$\mathbf{f} = \rho_v(\mathbf{E} + \mathbf{v} \times \mathbf{B}) = \rho_v \mathbf{E} + \mathbf{J} \times \mathbf{B}. \tag{4}$$

3. Maxwell integral equations in free space

The connection of the electric and magnetic fields to their charge and current sources is provided by Maxwell equations. Their integral forms in free space are

$$\Gamma = \partial\Omega \qquad \int_\Gamma \varepsilon_0 \mathbf{E} \cdot \mathbf{n} \mathrm{d}A = \mathcal{Q}_\Omega, \qquad (C) \quad (5)$$

$$\Gamma = \partial\Omega \qquad \int_\Gamma \mathbf{B} \cdot \mathbf{n} \mathrm{d}A = 0, \qquad (Wb) \quad (6)$$

$$\mathrm{l} \text{ border of } \mathcal{S} \qquad \int_\mathrm{l} \mathbf{E} \cdot \boldsymbol{\tau} \mathrm{dl} = -\frac{\mathrm{d}}{\mathrm{dt}} \int_\mathcal{S} \mathbf{B} \cdot \mathbf{n} \mathrm{d}A, \qquad (V) \quad (7)$$

$$\mathrm{l} \text{ border of } \mathcal{S} \qquad \int_\mathrm{l} \frac{\mathbf{B}}{\mu_0} \cdot \boldsymbol{\tau} \mathrm{dl} = \mathcal{I}_\mathcal{S} + \frac{\mathrm{d}}{\mathrm{dt}} \int_\mathcal{S} \varepsilon_0 \mathbf{E} \cdot \mathbf{n} \mathrm{d}A, \qquad (A), \quad (8)$$

where $\mathcal{Q}_\Omega$ and $\mathcal{I}_\mathcal{S}$ denotes, respectively, the total charge in volume Ω and the total current flux through surface $\mathcal{S}$, and V (volts) is Joule over Coulomb.

Equation (5) and (6) are the Gauss laws for electric charge and magnetic field, respectively.

Equation (7) is Faraday's law and equation (8) is Ampére's law.

Constant ε_0 is called **electric permittivity** of the free space. Its value is approximately $\varepsilon_0 \simeq \dfrac{10^{-9}}{36\pi}$ in the MKS system, i.e., in C^2/Nm^2 or Faraday (F).

Constant μ_0 is called **magnetic permeability** of the free space. Its value is $\mu_0 = 4\pi 10^{-7}$ in the MKS system, i.e. in H/m.

4. Maxwell's equations in differential form in free space

In this section we deduce local or differential forms of Maxwell's equations from the integral ones given above.

4.1. Obtaining the partial differential equations.

For the sake of simplicity, let us assume that $\mathcal{Q}_\Omega$ and $\mathcal{I}_\mathcal{S}$ are given by

$$\mathcal{Q}_\Omega = \int_\Omega \rho_v \mathrm{d}V, \tag{9}$$

$$\mathcal{I}_\mathcal{S} = \int_\mathcal{S} \mathbf{J} \cdot \mathbf{n} \mathrm{d}A, \tag{10}$$

where ρ_v is the charge density and $\mathbf{J}$ is the current density.

Since the integral forms of Maxwell's equations must hold for any volume Ω and any surface $\mathcal{S}$, we can deduce local forms which are partial differential equations.

Indeed, by using the Gauss theorem in (5) and (6) we obtain

$$\int_\Omega (\varepsilon_0 \operatorname{div} \mathbf{E} - \rho_v)\, \mathrm{d}V = 0,$$

$$\int_\Omega \operatorname{div} \mathbf{B} \mathrm{d}V = 0,$$

and then,

$$\varepsilon_0 \operatorname{div} \mathbf{E} = \rho_v, \tag{11}$$

$$\operatorname{div} \mathbf{B} = 0. \tag{12}$$

Now, transforming (7) and (8) by Stokes theorem. We get

$$\int_{\mathcal{S}} [\mathbf{curl}\,\mathbf{E} \cdot \mathbf{n} + \frac{\partial \mathbf{B}}{\partial t} \cdot \mathbf{n}] \mathrm{d}A = 0,$$

and hence

$$\frac{\partial \mathbf{B}}{\partial t} + \mathbf{curl}\,\mathbf{E} = 0. \tag{13}$$

Similarly,

$$\int_{\mathcal{S}} \left[\frac{1}{\mu_0} \mathbf{curl}\, \mathbf{B} \cdot \mathbf{n} - \mathbf{J} \cdot \mathbf{n} - \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \cdot \mathbf{n} \right] \mathrm{d}A = 0,$$

and then

$$\varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} - \frac{1}{\mu_0} \mathbf{curl}\, \mathbf{B} = -\mathbf{J}. \tag{14}$$

4.2. Harmonic regime

Time harmonic fields $\mathbf{E}$ and $\mathbf{B}$ are generated whenever their charges and current sources have densities varying sinusoidally in time. More generally, let us assume that ρ_v and $\mathbf{J}$ are of the form,

$$\rho_v(x,t) = Re(e^{i\omega t}\widehat{\rho}_v(x)),$$
$$\mathbf{J}(x,t) = Re(e^{i\omega t}\widehat{\mathbf{J}}(x)),$$

where $\widehat{\rho}_v$ and $\widehat{\mathbf{J}}$ are complex valued fields independent of time.

Then the solutions of the Maxwell equations also have the form,

$$\mathbf{E}(x,t) = Re(e^{i\omega t}\widehat{\mathbf{E}}(x)),$$

$$\mathbf{B}(x,t) = Re(e^{i\omega t}\widehat{\mathbf{B}}(x)),$$

where, again, $\widehat{\mathbf{E}}$ and $\widehat{\mathbf{B}}$ are complex valued fields independent of time.

By replacing the above expressions in the differential forms of Maxwell's equations we obtain

$$\begin{aligned}
\operatorname{div}(\varepsilon_0 \widehat{\mathbf{E}}) &= \widehat{\rho}_v, && (15)\\
\operatorname{div} \widehat{\mathbf{B}} &= 0, && (16)\\
\mathbf{curl}\, \widehat{\mathbf{E}} &= -i\omega \widehat{\mathbf{B}}, && (17)\\
\mathbf{curl} \left(\frac{\widehat{\mathbf{B}}}{\mu_0} \right) &= -\widehat{\mathbf{J}} + i\omega\varepsilon_0 \widehat{\mathbf{E}}. && (18)
\end{aligned}$$

These equations are called Maxwell's equations for **harmonic regime** in the frequency domain. We notice that all fields are complex valued. However, they do not depend on the time variable.

5. Conductors and insulators

In terms of their charge conduction properties materials may be classified as **insulators** (or **dielectrics**), which possesses essentially no free electrons to provide currents under an electric field, and **conductors**, in which free, outer orbit electrons are readily available to produce a conduction current when an electric field is present.

5.1. Electrical conductivity. Ohm's Law

Many conductors exhibit a linear dependence of $\mathbf{J}$ on the applied electric field $\mathbf{E}$, namely,

$$\mathbf{J} = \sigma \mathbf{E}. \tag{19}$$

This is the Ohm's law. The factor σ is called the **electric conductivity** of the material.

In the MKS System σ units are $(\Omega\, m)^{-1}$ or mho/m $(\mho/m)$. The mho is also called siemens.

Electric conductivity depends on temperature. It is of the order of 10^8 $(\mho/m)$ for the best conductors at room temperature to 10^{-16} $(\mho/m)$ for the best insulators.

6. Electric polarization and electric displacement

- Under the action of an electric field, a microscopic distribution of electric dipoles arises in the dielectric.
- The density of these dipoles is denoted by $\mathbf{P}$ and called **density of dipole momentum** or **polarization vector**.
- Let us recall that the momentum of an electric dipole $\mathbf{m}$ has units $C \cdot m$. Hence $\mathbf{P}$ has units C/m^2.

- The electric field produced by the polarization is equivalent to the one created by the charge density

$$\rho_{\mathbf{P}} = -\operatorname{div}\mathbf{P} \quad (C/m^3).$$

Let us assume that a free charge of density ρ_v exists in the polarized material. According to the Gauss's law for electric charge,

$$\operatorname{div} \varepsilon_0 \mathbf{E} = \rho_v + \rho_{\mathbf{P}} = \rho_v - \operatorname{div} \mathbf{P}, \tag{20}$$

and then,

$$\operatorname{div}(\varepsilon_0 \mathbf{E} + \mathbf{P}) = \rho_v.$$

The vector field

$$\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P}, \tag{21}$$

is called **electric displacement**.

In terms of $\mathbf{D}$, the Gauss's law for electric charge (20) becomes

$$\operatorname{div} \mathbf{D} = \rho_v. \tag{22}$$

6.1. Electric susceptibility and electric permittivity

Experiments reveal that many dielectric substances are essentially linear, meaning that $\mathbf{P}$ is proportional to the applied electric field, $\mathbf{E}$. For such materials,

$$\mathbf{P} = \chi_{\mathrm{e}} \varepsilon_0 \mathbf{E}, \tag{23}$$

where parameter χ_{e} is called **electric susceptibility** of the dielectric. The factor ε_0 is retained to make χ_{e} dimensionless.

According to the definition of $\mathbf{D}$ we have

$$\mathbf{D} = (1 + \chi_{\mathrm{e}})\, \varepsilon_0 \mathbf{E}. \tag{24}$$

It is usual to denote $1 + \chi_{\mathrm{e}}$ by the dimensionless symbol

$$\varepsilon_{\mathrm{r}} = 1 + \chi_{\mathrm{e}}, \tag{25}$$

which is called **relative (electric) permittivity** of the region.

The **electric permittivity** of the material is defined by

$$\varepsilon = \varepsilon_{\mathrm{r}}\varepsilon_0 = (1 + \chi_{\mathrm{e}})\varepsilon_0 \quad (F/m). \tag{26}$$

Thus $\mathbf{D} = \varepsilon\mathbf{E}$.

7. Electric Gauss's law for materials

It is

$$\operatorname{div} \mathbf{D} = \rho_v, \tag{27}$$

in differential form, and in integral form

$$\int_\Gamma \mathbf{D} \cdot \mathbf{n} \mathrm{d}A = \mathcal{Q}_\Omega, \tag{28}$$

where Γ is the boundary of any regular volume Ω and $\mathcal{Q}_\Omega$ is the total charge enclosed in Ω.

8. Magnetic polarization for materials (magnetization)

- Magnetic materials are those that exhibit magnetic polarization when they are subjected to an applied magnetic field.
- The magnetization phenomenon is represented by the alignment of the magnetic dipoles of the material with the applied magnetic field.

- A magnetic material has a number of magnetic dipoles and thus many magnetic moments.
- In the absence of an applied magnetic field the magnetic dipoles and their corresponding electric loops are oriented in a random way so that, on a macroscopic scale, the vector sum of the magnetic moments is equal to zero.

- When the magnetic material is subjected to an applied magnetic field represented by the magnetic flux density $\mathbf{B}$, then a torque is exerted on the magnetic dipole given by $\mathbf{m} \times \mathbf{B}$.
- As a consequence, they will tend to align in the direction of $\mathbf{B}$. Thus the resultant magnetic field at every point in the material would be grater that its corresponding value at the same point when the material is absent.

The macroscopic density of magnetic dipoles is represented by a vector field $\mathbf{M}$ called **magnetization density**. Its units are $(\frac{Am^2}{m^3} = A/m)$.

Let us introduce the **magnetic intensity** field $\mathbf{H}$ defined by

$$\mathbf{H} = \frac{\mathbf{B}}{\mu_0} - \mathbf{M} \quad (A/m). \tag{29}$$

8.1. Magnetic susceptibility. Magnetic permeability.

For linear magnetic materials $\mathbf{M}$ is proportional to $\mathbf{H}$:

$$\mathbf{M} = \chi_{\mathrm{m}} \mathbf{H}, \tag{30}$$

where the dimensionless parameter χ_{m} is called **magnetic susceptibility** of the material.

Inserting this equality in the definition of $\mathbf{H}$ we have

$$\mathbf{H} = \frac{\mathbf{B}}{\mu_0} - \chi_{\mathrm{m}}\mathbf{H},$$

from which it follows that

$$\mathbf{B} = (1 + \chi_{\mathrm{m}})\mu_0\mathbf{H}.$$

The parameter

$$\mu_{\mathrm{r}} = 1 + \chi_{\mathrm{m}} \tag{31}$$

is called the **relative (magnetic) permeability** of the material.

Further, the **(magnetic) permeability** is defined by

$$\mu = \mu_r \mu_0, \tag{32}$$

so, finally,

$$\mathbf{B} = \mu \mathbf{H}. \tag{33}$$

9. Magnetic Gauss's law for materials

Recall that the electric Gauss's law for material was developed by adding the effect of electric polarization charge density to the corresponding law for free-space.

The magnetic Gauss's law for materials can be developed analogously. However, no additive term is required in this case because no free magnetic charges exist physically in any known material.

Thus $\mathbf{B}$ remains divergence-free in materials, that is,

$$\operatorname{div} \mathbf{B} = 0,$$

or

$$\int_{\Gamma} \mathbf{B} \cdot \mathbf{n} \mathrm{d}A = 0,$$

where $\Gamma = \partial\Omega$, for all regular domain Ω.

10. Ampère's law for materials

In free-space the $\mathbf{curl}$ of $\mathbf{B}/\mu_0$ has been expressed as the sum of a conduction current density $\mathbf{J}$ plus a displacement current density $\dfrac{\partial}{\partial t}(\varepsilon_0\mathbf{E})$ at any point, i.e.,

$$\mathbf{curl}\left(\frac{\mathbf{B}}{\mu_0}\right) = \mathbf{J} + \frac{\partial}{\partial t}(\varepsilon_0\mathbf{E}).$$

For materials two additional types of currents occur: $\mathbf{J_P} = \dfrac{\partial \mathbf{P}}{\partial t}$ and $\mathbf{J_M} = \mathbf{curl}\,\mathbf{M}$, arising from dielectric and magnetic polarization effects, respectively.

Adding these currents together leads to a revision of the Ampère's law for a material region, namely,

$$\mathbf{curl}\left(\frac{\mathbf{B}}{\mu_0}\right) = \mathbf{J} + \frac{\partial}{\partial t}(\varepsilon_0 \mathbf{E}) + \frac{\partial \mathbf{P}}{\partial t} + \mathbf{curl}\,\mathbf{M}.$$

Grouping the $\mathbf{curl}$ terms and the time derivative terms together we obtain

$$\mathbf{curl}\left(\frac{\mathbf{B}}{\mu_0} - \mathbf{M}\right) = \mathbf{J} + \frac{\partial}{\partial t}\left(\varepsilon_0 \mathbf{E} + \mathbf{P}\right).$$

Recalling that

$$\mathbf{H} := \frac{\mathbf{B}}{\mu_0} - \mathbf{M} \text{ and } \mathbf{D} := \varepsilon_0 \mathbf{E} + \mathbf{P},$$

we finally get

$$\frac{\partial \mathbf{D}}{\partial t} - \mathbf{curl}\,\mathbf{H} = -\mathbf{J}, \tag{34}$$

which is the Ampère's law for materials.

11. Maxwell's equations in differential forms for materials

We summarize the Maxwell's equations describing the electromagnetic behaviour of materials:

$$
\begin{aligned}
\operatorname{div} \mathbf{D} &= \rho_v, \\
\operatorname{div} \mathbf{B} &= 0, \\
\frac{\partial \mathbf{D}}{\partial t} - \mathbf{curl}\, \mathbf{H} &= -\mathbf{J}, \\
\frac{\partial \mathbf{B}}{\partial t} + \mathbf{curl}\, \mathbf{E} &= 0, \\
\mathbf{D} &= \varepsilon \mathbf{E}, \\
\mathbf{B} &= \mu \mathbf{H}, \\
\mathbf{J} &= \sigma \mathbf{E}.
\end{aligned}
$$

12. Properties of materials: homogeneity, isotropy, linearity.

A material having parameter μ, ε and σ independent of the position is termed **homogeneous**.

Conversely, if one or more of these parameters is space-dependent then the material is said **inhomogeneous**.

In some physical materials such as crystalline substances possessing a well-ordered atomic or molecular lattice , the polarization fields $\mathbf{P}$ or $\mathbf{M}$ resulting from the application of electric or magnetics fields may not necessarily have the same directions as the applied fields.

Such material are called **anisotropic**.

This fact can be modelled by taking tensors rather than scalars for the characteristic parameters ε, μ and σ. As an example, the constitutive law $\mathbf{D} = \varepsilon \mathbf{E}$ should be replaced by (in coordinates),

$$\begin{pmatrix} \mathrm{D}_1 \\ \mathrm{D}_2 \\ \mathrm{D}_3 \end{pmatrix} = \begin{pmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} \end{pmatrix} \begin{pmatrix} \mathrm{E}_1 \\ \mathrm{E}_2 \\ \mathrm{E}_3 \end{pmatrix}$$

Finally, a material is termed nonlinear if one or more of the parameters ε, μ or σ are dependent on the level of the applied fields. In that case one would write,

$$\begin{aligned}
\mathbf{D} &= \varepsilon(|\mathbf{E}|)\mathbf{E}, \\
\mathbf{B} &= \mu(|\mathbf{H}|)\mathbf{H}, \\
\mathbf{J} &= \sigma(|\mathbf{E}|)\mathbf{E}.
\end{aligned}$$

13. Electromagnetic energy

$$\frac{1}{2}\int_{\mathcal{E}} \mathbf{D}(x,t_1)\cdot\mathbf{E}(x,t_1)\,\mathrm{d}V_x + \int_{\mathcal{E}} \mathbf{B}(x,t_1)\cdot\mathbf{H}(x,t_1)\,\mathrm{d}V_x$$

$$= \frac{1}{2}\int_{\mathcal{E}} \mathbf{D}(x,t_2)\cdot\mathbf{E}(x,t_2)\,\mathrm{d}V_x + \int_{\mathcal{E}} \mathbf{B}(x,t_2)\cdot\mathbf{H}(x,t_2)\,\mathrm{d}V_x$$

$$+ \int_{t_1}^{t_2}\int_{\mathcal{E}} \mathbf{J}\cdot\mathbf{E}\,\mathrm{d}V_x\,\mathrm{d}t = 0. \qquad (35)$$

Terms

$$\mathfrak{E}_{\mathrm{E}}(t) = \frac{1}{2}\int_{\mathcal{E}} \mathbf{D}(x,t)\cdot\mathbf{E}(x,t)\,\mathrm{d}V_x$$

and

$$\mathfrak{E}_{\mathrm{M}}(t) = \frac{1}{2}\int_{\mathcal{E}} \mathbf{B}(x,t)\cdot\mathbf{H}(x,t)\,\mathrm{d}V_x$$

represents the electric and magnetic energy at time t, so the equality (35) can be rewritten as

$$\mathfrak{E}(t_1) = \mathfrak{E}(t_2) + \int_{t_1}^{t_2}\int_{\mathcal{E}} \mathbf{J}\cdot\mathbf{E}\,\mathrm{d}V_x\,\mathrm{d}t,$$

where $\mathfrak{E}(t) = \mathfrak{E}_{\mathrm{E}}(t) + \mathfrak{E}_{\mathrm{M}}(t)$ is the energy of the electromagnetic field at time t.

- This equality say that the electromagnetic energy is conserved as far as no conductors exist in the space.
- Otherwise, it says that the electromagnetic energy decreases along time due to the (increasing with time) dissipation term

$$\int_{t_1}^{t_2} \int_{\mathcal{E}} \mathbf{J} \cdot \mathbf{E} \, \mathrm{d}V_x \, \mathrm{d}t$$

representing the electromagnetic energy transformed in heat by the conductors existing in the space (*Joule effect*).

13.1. Energy balance in a bounded domain

- $$\mathfrak{E}^{\Omega}(t_1) = \mathfrak{E}^{\Omega}(t_2) + \int_{t_1}^{t_2} \int_{\mathcal{E}} \mathbf{J} \cdot \mathbf{E} \, \mathrm{d}V_x \, \mathrm{d}t + \int_{t_1}^{t_2} \int_{\Gamma} \mathbf{E} \times \mathbf{H} \cdot \mathbf{n} \, \mathrm{d}A_x \, \mathrm{d}t,$$

  where $\mathfrak{E}^{\Omega}(t)$ denotes the electromagnetic energy in domain Ω at time t.

- Vector field $\mathcal{P} := \mathbf{E} \times \mathbf{H}$ is called the *Poynting vector*.

- The last term in this equality represents the outgoing electromagnetic flux from Ω through boundary Γ along time interval (t_1, t_2).

14. Maxwell's equations for electrostatics

In electrostatics charges do not move, so currents do not exist and then magnetic field is null. In this situation Maxwell's equations becomes,

$$\left.\begin{aligned} \operatorname{div}\mathbf{D} &= \rho_v, \\ \mathbf{curl}\,\mathbf{E} &= 0, \end{aligned}\right\} \tag{36}$$

with $\mathbf{D} = \varepsilon\mathbf{E}$.

14.1. Electrostatic potential

As in empty space, the electric fields is curl-free which implies existence of a potential fields V such that,,

$$\mathbf{E} = -\nabla \mathrm{V}.$$

By replacing in the Gauss's law for electric charge we get,

$$-\operatorname{div}(\varepsilon \nabla \mathrm{V}) = \rho_v, \tag{37}$$

which is an elliptic partial differential equation similar to Poisson's equation.

The electric permittivity needs not to be constat in the whole space but if this is the case the equation [37] is equivalent to

$$-\Delta \mathrm{V} = -\frac{\rho_v}{\varepsilon},$$

which is a Poisson's equation similar to that obtained for empty space.

Let us recall that, in general, the solution of the problem

$$\left.\begin{aligned} -\Delta \mathrm{V} &= \mathrm{S}, \\ \mathrm{V}(x) \to 0 \quad \text{as} \quad r(x) &\to \infty, \end{aligned}\right\} \tag{38}$$

is the distribution obtained by convolution,

$$\mathrm{V} = \mathrm{T} * \mathrm{S},$$

where T is the fundamental solution, i.e., $\mathrm{T} = \mathrm{T_v}$ with

$$\left.\begin{aligned} -\Delta \mathrm{V} &= \delta_{\mathrm{o}}, \\ \mathrm{V}(x) \to 0 \quad &\text{as} \quad r(x) \to \infty. \end{aligned}\right\}$$

If the distribution S is defined through a compact support function ρ_v/ε, i.e.,

$$\mathrm{S}(\psi) = \int_{\mathcal{E}} \frac{\rho_v(x)}{\varepsilon} \psi(x) \mathrm{d}V_x,$$

then V is defined by a $\mathrm{L}^1_{\mathrm{loc}}(\mathcal{E})$ function given by (see subsection **??**)

$$\mathrm{V}(x) = \int_{\mathcal{E}} \frac{1}{4\pi\varepsilon} \frac{\rho_v(y)}{|x-y|} \mathrm{d}V_y.$$

Also in this case the electric field $\mathbf{E} = -\nabla \mathrm{V}$ is given by

$$\mathbf{E}(x) = \frac{1}{4\pi\varepsilon} \int_{\mathcal{E}} \rho_v(y) \frac{x-y}{|x-y|^3} \mathrm{d}V_y.$$

14.2. Electric field created by a set of charged conductors. Capacitance matrix

This is a case where we do not need to know the charge distribution in order to determine the electric field.

- Let us consider N bounded conductors $\Omega_1, \ldots, \Omega_\mathrm{N}$ having total charges $\mathrm{Q}_1, \ldots, \mathrm{Q}_\mathrm{N}$, respectively.
- Let us assume that the complementary set $\Omega = \mathcal{E} \setminus (\Omega_1 \cup \ldots \Omega_\mathrm{N})$ is filled with a dielectric with electric permittivity, $\varepsilon(x)$, which can be dependent on the position (non-homogenous media).

According to the local form of the Gauss' law, we have

$$-\operatorname{div}(\varepsilon \nabla \mathrm{V}) = 0 \text{ in } \Omega,$$

because the dielectric is assumed to be charge-free.

In order to get a well-posed problem we need to write boundary conditions on boundaries $\Gamma_1, \ldots, \Gamma_{\mathrm{N}}$ of $\Omega_1, \ldots, \Omega_{\mathrm{N}}$, respectively.

For this we observe that, in electrostatics, the electric field in conductors must be null because otherwise we would have an electric current different from zero by Ohm's law.

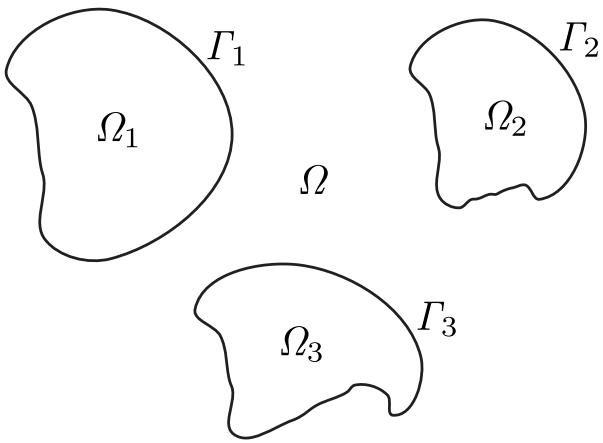


Figura 1: The domain

Furthermore, $\mathbf{E} = 0$ in Ω_i implies the potential is constant in each conductor. Let $\mathrm{V} = \mathrm{V}_i$ in Ω_i, for $i = 1, \ldots, \mathrm{N}$.

If these potentials were known, then they could be Dirichlet data for the above Poisson's equation and we were lead to solve

$$\left.\begin{aligned} -\operatorname{div}(\varepsilon \nabla \mathrm{V}) = 0 \quad &\text{in} \quad \Omega, \\ \mathrm{V} = \mathrm{V}_i \quad &\text{on} \quad \Gamma_i, \\ \mathrm{V}(x) \to 0 \quad &\text{as} \quad r(x) \to \infty. \end{aligned}\right\} \tag{39}$$

Moreover, since $\mathbf{E} = 0$ in Ω_i, then $\mathrm{div}(\varepsilon\mathbf{E}) = 0$ in Ω_i so the charge density must be null in the interior of each conductor. This implies that the charge must be concentrated on the surface Γ_i and surface density, ρ_{Γ_i}, may depend on the particular point on Γ_i. Furthermore,

$$\varepsilon^+\mathbf{E}^+ \cdot \mathbf{n}^+ + \varepsilon^-\mathbf{E}^- \cdot \mathbf{n}^- = -\rho_{\Gamma_i}.$$

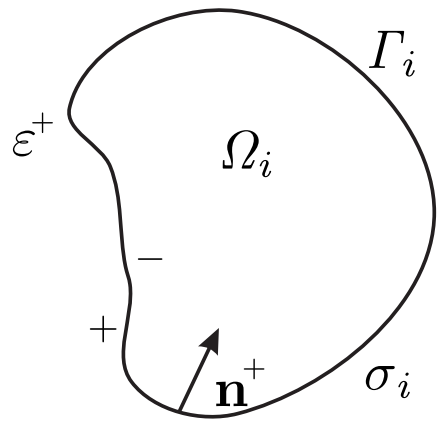


Figura 2: Conductor Ω_i

Let the region "minus" be the interior of Ω_i and the region "plus" be its complementary set. Then the previous equation yields

$$\varepsilon^+ \mathbf{E}^+ \cdot \mathbf{n}^+ = -\rho_{\Gamma_i},$$

or

$$\varepsilon^+ \frac{\partial \mathrm{V}^+}{\partial \mathbf{n}^+} = \rho_{\Gamma_i}.$$

This is a Neumann boundary condition for the Poisson equation.

The main difficulty is that the σ_i are not known (recall that the data are charges $\mathrm{Q}_1, \ldots, \mathrm{Q}_\mathrm{N}$). Thus, in order to solve the problem we observe that the mapping giving the total charges from the potentials is linear:

$$\begin{array}{ccc} \mathbb{R}^\mathrm{N} & \longrightarrow & \mathbb{R}^\mathrm{N} \\ (\mathrm{V}_1, \ldots, \mathrm{V}_\mathrm{N}) & \longrightarrow & \left(\mathrm{Q}_1 = \displaystyle\int_{\Gamma_1} \varepsilon \frac{\partial \mathrm{V}}{\partial \mathrm{n}} \mathrm{d}A_x, \ldots, \mathrm{Q}_\mathrm{N} = \int_{\Gamma_\mathrm{N}} \varepsilon \frac{\partial \mathrm{V}}{\partial \mathrm{n}} \mathrm{d}A_x \right), \end{array}$$

where V is the solution of problem (39).

Then we can find a matrix $\mathcal{C}$ such that

$$\mathcal{C}\overrightarrow{\mathrm{V}} = \overrightarrow{\mathrm{Q}},$$

where

$$\overrightarrow{\mathrm{V}} = \begin{pmatrix} \mathrm{V}_1 \\ \vdots \\ \mathrm{V}_\mathrm{N} \end{pmatrix}$$

and

$$\overrightarrow{\mathrm{Q}} = \begin{pmatrix} \mathrm{Q}_1 \\ \vdots \\ \mathrm{Q}_\mathrm{N} \end{pmatrix}.$$

We notice that the columns of $\mathcal{C}$ are simply the charge vectors corresponding to potentials $(1, 0, \ldots, 0)$, $(0, 1, 0, \ldots, 0)$, $(0, 0 \ldots, 0, 1)$ so it can be easily computed by solving N Poisson problems with the same differential operator.

Matrix $\mathcal{C}$ does not depend on the particular electric charges. It only depends on ε and the geometry of conductors. It is called **capacitance matrix** and its entries have units coulomb per volt or *farad*.

Moreover, once $\mathcal{C}$ is known, the problem of determining the potentials in the conductors, and thereby the electric field in the dielectrics, is simply solved by taking the inverse matrix of $\mathcal{C}$. Indeed,

$$\overrightarrow{\mathrm{V}} = \mathcal{C}^{-1}\overrightarrow{\mathrm{Q}}.$$

Direct current

In this case, sources, i.e., **charges and currents** do not depend on time but, unlike electrostatics, currents are not null.

Since all fields must be independent on time, Maxwell's equations becomes,

$$\operatorname{div}\mathbf{D} = \rho, \qquad \operatorname{div}\mathbf{B} = 0,$$
$$\mathbf{curl}\,\mathbf{E} = 0, \qquad \mathbf{curl}\,\mathbf{H} = \mathbf{J},$$

with, $\mathbf{D} = \varepsilon\mathbf{E}$, $\mathbf{B} = \mu\mathbf{H}$ and $\mathbf{J} = \sigma\mathbf{E}$.

The interesting feature of this case is that we can compute first the electric field and the current density, and then the magnetic field.

For the former we notice that, by taking the divergence, Ampère's law yields

$$\operatorname{div} \mathbf{J} = 0, \tag{46}$$

and using Ohm's law,

$$\operatorname{div}(\sigma \mathbf{E}) = 0. \tag{47}$$

15. Electric potential. Boundary conditions

Moreover $\mathbf{curl}\,\mathbf{E} = 0$ implies the existence of an electric potential V such that $\mathbf{E} = -\nabla\mathrm{V}$. By replacing in (46) we get

$$-\operatorname{div}(\sigma\nabla\mathrm{V}) = 0,$$

which is a Laplace-like partial differential equation similar to the one in electrostatics.

In order to solve this equation, which is only valid in conductors (otherwise $\sigma = 0$), we consider boundary conditions.

They can be either Dirichlet or Neumann conditions:

- Dirichlet: $\mathrm{V} = \mathrm{V_d}$ on Γ_d,
- Neumann: $-\sigma \dfrac{\partial \mathrm{V}}{\partial \mathrm{n}} = \sigma \mathbf{E} \cdot \mathbf{n} = \mathbf{J} \cdot \mathbf{n} = j_\mathrm{n}$ on Γ_n.

16. Numerical solution

The weak formulation of the above boundary value problem can be obtained by standard procedures. It reads as follows:

To find $\mathrm{V} \in \mathrm{H}^1(\Omega)$ *with* $\mathrm{V} = \mathrm{V_d}$ *on* Γ_d *such that*

$$\int_\Omega \sigma \nabla \mathrm{V} \cdot \nabla z \mathrm{d}V_x = -\int_{\Gamma_\mathrm{n}} j_\mathrm{n} z \mathrm{d}A_x \quad \forall z \in \mathrm{H}^1(\Omega) \text{ with } z|_{\Gamma_d} = 0.$$

This weak formulation can be discretized by using a finite element approximation of the Sobolev space $\mathrm{H}^1(\Omega)$.

This approximation can be done from a tetrahedral mesh, $\mathfrak{T}_h$, h being the mesh-size. Let us consider the finite-dimensional space

$$\mathcal{V}_h = \left\{ z \in \mathrm{C}^0(\Omega) : \; z|_{\mathrm{K}} \in \mathrm{P}_1 \; \forall \mathrm{K} \in \mathfrak{T}_h \right\},$$

where P_1 denotes the space of polynomials of degree less or equal than one.

The approximated problem is defined by

To find $\mathrm{V}_h \in \mathcal{V}_h$ *such that* $\mathrm{V}_h(p) = \mathrm{V}_\mathrm{d}(p)\ \forall p$ *vertex in* Γ_d *and*

$$\int_\Omega \sigma \nabla \mathrm{V}_h \cdot \nabla z_h \mathrm{d}V_x = -\int_{\Gamma_\mathrm{N}} j_\mathrm{d} z_h \mathrm{d}A_x,$$

$\forall z_h \in \mathcal{V}_h$ such that $z_h(p) = 0\ \forall p$ vertex in Γ_d.

17. Conductance matrix. Resistance matrix

Let us consider a conducting domain Ω with boundary Γ. Let $\Gamma_0, \ldots, \Gamma_{\mathrm{N}}$ the ports of Ω, i.e. the connected subsets of Γ through which direct currents enter or leave Ω.

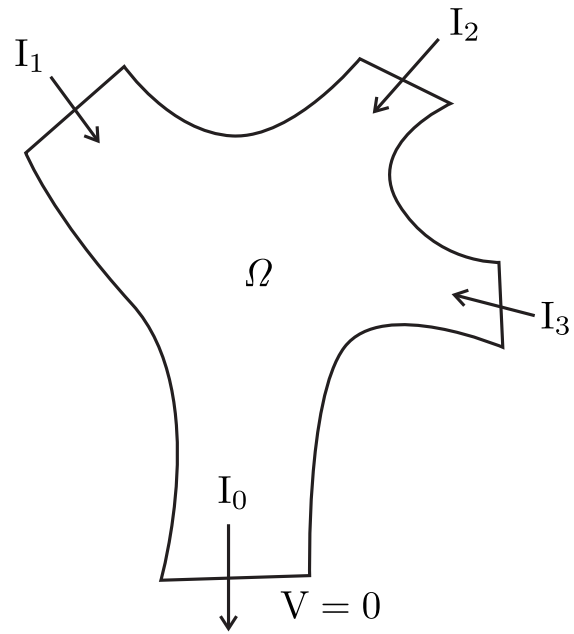


Figura 3: Domain

We assume that no current flux exists across the rest of the boundary $\Gamma_{\mathrm{n}} = \Gamma - (\Gamma_0 \cup \ldots \cup \Gamma_{\mathrm{N}})$, i.e.

This assumption leads to the boundary condition

$$\sigma \frac{\partial \mathrm{V}}{\partial \mathrm{n}} = -\mathbf{J} \cdot \mathbf{n} = 0 \text{ on } \Gamma_{\mathrm{n}}.$$

We also suppose that current enters or leaves the domain Ω perpendicularly to the boundary (i.e., $\mathbf{J} \times \mathbf{n} = 0$) which yields

$$\mathbf{E} \times \mathbf{n} = 0 \quad \text{on } \Gamma_0 \cup \ldots \cup \Gamma_{\mathrm{N}}.$$

Since $\mathbf{E} = -\nabla \mathrm{V}$, condition $\mathbf{E} \times \mathbf{n} = 0$ on $\Gamma_0 \cup \ldots \cup \Gamma_\mathrm{N}$ gives the Dirichlet boundary conditions

$$\mathrm{V} = \mathrm{V}_i \text{ (constant) on } \Gamma_i, \; i = 0, \ldots, \mathrm{N},$$

because each Γ_i is connected.

If we know the values V_i, $i = 1, \ldots, \mathrm{N}$, we can state the well-posed boundary-value problem

$$\left.\begin{aligned} -\operatorname{div}(\sigma \nabla \mathrm{V}) &= 0, \\ \sigma \frac{\partial \mathrm{V}}{\partial \mathrm{n}} &= 0 \qquad \text{on } \Gamma_{\mathrm{n}}, \\ \mathrm{V} &= \mathrm{V}_i \qquad \text{on } \Gamma_i,\ i = 0, \ldots, \mathrm{N}, \end{aligned}\right\} \tag{48}$$

which can be solved by numerical methods.

However, there are many cases where we know the intensities through each Γ_i instead of the potentials.

Firstly, we notice that once problem (48) has been solved, intensities entering the domain are given by

$$\mathrm{I}_i = -\int_{\Gamma_i} \mathbf{J} \cdot \mathbf{n} \mathrm{d}A = \int_{\Gamma_i} \sigma \frac{\partial \mathrm{V}}{\partial \mathrm{n}} \mathrm{d}A, \quad i = 1, \ldots, \mathrm{N}.$$

Moreover,

$$\sum_{i=0}^{\mathrm{N}} -\int_{\Gamma_i} \mathbf{J}\cdot\mathbf{n}\mathrm{d}A = -\int_{\Gamma} \mathbf{J}\cdot\mathbf{n}\mathrm{d}A = -\int_{\Omega} \mathrm{div}\,\mathbf{J}\mathrm{d}V = 0.$$

Furthermore, since V is only defined up to a constant, we can choose $\mathrm{V}_0 = 0$, i.e.,

$$\mathrm{V}(x) = 0 \text{ on } \Gamma_0.$$

Let us consider the mapping giving intensities from potentials, i.e.,

$$(\mathrm{V}_1, \ldots, \mathrm{V}_\mathrm{N}) \in \mathbb{R}^\mathrm{N} \longrightarrow (\mathrm{I}_1, \ldots, \mathrm{I}_\mathrm{N}) \in \mathbb{R}^\mathrm{N}, \tag{49}$$

with

$$\mathrm{I}_i = \int_{\Gamma_i} \sigma \frac{\partial \mathrm{V}}{\partial \mathrm{n}} \mathrm{d}A, \; i = 1, \ldots, \mathrm{N},$$

and V being the solution of (48). Let

$$\mathrm{I}_0 = -\sum_{i=1}^{\mathrm{N}} \mathrm{I}_i.$$

Since this mapping is linear, there exists a matrix $\mathcal{G}$, called **conductance matrix**, such that

$$\mathcal{G}\overrightarrow{\mathrm{V}} = \overrightarrow{\mathrm{I}},$$

where

$$\overrightarrow{\mathrm{V}} = \begin{pmatrix} \mathrm{V}_1 \\ \vdots \\ \mathrm{V}_\mathrm{N} \end{pmatrix} \quad \text{and} \quad \overrightarrow{\mathrm{I}} = \begin{pmatrix} \mathrm{I}_1 \\ \vdots \\ \mathrm{I}_\mathrm{N} \end{pmatrix}.$$

The inverse of $\mathcal{G}$, denoted by $\mathcal{R}$, is called **resistance matrix**. Since we have

$$\overrightarrow{\mathrm{V}} = \mathcal{G}^{-1}\overrightarrow{\mathrm{I}} = \mathcal{R}\,\overrightarrow{\mathrm{I}},$$

this matrix allows us to obtain potentials from intensities.

Similar to capacitance matrix in electrostatics, the conductance matrix can be obtained by solving N problems like (48). Indeed, the k-th column of $\mathcal{G}$ is the vector of intensities corresponding to the potentials $\mathrm{V}_i = \delta_{ik}$ on $\Gamma_i,\ i = 1, \ldots, \mathrm{N}$.

18. Maxwell's equations for magnetostatics

If we know the static current density $\mathbf{J}$, the problem of determining the corresponding magnetic field is called the magnetostatic problem.

Equations for magnetostatics are

$$\mathbf{curl}\,\mathbf{H} = \mathbf{J}, \tag{50}$$

$$\mathrm{div}\,\mathbf{B} = 0, \tag{51}$$

$$\mathbf{B} = \mu\mathbf{H}. \tag{52}$$

19. Vector magnetic potential

Since $\operatorname{div} \mathbf{B} = 0$, there exists a vector field $\mathbf{A}$ such that

$$\mathbf{B} = \mathbf{curl}\, \mathbf{A}.$$

Actually, there exist many of such vector fields.

Indeed, if $\mathbf{curl}\, \mathbf{A} = \mathbf{B}$, then

$$\mathbf{curl}(\mathbf{A} + \nabla\varphi) = \mathbf{B}$$

for all scalar field φ.

In order to uniquely determine $\mathbf{A}$ a so-called *gauge condition* should be added.

An example is the Coulomb's gauge:

$$\operatorname{div} \mathbf{A} = 0.$$

Moreover, by using the constitutive law $\mathbf{B} = \mu \mathbf{H}$, Ampère's law yields

$$\mathbf{curl}\left(\frac{1}{\mu}\,\mathbf{curl}\,\mathbf{A}\right) = \mathbf{J}.$$

Now we have to solve this equation together with the gauge condition.

Things become much simpler if μ is constant in the whole space, let say, $\mu = \mu_0$. In this case we have,

$$\mathbf{curl}(\mathbf{curl}\,\mathbf{A}) = \mu_0 \mathbf{J}. \tag{53}$$

By adding the term $\nabla(\mathrm{div}\,\mathbf{A})$ (which is null by the Coulomb's gauge) to the left-hand side, and using the vector equality,

$$-\Delta\mathbf{A} = \mathbf{curl}\,\mathbf{curl}\,\mathbf{A} - \nabla(\mathrm{div}\,\mathbf{A}),$$

equation (53) yields

$$-\Delta\mathbf{A} = \mu_0\mathbf{J}.$$

In a fixed cartesian system of coordinates, this equation is equivalent to

$$-\Delta \mathrm{A}_i = \mu_0 \mathbf{J}_i, \ i = 1, 2, 3. \tag{54}$$

Then we can use the fundamental solution of the Poisson's equation to write the solutions of (54) by convolution with their right-hand sides. We have

$$\mathrm{A}_i(x) = \int_\Omega \frac{\mu_0 \mathrm{J}_i(y)}{4\pi|x-y|} \mathrm{d}V_y, \ i = 1, 2, 3,$$

and hence,

$$\mathbf{A}(x) = \int_\Omega \frac{\mu_0 \mathbf{J}(y)}{4\pi|x-y|} \mathrm{d}V_y \ \ (Wb/m). \tag{55}$$

Remark 19.1 *In the previous integrals Ω denotes any bounded domain in the affine space containing the support of* $\mathbf{J}$.

Of course, $\mathbf{J}$ *may also be a distribution supported on a surface* $\mathcal{S}$ *or on a line* l *in which case* Ω *should be replaced in the above formula by* $\mathcal{S}$ *or* l*, respectively.*

By taking the $\mathbf{curl}$ in (55), we get

$$\begin{aligned}\mathbf{B}(x) = \mathbf{curl}_x\, \mathbf{A}(x) &= \mathbf{curl}_x \int_\Omega \frac{\mu_0 \mathbf{J}(y)}{4\pi|x-y|} \mathrm{d}V_y \\ &= \int_\Omega \frac{\mu_0}{4\pi}\, \mathbf{curl}_x \left(\frac{\mathbf{J}(y)}{|x-y|} \right) \mathrm{d}V_y.\end{aligned}$$

But $\mathbf{curl}(\phi\mathbf{u}) = \nabla\phi \times \mathbf{u} + \phi\,\mathbf{curl}\,\mathbf{u}$. In our case $\mathbf{J}(y)$ does not depend on x, so

$$\mathbf{curl}_x\left(\frac{\mathbf{J}(y)}{|x-y|}\right) = \nabla_x\left(\frac{1}{|x-y|}\right) \times \mathbf{J}(y) = -\frac{x-y}{|x-y|^3} \times \mathbf{J}(y).$$

Hence,

$$\mathbf{B}(x) = \int_\Omega \frac{\mu_0}{4\pi} \frac{\mathbf{J}(y) \times (x-y)}{|x-y|^3} \mathrm{d}V_y. \tag{56}$$

This integral for $\mathbf{B}$, expressed directly in terms of the static current distribution $\mathbf{J}$ in free space, is known as the *Biot-Savart law*.

In the general case, when μ is not constant, introducing the gauge condition and solving the problem is more difficult. A way to do it is described below.

For the sake of simplicity, let us suppose that Ω is a bounded domain containing the support of the current density $\mathbf{J}$ and such that we may assume the boundary condition

$$\mathbf{H} \times \mathbf{n} = \mathbf{0} \text{ in } \Gamma = \partial\Omega \tag{57}$$

is satisfied.

Firstly, we notice that $\operatorname{div} \mathbf{B} = 0$ implies

$$\int_\Gamma \mathbf{B} \cdot \mathbf{n} \, \mathrm{d}A = 0,$$

by using Gauss's theorem.

Let us assume Ω is connected and simply connected. Then Theorem 3.5 in the book by Girault and Raviart affirms that, if $\mathbf{B} \in \mathbf{L}^2(\Omega)$, there exists a unique vector field $\mathbf{A} \in \mathbf{H}(\mathbf{curl}, \Omega) \cap \mathbf{H}(\operatorname{div}, \Omega)$ such that

$$\mathbf{curl}\, \mathbf{A} = \mathbf{B} \text{ in } \Omega, \tag{58}$$

$$\operatorname{div} \mathbf{A} = 0 \text{ in } \Omega, \tag{59}$$

$$\mathbf{A} \cdot \mathbf{n} = 0 \text{ on } \Gamma. \tag{60}$$

Furthermore, if Ω is smooth (in particular $\mathrm{C}^{1,1}$) then $\mathbf{A} \in \mathbf{H}^1(\Omega)$.

By using Ampère's law we have

$$\mathbf{curl}(\frac{1}{\mu}\,\mathbf{curl}\,\mathbf{A}) = \mathbf{J}, \tag{61}$$

and boundary condition (57) becomes

$$\frac{1}{\mu}\,\mathbf{curl}\,\mathbf{A} \times \mathbf{n} = \mathbf{0} \text{ on } \Gamma. \tag{62}$$

One can show that problem (61), (62) together with (59) and (60) has a unique solution $\mathbf{A} \in \mathbf{H}(\mathbf{curl}, \Omega) \cap \mathbf{H}(\mathrm{div}, \Omega)$.

Let us consider the variational problem:

$$
\begin{aligned}
&\textit{Find} \;\; \mathbf{A} \in \mathfrak{V} \textit{ such that} \\
&\int_\Omega \frac{1}{\mu} \mathbf{curl}\, \mathbf{A} \cdot \mathbf{curl}\, \boldsymbol{\phi} \, \mathrm{d}V + \int_\Omega \mathrm{div}\, \mathbf{A} \; \mathrm{div}\, \boldsymbol{\phi} \, \mathrm{d}V = \int_\Omega \mathbf{J} \cdot \boldsymbol{\phi} \, \mathrm{d}V \;\; \forall \boldsymbol{\phi} \in \mathfrak{V}, \\
&\textit{where } \mathfrak{V} \textit{ is the functional space,} \\
&\mathfrak{V} = \left\{ \mathbf{A} \in \mathbf{H}(\mathbf{curl}, \Omega) \cap \mathbf{H}(\mathrm{div}, \Omega) / \mathbf{A} \cdot \mathbf{n} = 0 \text{ on } \Gamma \right\}.
\end{aligned}
\tag{63}
$$

Since the bilinear form,

$$\mathrm{a}(\mathbf{A}, \boldsymbol{\phi}) = \int_\Omega \frac{1}{\mu} \, \mathbf{curl}\, \mathbf{A} \cdot \mathbf{curl}\, \boldsymbol{\phi} \, \mathrm{d}V + \int_\Omega \mathrm{div}\, \mathbf{A} \; \mathrm{div}\, \boldsymbol{\phi} \, \mathrm{d}V,$$

is continuous and coercive in $\mathfrak{V}$, the above problem has unique solution $\mathbf{A}$.

If Ω is smooth, then $\mathfrak{V} \subset \mathbf{H}^1(\Omega)$ and one can use continuous piecewise linear finite elements to discretize each component of $\mathbf{A}$.

Otherwise, if the boundary of Ω is only Lipschitz-continuous, then $\mathbf{A}$ is singular at reentrant corners and these standard Lagrangian finite elements do not converge. This drawback can be overcomed by adding a singular space of approximation as it it done by A. S. Bonnet, P. Ciarlet Jr. and co-workers.

Another alternative consists of using edge Nédélec finite elements for the mixed formulation:

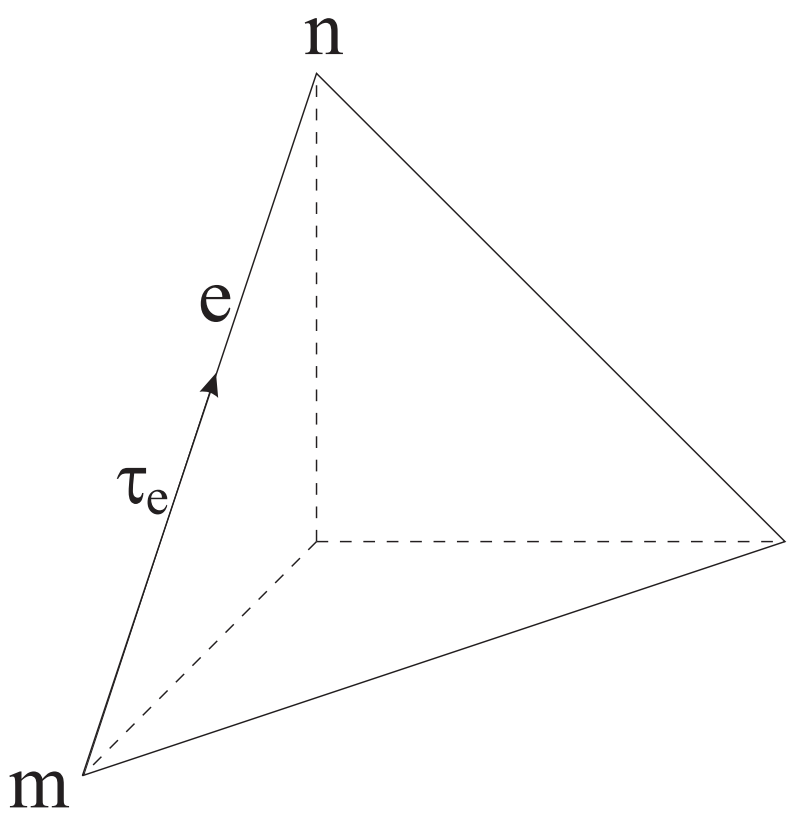
Find $\mathbf{A} \in \mathfrak{W}$ *and* $\varphi \in \mathfrak{H}$ *such that*

$$\int_\Omega \frac{1}{\mu}\,\mathbf{curl}\,\mathbf{A}\cdot\mathbf{curl}\,\boldsymbol{\phi}\,\mathrm{d}V + \int_\Omega \nabla\psi\cdot\boldsymbol{\phi}\,\mathrm{d}V = \int_\Omega \mathbf{J}\cdot\boldsymbol{\phi}\,\mathrm{d}V \;\forall\boldsymbol{\phi}\in\mathfrak{W},$$

$$\int_\Omega \mathbf{A}\cdot\nabla\varphi\,\mathrm{d}V = 0 \;\forall\varphi\in\mathfrak{H},$$

$$\textit{where } \mathfrak{W} = \mathbf{H}(\mathbf{curl},\Omega) \textit{ and } \mathfrak{H} = \left\{\varphi\in\mathrm{H}^1(\Omega):\ \int_\Omega \varphi\,\mathrm{d}V = 0\right\}. \tag{64}$$

This formulation can be discretized by using edge Nédélec finite elements to approximate $\mathbf{A}$ and piecewise linear continuos finite elements to approximate ψ. Moreover, by taking $\boldsymbol{\phi} = \nabla\psi$ in the first equation it is easy to see that $\psi \equiv 0$ in Ω.


n
e
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m

- Lowest-order *Nédélec* tetrahedral edge elements for the magnetic field $\mathbf{H}_h$:

$$\boldsymbol{\mathcal{X}}_h := \{\mathbf{G}_h \in \mathrm{H}(\mathbf{rot}, \Omega) : \ \mathbf{G}_h|_T \in \boldsymbol{\mathcal{N}}(T) \ \forall T \in \mathcal{T}_h\},$$

$$\mathbf{G}_h|_T \in \boldsymbol{\mathcal{N}}(T) \quad \text{iff} \quad \mathbf{G}_h(\mathbf{x}) = \mathbf{a} \times \mathbf{x} + \mathbf{b}, \quad \mathbf{a}, \mathbf{b} \in \mathrm{C}^3.$$

  - Basis functions:

$$\mathbf{\Phi_e} = \lambda_m \nabla \lambda_n - \lambda_n \nabla \lambda_m.$$

  - $\mathbf{curl}\, \mathbf{\Phi_e} = 2\, \nabla \lambda_m \times \nabla \lambda_n$.
  - $\mathbf{H}_h \cdot \boldsymbol{\tau}_\mathbf{e}$ continuous.

20. Scalar magnetic potential

Another technique to solve magnetostatic problems consists in introducing a scalar potential. We recall that we have to solve the system

$$\left.\begin{aligned} \mathbf{curl}\,\mathbf{H} &= \mathbf{J}, \\ \operatorname{div}\mathbf{B} &= 0, \\ \mathbf{B} &= \mu\mathbf{H}. \end{aligned}\right\}$$

Let $\boldsymbol{\mathcal{H}}$ be a vector field such that $\mathbf{curl}\,\boldsymbol{\mathcal{H}} = \mathbf{J}$. Notice that $\boldsymbol{\mathcal{H}}$ can be obtained, for instance, by the Biot and Savart's law, namely,

$$\boldsymbol{\mathcal{H}}(x) = \int_\Omega \frac{1}{4\pi} \frac{\mathbf{J}(y) \times (x-y)}{|x-y|^3} \mathrm{d}V_y. \tag{65}$$

In general, $\mathbf{H}(x) \neq \boldsymbol{\mathcal{H}}(x)$ because $\mathrm{div}(\mu\boldsymbol{\mathcal{H}})$ need not to be null. However, $\mathbf{curl}\,\mathbf{H} = \mathbf{curl}\,\boldsymbol{\mathcal{H}}$ and then there exists a scalar field φ_{R} such that

$$\mathbf{H} = \boldsymbol{\mathcal{H}} - \nabla\varphi_{\mathrm{R}},$$

which is called the **reduced scalar magnetic potential**.

Then we seek for φ_{R} such that,

$$\operatorname{div}(\mu(\boldsymbol{\mathcal{H}} - \nabla\varphi_{\mathrm{R}})) = 0,$$

or

$$-\operatorname{div}(\mu\nabla\varphi_{\mathrm{R}}) = -\operatorname{div}(\mu\boldsymbol{\mathcal{H}}) \text{ in } \mathcal{E},$$

and satisfying $\varphi_{\mathrm{R}} \to 0$ at infinity.

The inconvenient of this method is that it suffers from the so called *cancellation error* because the two terms $\boldsymbol{\mathcal{H}}$ and $\nabla\varphi_{\mathrm{R}}$ are of the same order of magnitude and opposite direction in the magnetic materials.

Now we present an alternative approach.

Firstly, we split the affine space $\mathcal{E}$ into three sub-domains:

- the support of the current density $\mathbf{J}$, Ω_{J},
- the set occupied by air and other non-magnetic materials (i.e., for which $\mu = \mu_0$, the permeability of the empty space) to be called Ω_{A},
- and the domain filled with magnetic materials ($\mu \neq \mu_0$) where $\mathbf{J} = \mathbf{0}$, to be denoted by Ω_{M}.

Let us assume that $\Omega_{\mathrm{R}} =: \mathrm{int}(\overline{\Omega}_{\mathrm{A}} \cup \overline{\Omega}_{\mathrm{J}})$ and Ω_{M} are simply connected.

Since $\mathbf{curl}\,\mathbf{H} = \mathbf{0}$ in Ω_{M} there exists a scalar field φ, defined in Ω_{M}, such that

$$\mathbf{H}|_{\Omega_{\mathrm{M}}} = -\nabla\varphi.$$

Moreover, in Ω_{R} we have as before,

$$\mathbf{H} = -\nabla\varphi_{\mathrm{R}} + \boldsymbol{\mathcal{H}}.$$

Now equations $\operatorname{div}\mathbf{B} = 0$ and $\mathbf{B} = \mu_0\mathbf{H}$ yield,

$$
\begin{aligned}
-\operatorname{div}(\mu_0\nabla\varphi_{\mathrm{R}}) = -\operatorname{div}(\mu_0\boldsymbol{\mathcal{H}}) = 0 \quad &\text{in} \quad \Omega_{\mathrm{R}},\\
-\operatorname{div}(\mu\nabla\varphi) = 0 \quad &\text{in} \quad \Omega_{\mathrm{M}}
\end{aligned}
$$

to which we have to add the interface conditions on $\Gamma_I = \bar{\Omega}_{\mathrm{R}} \cap \bar{\Omega}_M$

$$\begin{aligned} \mathbf{B}^+ \cdot \mathbf{n}^+ + \mathbf{B}^- \cdot \mathbf{n}^- &= 0, \\ \mathbf{H}^+ \times \mathbf{n}^+ + \mathbf{H}^- \times \mathbf{n}^- &= 0, \end{aligned}$$

In terms of the scalar potentials they read

$$\mu \frac{\partial \varphi}{\partial \mathrm{n}} = \mu_0 \frac{\partial \varphi_{\mathrm{R}}}{\partial \mathrm{n}} - \mu_0 \boldsymbol{\mathcal{H}} \cdot \mathbf{n} \qquad \text{on} \quad \Gamma_I,$$

$$-\nabla \varphi \times \mathbf{n} = -\nabla \varphi_{\mathrm{R}} \times \mathbf{n} + \boldsymbol{\mathcal{H}} \times \mathbf{n} \quad \text{on} \quad \Gamma_I.$$

One can show that a weak formulation of this problem is the following:

$$\textit{Find } \varphi_{\mathrm{R}} \in \mathrm{W}^1(\Omega_{\mathrm{R}}),\ \varphi \in \mathrm{H}^1(\Omega_{\mathrm{M}}) \textit{ and } \boldsymbol{\lambda} \in \mathbf{H}^{1/2}(\Gamma_I) \textit{ such that}$$

$$\int_{\Omega_{\mathrm{R}}} \mu_0 \nabla\varphi_{\mathrm{R}} \cdot \nabla z_{\mathrm{R}} \,\mathrm{d}V + \int_{\Omega_{\mathrm{M}}} \mu \nabla\varphi \cdot \nabla z \,\mathrm{d}V$$

$$+ < \boldsymbol{\lambda},\ \nabla z \times \mathbf{n} - \nabla z_{\mathrm{R}} \times \mathbf{n} >_{\Gamma} = \int_{\Omega_{\mathrm{R}}} \mu \boldsymbol{\mathcal{H}} \cdot \nabla z_{\mathrm{R}} \,\mathrm{d}V,$$

$$< -\nabla\varphi_{\mathrm{R}} \times \mathbf{n} + \boldsymbol{\mathcal{H}} \times \mathbf{n} + \nabla\varphi \times \mathbf{n}, \boldsymbol{\beta} >_{\Gamma} = 0,$$

$$\forall (z_{\mathrm{R}}, z) \in \mathrm{W}^1(\Omega_{\mathrm{R}}) \times \mathrm{H}^1(\Omega_{\mathrm{M}}) \ \ \forall \boldsymbol{\beta} \in \mathbf{H}^{1/2}(\Gamma_I).$$

20.0.1. Remarks

1. Function $\boldsymbol{\lambda}$ is a Lagrange multiplier to impose the *continuity of the tangential component* of $\mathbf{H}$ on the interface.

2. If Ω_{M} is not simply connected we have to introduce cuts across which the potential φ jumps.

3. The above formulation can be discretized by using 3D piecewise linear finite elements on a tetrahedral mesh of Ω for φ_R and φ, and 2D piecewise linear finite elements on boundary Γ for the Lagrange multiplier $\boldsymbol{\lambda}$.

The eddy currents model

In this chapter we deal with the well known eddy currents model which is obtained from Maxwell equations by neglecting the electric displacement in the Ampère's law.

21. The time-harmonic eddy currents model in a bounded domain

The eddy currents model is obtained from Maxwell's equations when the electromagnetic field is slowly time-varying; more precisely, when the current carrying system is small compared with the electromagnetic wavelength associated with the dominant time scale of the problem.

In this case fields are propagated instantaneously so we are dealing with a $c \to \infty$ limit, c being the propagation velocity of waves. This leads to neglect field radiation effects. In other words, term $\frac{\partial \mathbf{D}}{\partial t}$ is suppressed in the Ampère's law.

Thus, the eddy current model is

$$\frac{\partial \mathbf{B}}{\partial t} + \mathbf{curl}\,\mathbf{E} = \mathbf{0}, \tag{66}$$

$$\mathbf{curl}\,\mathbf{H} = \mathbf{J}, \tag{67}$$

$$\mathrm{div}\,\mathbf{B} = 0, \tag{68}$$

$$\mathrm{div}\,\mathbf{D} = \rho_V, \tag{69}$$

$$\mathbf{B} = \mu\mathbf{H}, \tag{70}$$

$$\mathbf{D} = \varepsilon\mathbf{E}, \tag{71}$$

$$\mathbf{J} = \sigma\mathbf{E} + \mathbf{J}_S, \tag{72}$$

where $\mathbf{J}_S \in \mathrm{H}(\mathrm{div}, \Omega)$ is a divergence-free source current. We also suppose that σ is positive in conductors, null in dielectrics and

$$\mathrm{supp}(\mathbf{J}_S) \cap \mathrm{supp}(\sigma) = \emptyset. \tag{73}$$

Since we have neglected the term $\frac{\partial \mathbf{D}}{\partial t}$ in Ampère's law, we do not keep the Gauss' law for electric charge, $\mathrm{div}\, \mathbf{D} = 0$, in the conductors. Instead, from (67) and (72) we get

$$\mathrm{div}(\sigma \mathbf{E}) = 0 \ \text{ in the conductors .}$$

We are interested in obtaining the magnetic field everywhere but the electric field only in the conductors. If we wanted to get this field also in the dielectrics then equation $\operatorname{div} \mathbf{D} = 0$ would be retained in the model, in the dielectrics.

In what follows we consider the harmonic case. This means that all fields are of the following form

$$\mathbf{G}(x,t) = \operatorname{Re}(e^{i\omega t}\hat{\mathbf{G}}(x)).$$

By replacing in (66)-(72) we obtain the electromagnetic Helmholtz equations for the complex phasors $\hat{\mathbf{G}}$ (for the sake of simplicity we will drop the "hat" in the notation of phasors):

$$i\omega\mathbf{B} + \mathbf{curl}\,\mathbf{E} = \mathbf{0}, \tag{74}$$

$$\mathbf{curl}\,\mathbf{H} = \mathbf{J}, \tag{75}$$

$$\mathrm{div}\,\mathbf{B} = 0, \tag{76}$$

$$\mathbf{B} = \mu\mathbf{H}, \tag{77}$$

$$\mathbf{J} = \mathbf{J}_S + \sigma\mathbf{E}. \tag{78}$$

To solve these equations, we restrict them to a simply connected 3D bounded domain Ω consisting of two parts, Ω_{C} and Ω_{D}, occupied by conductors and dielectrics, respectively. The mathematical framework we are going to analyze covers eddy current problems posed on different geometrical settings. We sketch a particular case in Figure 4 including several connected components of the conducting domain with different topological properties.

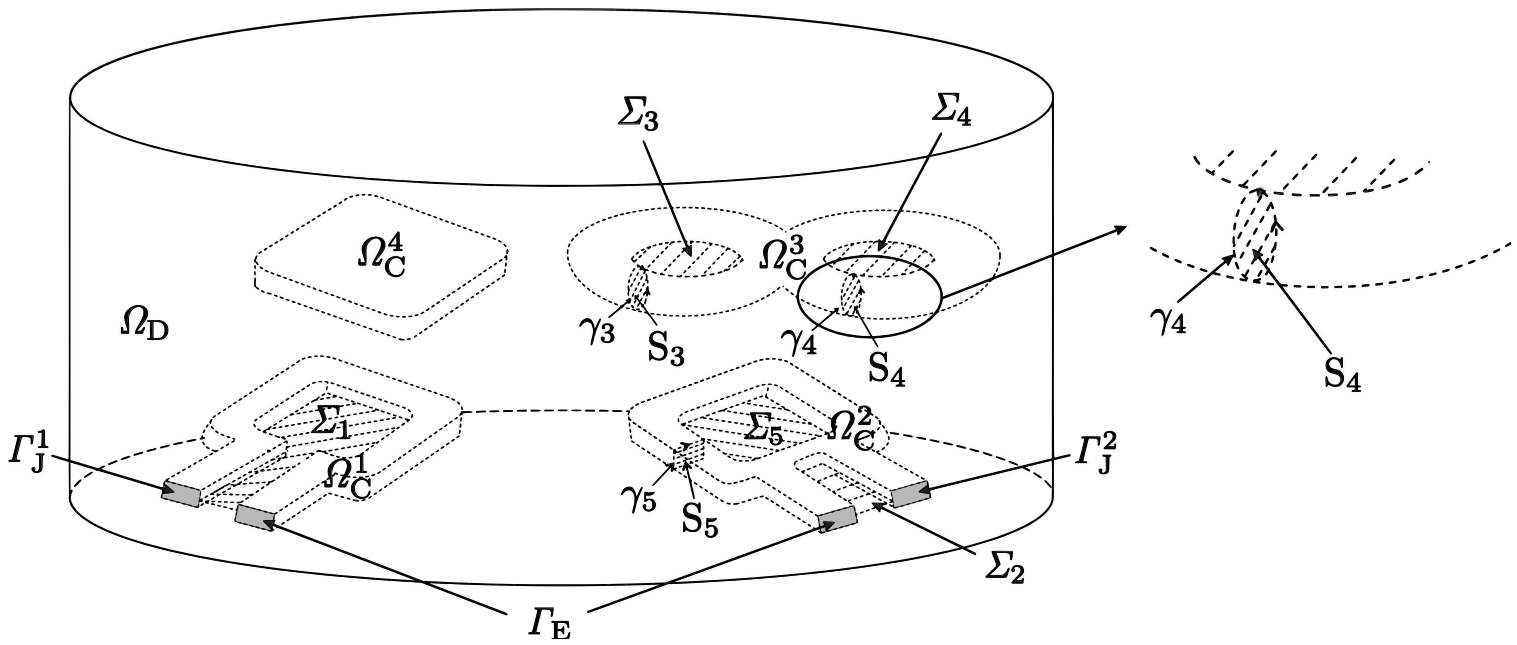

Figura 4: Sketch of the domain.

The domain Ω is assumed to have a Lipschitz-continuous connected boundary $\partial\Omega$. We denote by $\Gamma_{\mathbf{C}}$, Γ_D and Γ_{I} the open surfaces such that $\bar{\Gamma}_{\mathrm{C}} := \partial\Omega_{\mathrm{C}} \cap \partial\Omega$ is the outer boundary of the conducting domain, $\bar{\Gamma}_{\mathrm{D}} := \partial\Omega_{\mathrm{D}} \cap \partial\Omega$ that of the dielectric domain and $\bar{\Gamma}_{\mathrm{I}} := \partial\Omega_{\mathrm{C}} \cap \partial\Omega_{\mathrm{D}}$ the interface between both domains. We also denote by $\mathbf{n}$ a unit normal vector to a given surface.

As shown in Fig. 4, the connected components of the conducting domain are of two types: "inductors" which cross the boundary of Ω, and "workpieces" which have their closure included in Ω. Let us denote $\Omega_{\mathrm{C}}^{1}, \ldots, \Omega_{\mathrm{C}}^{L}$ the former and $\Omega_{\mathrm{C}}^{L+1}, \ldots, \Omega_{\mathrm{C}}^{M}$ the latter.

We assume that the outer boundary of each inductor, $\partial\Omega_{\mathrm{C}}^{n} \cap \partial\Omega$ $(n = 1, \ldots, L)$, has two disjoint connected components, both being the closure of open surfaces: the current entrance $\Gamma_{\mathbf{J}}^{n}$, where the inductor is connected to an alternate electric current source, and the current exit $\Gamma_{\mathbf{E}}^{n}$. We denote $\Gamma_{\mathbf{J}} := \Gamma_{\mathbf{J}}^{1} \cup \cdots \cup \Gamma_{\mathbf{J}}^{L}$ and $\Gamma_{\mathbf{E}} := \Gamma_{\mathbf{E}}^{1} \cup \cdots \cup \Gamma_{\mathbf{E}}^{L}$.
Furthermore, we assume that $\bar{\Gamma}_{\mathrm{J}}^{n} \cap \bar{\Gamma}_{\mathrm{J}}^{m} = \emptyset$, $\bar{\Gamma}_{\mathrm{E}}^{n} \cap \bar{\Gamma}_{\mathrm{E}}^{m} = \emptyset$, $1 \leq m, n \leq L$, $m \neq n$, and $\bar{\Gamma}_{\mathrm{J}} \cap \bar{\Gamma}_{\mathrm{E}} = \emptyset$.

Let us assume that μ and σ are frequency-independent and there exist constants $\underline{\mu}$, $\overline{\mu}$, $\overline{\sigma}$ and $\underline{\sigma}$ such that

$$0 < \underline{\mu} \leq \mu(x) \leq \overline{\mu}, \quad \text{a.e. } x \in \Omega,$$

$$0 < \underline{\sigma} \leq \sigma(x) \leq \overline{\sigma}, \quad \text{a.e. } x \in \Omega_{\mathrm{C}} \quad \text{and} \quad \sigma \equiv 0 \text{ in } \Omega_{\mathrm{D}}.$$

We have to complete the model with suitable boundary conditions. For the moment, let us consider the following ones:

$$\mathbf{E} \times \mathbf{n} = \mathbf{0} \quad \text{on } \Gamma_{\mathbf{E}}, \tag{79}$$

$$\mathbf{E} \times \mathbf{n} = \mathbf{0} \quad \text{on } \Gamma_{\mathbf{J}}, \tag{80}$$

$$\mu \mathbf{H} \cdot \mathbf{n} = 0 \quad \text{on } \partial\Omega. \tag{81}$$

Conditions (79) and (80) mean that the electric current enters and exits domain Ω perpendicularly to the boundary whereas condition (81) implies that the magnetic field is tangential to the boundary.

Formal calculations allow us to show that boundary condition (81) implies that the tangential component of the electric field $\mathbf{E}$ is a gradient. Indeed, after integrating $i\omega\mu\mathbf{H}\cdot\mathbf{n}$ on any surface S contained in $\partial\Omega$, by using (74), (77) and Stokes' Theorem, we obtain

$$0 = i\omega\int_S \mu\mathbf{H}\cdot\mathbf{n}\,\mathrm{dS} = -\int_S \mathbf{curl}\,\mathbf{E}\cdot\mathbf{n}\,\mathrm{dS} = -\int_{\partial S}\mathbf{E}\cdot\mathbf{t}\,\mathrm{d}l$$
$$= -\int_{\partial S}\mathbf{n}\times(\mathbf{E}\times\mathbf{n})\cdot\mathbf{t}\,\mathrm{d}l,$$

with $\mathbf{t}$ being a unit vector tangent to ∂S.

Therefore, since $\partial\Omega$ is simply connected, we can assert that there exists a sufficiently smooth function V defined in Ω up to a constant, such that $\mathrm{V}|_{\partial\Omega}$ is a surface potential of the tangential component of $\mathbf{E}$, namely, $\mathbf{E} \times \mathbf{n} = -\operatorname{grad} \mathrm{V} \times \mathbf{n}$ on $\partial\Omega$. On the other hand, (79) and (80) imply that V must be constant on each connected component of $\Gamma_{\mathbf{J}}$ and $\Gamma_{\mathbf{E}}$. The complex number $\mathrm{V}_n := \mathrm{V}|_{\Gamma_{\mathbf{E}}^n} - \mathrm{V}|_{\Gamma_{\mathbf{J}}^n}$ is the voltage drop along conductor Ω_{C}^n.

Many physical applications involve current intensities and voltage drops as boundary data. Let us suppose that the boundary data consist of the voltage drops V_n, for $n = 1, \ldots, \widehat{L}$, and the input current intensities through each surface $\Gamma_{\mathbf{J}}^n$, I_n, for $n = \widehat{L} + 1, \ldots, L$. We notice that the latter can be written as

$$\int_{\Gamma_{\mathbf{J}}^n} \mathbf{J} \cdot \mathbf{n} \, \mathrm{d}A = \mathrm{I}_n, \quad n = \widehat{L} + 1, \ldots, L.$$

Moreover, equation (75) yields $\operatorname{div} \mathbf{J} = 0$ and hence, by Gauss' theorem,

$$0 = \int_{\Omega_{\mathrm{C}}^n} \operatorname{div} \mathbf{J} \, \mathrm{d}V = \int_{\partial\Omega_{\mathrm{C}}^n} \mathbf{J} \cdot \mathbf{n} \, \mathrm{d}A = \int_{\Gamma_{\mathbf{J}}^n} \mathbf{J} \cdot \mathbf{n} \, \mathrm{d}A + \int_{\Gamma_{\mathbf{E}}^n} \mathbf{J} \cdot \mathbf{n} \, \mathrm{d}A,$$

because $\mathbf{J} \cdot \mathbf{n} = 0$ on $\partial\Omega_{\mathrm{C}}^n \setminus (\Gamma_{\mathbf{E}}^n \cup \Gamma_{\mathbf{J}}^n)$. Thus,

$$\int_{\Gamma_{\mathbf{J}}^n} \mathbf{J} \cdot \mathbf{n} \, \mathrm{d}A = -\int_{\Gamma_{\mathbf{E}}^n} \mathbf{J} \cdot \mathbf{n} \, \mathrm{d}A, \quad n = 1, \ldots, L. \tag{82}$$

These equalities simply means that the input current intensity coincides with the output one for each conductor $\Omega_{\mathrm{C}}^n, \quad n = 1, \ldots, L$.

We summarize the strong problem defined in Ω to be considered in the next section:

$$i\omega\mathbf{B} + \mathbf{curl}\,\mathbf{E} = \mathbf{0}, \tag{83}$$

$$\mathbf{curl}\,\mathbf{H} = \mathbf{J}, \tag{84}$$

$$\mathrm{div}\,\mathbf{B} = 0, \tag{85}$$

$$\mathbf{B} = \mu\mathbf{H}, \tag{86}$$

$$\mathbf{J} = \mathbf{J}_S + \sigma\mathbf{E}, \tag{87}$$

$$\mathbf{E} \times \mathbf{n} = \mathbf{0} \quad \text{on } \Gamma_{\mathbf{E}}, \tag{88}$$

$$\mathbf{E} \times \mathbf{n} = \mathbf{0} \quad \text{on } \Gamma_{\mathbf{J}}, \tag{89}$$

$$\mu \mathbf{H} \cdot \mathbf{n} = 0 \quad \text{on } \partial\Omega, \tag{90}$$

$$\mathrm{V} = \mathrm{V}_{\mathrm{J}}^{n} \quad \text{on } \Gamma_{\mathbf{J}}^{n}, \; n = 1, \ldots, \widehat{L}, \tag{91}$$

$$\mathrm{V} = \mathrm{V}_{\mathrm{E}}^{n} \quad \text{on } \Gamma_{\mathbf{E}}^{n}, \; n = 1, \ldots, \widehat{L}, \tag{92}$$

$$\int_{\Gamma_{\mathbf{J}}^{n}} \mathbf{J} \cdot \mathbf{n} \, \mathrm{d}A = \mathrm{I}_{n}, \quad n = \widehat{L} + 1, \ldots, L. \tag{93}$$

22. Magnetic vector potential/scalar electric potential formulation

We want to formulate and solve the above problem in terms of a magnetic vector potential $\mathbf{A}$ and an electric scalar potential V.

Firstly, from (85), (90) and Theorem 1.3.6 from Girault and Raviart book we deduce the existence of a vector field $\mathbf{A} \in \mathbf{H}(\mathbf{curl}, \Omega)$ such that

$$\mathbf{curl}\,\mathbf{A} = \mathbf{B} \text{ in } \Omega, \tag{94}$$

$$\mathrm{div}\,\mathbf{A} = 0 \text{ in } \Omega, \tag{95}$$

$$\mathbf{A} \times \mathbf{n} = \mathbf{0} \text{ on } \Gamma. \tag{96}$$

We notice that the latter equality guarantees boundary condition (90). Indeed, from Stokes' Theorem we have

$$\int_S \mathbf{B}\cdot\mathbf{n}\,\mathrm{dS} = \int_{\partial S} \mathbf{A}\cdot\mathbf{t}\,\mathrm{d}l = \int_{\partial S} \mathbf{n}\times\mathbf{A}\times\mathbf{n}\cdot\mathbf{t}\,\mathrm{d}l = 0 \quad \text{for all surface } S \subset \Gamma.$$

By replacing (94) in (83) we obtain

$$\mathbf{curl}(i\omega\mathbf{A} + \mathbf{E}) = \mathbf{0} \text{ in } \Omega$$

and then, in particular,

$$i\omega\mathbf{A} + \mathbf{E} = -\operatorname{grad} \mathrm{V} \text{ in } \Omega, \tag{97}$$

for some $\mathrm{V} \in \mathrm{H}^1(\Omega)$.

From (88), (89) and (96) we deduce

$$\mathrm{grad}_\Gamma \mathrm{V} := \mathbf{n} \times \mathrm{grad}\, \mathrm{V} \times \mathbf{n} = -\mathbf{n} \times \mathbf{E} \times \mathbf{n} = \mathbf{0} \quad \text{on } \Gamma_{\mathbf{J}} \cup \Gamma_{\mathbf{E}},$$

which implies that V must be constant on each connected component of $\Gamma_{\mathbf{J}}$ and $\Gamma_{\mathbf{E}}$.We notice that the complex number $\mathrm{V}_n := \mathrm{V}_{\mathrm{E}}^n - \mathrm{V}_{\mathrm{J}}^n$ is the voltage drop along conductor Ω_{C}^n.

From (83), (86), (87) and (97) we deduce

$$\sigma(i\omega\mathbf{A} + \mathrm{grad}\,\mathrm{V}) + \mathbf{curl}(\frac{1}{\mu}\,\mathbf{curl}\,\mathbf{A}) = \mathbf{J}_S \text{ in } \Omega. \tag{98}$$

We notice that, since $\sigma = 0$ in Ω_{D}, we only need to compute V in Ω_{C}.

Finally, from (87) and (97), boundary conditions (93) become

$$\int_{\Gamma_{\mathbf{J}}^{n}} \sigma(i\omega\mathbf{A} + \operatorname{grad}\mathrm{V}) \cdot \mathbf{n}\,\mathrm{dS} = -\mathrm{I}_n, \quad n = \widehat{L} + 1, \ldots, L. \tag{99}$$

Summarizing, the problem to be solved reads as follows:

Given a solenoidal field $\mathbf{J}_S$ *with support* Ω_{S} *included in* Ω_{D}*, and complex numbers* $\mathrm{V}_{\mathrm{J}}^n$*,* $\mathrm{V}_{\mathrm{E}}^n$*,* $n = 1, \ldots, \widehat{L}$ *and* $\mathrm{I}_n,\ n = \widehat{L} + 1, \ldots, L$*, find a vector field* $\mathbf{A}$ *defined in* Ω*, and a scalar field* V *defined in* Ω_{C} *and null on* $\Gamma_J^n,\ n = \widehat{L} + 1, \ldots, L$*, such that*

$$\sigma(i\omega\mathbf{A} + \operatorname{grad} \mathrm{V}) + \mathbf{curl}(\frac{1}{\mu}\,\mathbf{curl}\,\mathbf{A}) = \mathbf{J}_S \text{ in } \Omega, \tag{100}$$

$$\operatorname{div} \mathbf{A} = 0 \text{ in } \Omega, \tag{101}$$

$$\mathbf{A} \times \mathbf{n} = 0 \text{ on } \Gamma, \tag{102}$$

$$\sigma(i\omega\mathbf{A} + \operatorname{grad} \mathrm{V}) \cdot \mathbf{n} = 0 \text{ on } \partial\Omega^n_{\mathrm{C}} \setminus (\Gamma^n_{\mathbf{E}} \cup \Gamma^n_{\mathbf{J}}), \tag{103}$$

$$n = 1, \ldots L,$$

$$\mathrm{V} = \mathrm{V}^n_{\mathrm{J}} \text{ on } \Gamma^n_{\mathbf{J}},\ n = 1, \ldots, \widehat{L}, \tag{104}$$

$$\mathrm{V} = \mathrm{V}^n_{\mathrm{E}} \text{ on } \Gamma^n_{\mathbf{E}},\ n = 1, \ldots, \widehat{L}, \tag{105}$$

$$\int_{\Gamma^n_{\mathbf{J}}} \sigma(i\omega\mathbf{A} + \operatorname{grad} \mathrm{V}) \cdot \mathbf{n}\,\mathrm{dS} = -\mathrm{I}_n,\ n = \widehat{L} + 1, \ldots, L. \tag{106}$$

In what follows we write a weak formulation of this problem. Firstly, let us multiply (100) by the conjugate of a test function $\mathbf{G} \in \mathrm{H}(\mathbf{rot}, \Omega)$, integrate in Ω and use a Green's formula. We get

$$\int_{\Omega_{\mathrm{C}}} \sigma(i\omega\mathbf{A} + \mathrm{grad}\,\mathrm{V}) \cdot \bar{\mathbf{G}}\ \mathrm{d}V + \int_{\Omega} \frac{1}{\mu}\,\mathbf{curl}\,\mathbf{A} \cdot \mathbf{curl}\,\bar{\mathbf{G}}\ \mathrm{d}V = \int_{\Omega_{\mathrm{S}}} \mathbf{J}_s \cdot \bar{\mathbf{G}}\,\mathrm{d}V.$$

Next, we notice that, in principle, the traces appearing in (102) and (106) are not well defined for $\mathbf{A} \in \mathrm{H}(\mathbf{rot}, \Omega)$ and $\mathrm{V} \in \mathrm{H}^1(\Omega_{\mathrm{c}})$ so we write a weak formulation of them. For this purpose, we first notice that, by taking the divergence operator of (100), we deduce

$$\mathrm{div}\left(\sigma(i\omega\mathbf{A} + \mathrm{grad}\,\mathrm{V})\right) = 0 \text{ in } \Omega_{\mathrm{c}}. \tag{107}$$

Now, given two vectors $\overrightarrow{\mathrm{V}}_{\mathrm{J}},\ \overrightarrow{\mathrm{V}}_{\mathrm{E}} \in \mathbb{R}^{\widehat{L}}$ let us define the affine space

$$\mathcal{L}(\overrightarrow{\mathrm{V}}_{\mathrm{J}}, \overrightarrow{\mathrm{V}}_{\mathrm{E}}) = \{\mathrm{V} \in \mathrm{H}^1(\Omega_{\mathrm{C}}) :\ \mathrm{V} = \mathrm{V}_{\mathrm{J}}^n \text{ on } \Gamma_{\mathbf{J}}^n,\ \mathrm{V} = \mathrm{V}_{\mathrm{E}}^n \text{ on } \Gamma_{\mathbf{E}}^n,$$

$$n = 1, \ldots, \widehat{L},$$

$$\mathrm{V} = \mathit{constant} \text{ on } \Gamma_{\mathbf{J}}^n \text{ and on } \Gamma_{\mathbf{E}}^n,\ n = \widehat{L} + 1, \ldots, L\}.$$

We denote $\mathcal{L}_0$ the above space corresponding to null vectors $\overrightarrow{\mathrm{V}}_{\mathrm{J}}$ and $\overrightarrow{\mathrm{V}}_{\mathrm{E}}$. By multiplying equation (107) by the conjugate of a test function $Z \in \mathcal{L}_0$, integrating in Ω_{C}, using a Green's formula, (103) and (106) we get

$$\int_{\Omega_{\mathrm{C}}} \sigma(i\omega \mathbf{A} + \mathrm{grad}\,\mathrm{V}) \cdot \mathrm{grad}\,\bar{Z}\ \mathrm{d}V = \sum_{n=\widehat{L}+1}^{L} \mathrm{I}_n \bar{Z}_n, \qquad (108)$$

with $Z_n = Z_{|\Gamma_{\mathbf{E}}^n} - Z_{|\Gamma_{\mathbf{J}}^n},\ n = \widehat{L}+1, \ldots, L$. Finally, by multiplying the

gauge condition (101) by the conjugate of a test function $\Psi \in \mathrm{H}_0^1(\Omega)$ and using a Green's formula we obtain

$$\int_\Omega \mathbf{A} \cdot \operatorname{grad} \bar{\Psi} \, \mathrm{d}V = 0.$$

Let $\mathcal{W}_0$ be the function space

$$\mathcal{W}_0 = \{\mathbf{G} \in \mathrm{H}(\mathbf{rot}, \Omega) : \ \mathbf{G} \times \mathbf{n} = \mathbf{0} \text{ on } \Gamma\}.$$

Next, we summarize the whole weak problem:

Find $\mathbf{A} \in \mathcal{W}_0$, $\mathrm{V} \in \mathcal{L}(\overrightarrow{\mathrm{V}}_{\mathrm{J}}, \overrightarrow{\mathrm{V}}_{\mathrm{E}})$ *and* $\varphi \in \mathrm{H}_0^1(\Omega)$ *such that*

$$\int_{\Omega_{\mathrm{C}}} \sigma(i\omega\mathbf{A} + \operatorname{grad}\mathrm{V}) \cdot \bar{\mathbf{G}}\ \mathrm{d}V + \int_{\Omega} \frac{1}{\mu}\,\mathbf{curl}\,\mathbf{A} \cdot \mathbf{curl}\,\bar{\mathbf{G}}\ \mathrm{d}V$$

$$+ \int_{\Omega} \operatorname{grad}\varphi \cdot \bar{\mathbf{G}}\ \mathrm{d}V = \int_{\Omega_{\mathrm{S}}} \mathbf{J}_s \cdot \bar{\mathbf{G}}\,\mathrm{d}V \quad \forall \mathbf{G} \in \mathcal{W}_0,$$

$$\int_{\Omega_{\mathrm{C}}} \sigma(i\omega\mathbf{A} + \operatorname{grad}\mathrm{V}) \cdot \operatorname{grad}\bar{Z}\ \mathrm{d}V = \sum_{n=\widehat{L}+1}^{L} \mathrm{I}_n \bar{Z}_n \quad \forall Z \in \mathcal{L}_0,$$

$$\int_{\Omega} \mathbf{A} \cdot \operatorname{grad}\bar{\Psi}\ \mathrm{d}V = 0 \quad \forall \Psi \in \mathrm{H}_0^1(\Omega).$$

Remark 22.1 *We notice that in this formulation the scalar potential* V *only appears under the gradient operator. Since the gradient of* V *does not change if* V *is translated by a constant in each connect component* Ω^n_{C}*, we can replace the above problem by the following one:*

Find $\mathbf{A} \in \mathcal{W}_0$, $\mathrm{V} \in \mathcal{M}(\overrightarrow{\mathrm{V}})$ *and* $\varphi \in \mathrm{H}_0^1(\Omega)$ *such that*

$$\int_{\Omega_{\mathrm{C}}} \sigma(i\omega\mathbf{A} + \operatorname{grad}\mathrm{V}) \cdot \bar{\mathbf{G}}\ \mathrm{d}V + \int_{\Omega} \frac{1}{\mu}\,\mathbf{curl}\,\mathbf{A} \cdot \mathbf{curl}\,\bar{\mathbf{G}}\ \mathrm{d}V$$

$$\int_{\Omega} \operatorname{grad}\varphi \cdot \bar{\mathbf{G}}\ \mathrm{d}V = \int_{\Omega_{\mathrm{S}}} \mathbf{J}_s \cdot \bar{\mathbf{G}}\,\mathrm{d}V \quad \forall \mathbf{G} \in \mathcal{W}_0,$$

$$\int_{\Omega_{\mathrm{C}}} \sigma(i\omega\mathbf{A} + \operatorname{grad}\mathrm{V}) \cdot \operatorname{grad}\bar{Z}\ \mathrm{d}V = \sum_{n=\widehat{L}+1}^{L} \mathrm{I}_n \bar{Z}_n \quad \forall Z \in \mathcal{M}_0,$$

$$\int_{\Omega} \mathbf{A} \cdot \operatorname{grad}\bar{\Psi}\ \mathrm{d}V = 0 \quad \forall \Psi \in \mathrm{H}_0^1(\Omega),$$

where $\mathcal{M}(\overrightarrow{\mathrm{V}})$ *is the affine space*

$$\mathcal{M}(\overrightarrow{\mathrm{V}}) := \{\mathrm{V} \in \mathrm{H}^1(\Omega_{\mathrm{C}}) : \ \mathrm{V} = 0 \ \textit{on} \ \Gamma^n_{\mathbf{J}}, \ n = 1, \ldots, L, \ \mathrm{V} = \mathrm{V}_n \ \textit{on} \ \Gamma^n_{\mathbf{E}},$$
$$n = 1, \ldots, \widehat{L}, \mathrm{V} = \textit{constant} \ \textit{on} \ \Gamma^n_{\mathbf{E}}, \ n = \widehat{L} + 1, \ldots, L\}$$

and $\mathcal{M}_0$ *is the above space for the null vector* $\overrightarrow{\mathrm{V}}$.

Remark 22.2 *Existence and uniqueness of a solution to this problem can be shown by using the Babuska-Brezzi theory of mixed formulations.*

Remark 22.3 *Numerical discretization can be done by using Nédélec edge finite elements to approximate the vector potential and continuous piecewise linear nodal elements to approximate* φ *and* V.

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