Computers and Mathematics with Applications 65 (2013) 1200–1219

Contents lists available at SciVerse ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

Finite element approximation of nonlinear transient magnetic problems involving periodic potential drop excitations ☆

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ARTICLE INFO

Article history:
Received 25 August 2012
Received in revised form 29 December 2012
Accepted 19 February 2013

Keywords:
Transient magnetic
Nonlinear partial differential equations
Finite element methods
Periodic solutions
Voltage drops
Pulse-width modulation

ABSTRACT

This paper deals with the computation of nonlinear 2D transient magnetic fields when the data concerning the electric current sources involve potential drop excitations. In the first part, a mathematical model is stated, which is solved by an implicit time discretization scheme combined with a finite element method for space approximation. The second part focuses on developing a numerical method to compute periodic solutions by determining a suitable initial current which avoids large simulations to reach the steady state. This numerical method leads to solve a nonlinear system of equations which requires to approximate several nonlinear and linear magnetostatic problems. The proposed methods are first validated with an axisymmetric example and sinusoidal source having an analytical solution. Then, we show the saving in computational effort that this methodology offers to approximate practical problems specially with pulse-width modulation (PWM) voltage supply.

1. Introduction

This paper is devoted to the numerical computation of transient magnetic fields when the source data involve potential drops in some conductors. The case where the currents are given reduces to solve a nonlinear magnetostatic problem at each time in some interval, and hence time appears as a parameter. However, the case with potential drop excitations is more involved because the model becomes a system of degenerate parabolic nonlinear partial differential equations.

The engineering problem motivating our study is the numerical computation of heat losses in devices like electric machines. Nowadays, some of these devices are frequently supplied with pulse-width modulation (PWM) voltages (see, for instance, [1]) rather than harmonic ones. This leads to increased losses in the ferromagnetic cores and, consequently, to reduce the normal operating capacity of the devices. Therefore, the accurate prediction of losses caused by a PWM source is critical for electrical machine design operated by PWM inverters, in order to improve the efficiency of the device.

The numerical simulation provides an alternative tool to laboratory experiments to determine the electromagnetic losses. However, the nonlinear behavior of ferromagnetic materials and the laminated structure of the devices lead to difficult nonlinear problems; see, for instance, [2–6] and references therein. In this framework, a two-dimensional (2D) transient

☆ Partially supported by Ministerio de Ciencia e Innovación under research project MTM2008-02483 and by Xunta de Galicia (Spain) under research projects INCITE 09207047 PR and 2010/22.

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http://dx.doi.org/10.1016/j.camwa.2013.02.019

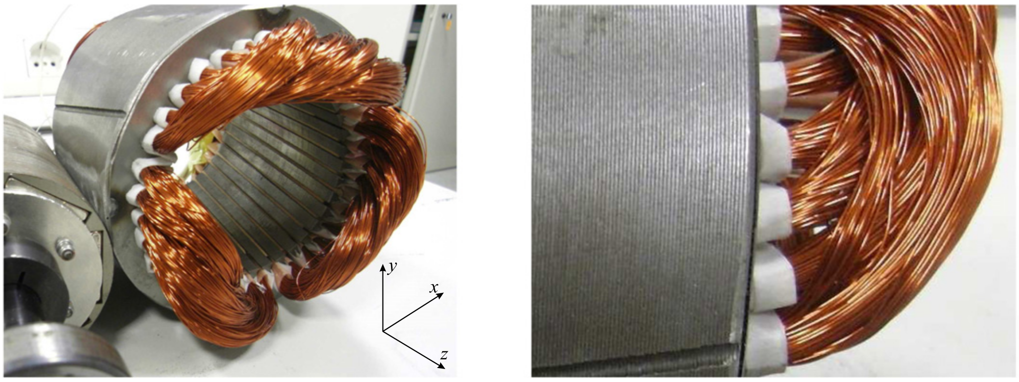

Fig. 1. Coils with magnetic core (left) and detail of the laminate (right).

nonlinear magnetic model is often used to compute the electromagnetic fields in a plane parallel to the plates and then the losses are estimated a posteriori [7,8]. The numerical solution of this 2D model with voltage drop excitations is the main objective of this paper. The mathematical model covers the case where the voltage drop per unit length is known and also the one where the data is the voltage drop between two conductors carrying the same current in opposite directions. Of particular interest and difficulty is the case of PWM voltage supply because these kinds of signals are discontinuous with a large number of discontinuities in each period, thus requiring the use of very small time steps and thereby a great computational effort. The problem is even more serious because the calculation of losses requires to obtain previously the steady-state electromagnetic field. In principle, this field is only reached after simulating a certain number of cycles, unless the initial current is properly chosen. Thus, one of the main contributions of the paper is the introduction of a new procedure to calculate the initial current corresponding to the periodic steady-state solution thus allowing to integrate the equations only along one single period.

In the literature there are several references dealing with the efficient computation of steady-state solutions of nonlinear transient eddy current problems. Let us mention, for instance, papers based on the time-periodic finite element method [1,9], on frequency domain approximations [10,11], on shooting-Newton method [12,13] or on another procedures [14]. All these methodologies try to avoid the solution of the transient problem along several cycles, which would require a large computational effort specially with signals like the PWM ones. In this paper, we focus on the transient magnetic model without considering eddy currents effects. To avoid the step by step procedure, we introduce a new method that essentially requires the solution of a nonlinear system of equations the unknowns of which are the initial currents. For this purpose, Newton's method is employed. At each iteration, a nonlinear magnetostatic problem and some linear ones have to be solved. Each of the latter provides the derivatives of the equations to be solved with respect to the initial currents. These derivatives are needed by the Newton's algorithm. The positive definite symmetric matrices of these linear problems coincide so they are assembled and factorized only once.

The paper is organized as follows. In Section 2 we state the transient 2D nonlinear model to be solved and write the equations in terms of the axial component of the magnetic vector potential. Section 3 is devoted to the numerical solution of this nonlinear transient problem. We propose a backward Euler scheme for time discretization and a standard finite element method for space approximation; at each time step, the nonlinearity is solved by means of a duality iterative algorithm. In Section 4 we introduce a new methodology to compute the initial conditions allowing us to obtain an electromagnetic field very close to the steady-state solution by solving the problem in one single period. In Section 5 we obtain the analytical solution of a nonlinear test problem in an axisymmetric geometry under different source conditions. This analytical solution will be employed in Section 6 to validate the numerical techniques proposed in the previous sections. Finally, we also illustrate the performance of the method from the point of view of applications.

2. Mathematical modeling

In this section we state a 2D transient magnetic problem which arises in the mathematical modeling of laminated magnetic media. We specially focus on providing different kinds of current sources to the electromagnetic system.

2.1. A two-dimensional transient magnetic model

In order to minimize the electromagnetic losses, the magnetic cores of electrical machines are laminated media consisting of a large number of stacked steel sheets, which are orthogonal to the direction of the currents traversing the coils (see Fig. 1).

In principle, in order to compute the electromagnetic losses in the device it would be necessary to solve a three-dimensional transient eddy current model in the laminated media considering hysteresis effects. However, the high number of sheets and its small thickness (less than one millimeter) would require to consider a very fine mesh within each sheet leading to high computational costs. To avoid this problem, we can find several approaches trying to simplify the modeling of laminated cores [3–6]. An economic strategy extensively used consists in solving a 2D-FEM transient magnetic model defined in the transversal section of the device by assuming that the magnetic flux lies on the xy-plane. In this context, losses are estimated *a posteriori* (see, for instance, [7,8]) by means of *loss separation models* based on semi-empirical formulas which give, separately, the hysteresis, the classical eddy currents and the excess eddy current losses. In this paper, and as a previous step to the losses computation, we will focus on the aforementioned 2D transient magnetic problem with special emphasis in the different kind of sources the device can be supplied with. As explained in the Introduction, we will pay particular attention to the case of PWM voltage drop excitations and the main problems appearing when dealing with the numerical simulation of these signals.

Let us assume that the current sources $\mathbf{J}$ have non-null component only in the z space direction and that this component does not depend on z, i.e., $\mathbf{J} = J_z\mathbf{e}_z$, with $J_z = J_z(x, y, t)$. We also assume that the laminated core is invariant along the z-direction and that, in the field equations, we neglect the effects of eddy currents in this direction. In this case, the core can be considered as a homogeneous medium and it is easy to see that the magnetic field $\mathbf{H}$, and then the magnetic induction, $\mathbf{B}$, have only components on the xy-plane and both are independent of z; namely,

$$\mathbf{H} = H_x(x, y, t)\mathbf{e}_x + H_y(x, y, t)\mathbf{e}_y, \tag{1}$$

$$\mathbf{B} = B_x(x, y, t)\mathbf{e}_x + B_y(x, y, t)\mathbf{e}_y. \tag{2}$$

Thus, for a given current density $\mathbf{J}$, the 2D transient magnetic problem in the x, y-plane transversal to the device reads:

$$\mathbf{curl}\ \mathbf{H} = \mathbf{J}, \tag{3}$$

$$\operatorname{div} \mathbf{B} = 0. \tag{4}$$

This model is completed with the constitutive law relating the magnetic field to the flux density. In linear materials this relation reads $\mathbf{B} = \mu\mathbf{H}$, where μ is the magnetic permeability, while in nonlinear media μ depends on $|\mathbf{H}|$, i.e.,

$$\mathbf{B} = \mu(|\mathbf{H}|)\mathbf{H}. \tag{5}$$

Eqs. (3)–(4) are defined in the whole space. However, in order to apply a standard finite element method, we will reduce the computations to a bounded domain with suitable boundary conditions to be defined in the sequel.

Let us consider a bounded domain Ω composed by several connected conductors, a ferromagnetic core and the air around. Let us denote by $\Omega_i, i = 1, \ldots, N$ the conductors in Ω representing the cross section of the coils. In particular, we will suppose that all of them are stranded conductors, which makes possible to assume that the current density is uniformly distributed and given by

$$J_{z,i}(t) = \frac{I_i(t)}{\operatorname{meas}(\Omega_i)}, \tag{6}$$

where $I_i(t)$ denotes the total current across Ω_i at time t, that is, the number of turns multiplied by the current along the coil. Actually, for each conductor Ω_i, we will see below that the source can be given in terms of either the current or the potential drop per unit length in the z-direction.

We also denote by Ω_{N+1} the complementary domain occupied by the non-conducting media without current source (in our case, the air and the ferromagnetic cores), i.e., $\Omega_{N+1} = \Omega \setminus \cup_{i=1}^{N} \Omega_i$. Then, we must solve the following system of equations:

$$\mathbf{curl}\ \mathbf{H} = \mathbf{J} \quad \text{in } \Omega_i, i = 1, \ldots, N, \tag{7}$$

$$\mathbf{curl}\ \mathbf{H} = \mathbf{0} \quad \text{in } \Omega_{N+1}, \tag{8}$$

$$\operatorname{div} \mathbf{B} = 0 \quad \text{in } \Omega, \tag{9}$$

$$\mathbf{B} = \mu(|\mathbf{H}|)\mathbf{H} \quad \text{in } \Omega. \tag{10}$$

Concerning the magnetic law (10), we will assume a linear behavior for the air while the coils and the laminated media may have a nonlinear behavior. We notice that our setting may include several laminated media. However, for the sake of simplicity, along the paper we will assume there is only one. On the other hand, the coils are usually made by a magnetically linear material but, for the sake of completeness, we will deal here with a more general case.

2.2. Magnetic vector potential formulation

In order to solve the two-dimensional model described above it is convenient to introduce a magnetic vector potential because it leads to solve a scalar problem instead of a vector one.

Since $\mathbf{B}$ is divergence free, there exists a so-called magnetic vector potential $\mathbf{A}$ such that $\mathbf{B} = \mathbf{curl}\ \mathbf{A}$. Under the assumptions above, we can choose a magnetic vector potential that does not depend on z and does not have either x or y components, i.e., $\mathbf{A} = A_z(x, y, t)\mathbf{e}_z$ (see, for instance, [15]).

Thus, in terms of $\mathbf{A}$, the transient magnetic model reads:

$$\mathbf{curl}(\nu_i(|\mathbf{curl}\ \mathbf{A}|)\mathbf{curl}\ \mathbf{A}) = \mathbf{J} \quad \text{in } \Omega_i,\ i = 1, \ldots, N \text{ (coils)}, \tag{11}$$

$$\mathbf{curl}(\nu_{N+1}(|\mathbf{curl}\ \mathbf{A}|)\mathbf{curl}\ \mathbf{A}) = \mathbf{0} \quad \text{in } \Omega_{N+1} \text{ (laminate and air)}, \tag{12}$$

where ν_i denotes the magnetic reluctivity of Ω_i. In the air $\nu_{N+1} = 1/\mu_0$, (where μ_0 denotes the magnetic permeability of the empty space), while in the ferromagnetic material ν_{N+1} is a nonlinear function of $|\mathbf{B}| = |\mathbf{curl}\ \mathbf{A}|$.

Next, we will describe how to impose different kinds of transient sources in the coils. Let σ_i be the electrical conductivity of domain Ω_i. Taking into account the assumptions on **J**, from the Ohm's law, $\mathbf{J} = \sigma_i \mathbf{E}$, we deduce that, in each conductor Ω_i, the electric field **E** has to be of the form

$$\mathbf{E} = E_z(x, y, t)\mathbf{e}_z. \tag{13}$$

However, as it is argued in [15], the electric field should have a more general form outside the conductors. Indeed, from Faraday's law,

$$\frac{\partial \mathbf{B}}{\partial t} + \mathbf{curl}\ \mathbf{E} = \mathbf{0}$$

a scalar potential V must exist such that

$$\frac{\partial \mathbf{A}}{\partial t} + \mathbf{E} = -\,\mathbf{grad}\, V.$$

Given the shape of **A**, we deduce from this equality that

$$E_x = -\frac{\partial V}{\partial x}, \tag{14}$$

$$E_y = -\frac{\partial V}{\partial y}, \tag{15}$$

$$\frac{\partial A_z}{\partial t} + E_z = -\frac{\partial V}{\partial z}. \tag{16}$$

If **E** also had the form (13) outside the conductors, i.e., if $E_x \equiv 0$ and $E_y \equiv 0$ then V would only be dependent on z and t, in contradiction to the fact that the conductors may have different electric potentials.

Anyway, in conductors, Eq. (16) and the fact that the left-hand side does not depend on z while the right-hand side is independent of x and y leads to

$$\frac{\partial V}{\partial z} = C_i(t) \quad \text{in } \Omega_i,\ i = 1, \ldots, N.$$

Function $C_i(t)$ represents the potential drop per unit length in direction z, in conductor Ω_i. Hence, from the previous equation and (16) one deduces

$$\sigma_i \frac{\partial A_z}{\partial t} + \sigma_i E_z = -\sigma_i C_i(t) \quad \text{in } \Omega_i,\ i = 1, \ldots, N. \tag{17}$$

By integrating this equation on each Ω_i we get

$$\frac{d}{dt} \int_{\Omega_i} \sigma_i A_z(x, y, t)\, \mathrm{d}x\mathrm{d}y + \int_{\Omega_i} \sigma_i E_z(x, y, t)\, \mathrm{d}x\mathrm{d}y = -C_i(t) \int_{\Omega_i} \sigma_i\, \mathrm{d}x\mathrm{d}y \tag{18}$$

and hence, from Ohm's law,

$$\frac{d}{dt} \int_{\Omega_i} \sigma_i A_z(x, y, t)\, \mathrm{d}x\mathrm{d}y + I_i(t) = -C_i(t) \int_{\Omega_i} \sigma_i\, \mathrm{d}x\mathrm{d}y.$$

Taking into account the previous discussion, we will assume that, for each conductor Ω_i, either the potential drop $C_i(t)$ or the current $I_i(t)$ is given. In particular, let us suppose there are N_C conductors of the first type and $N - N_C$ of the second one. Moreover, let us introduce the *resistance* of the i-th conductor per unit length in the z direction by

$$\alpha_i := \frac{1}{\int_{\Omega_i} \sigma_i\, \mathrm{d}x\mathrm{d}y}. \tag{19}$$

On the boundary $\partial\Omega$ of Ω, we will consider for simplicity a homogeneous Dirichlet boundary condition, $\mathbf{A} = \mathbf{0}$, which means that $\mathbf{B} \cdot \mathbf{n} = 0$ on $\partial\Omega$. Another classic boundary condition in magnetostatics is $\mathbf{H} \times \mathbf{n} = 0$. In this case, $1/\mu$ $\mathbf{curl}\ \mathbf{A} \times \mathbf{n} = 0$ and further development would be done without any difficulty.

Thus, the problem to be solved is the following:

Problem 2.1. Given functions $C_i(t),\ i = 1, \ldots, N_C, I_i(t),\ i = N_C + 1, \ldots, N$, and initial currents $I_i^0,\ i = 1, \ldots, N_C$, find a field $\mathbf{A} = A_z(x, y, t)\mathbf{e}_z$ and currents $I_i(t),\ i = 1, \ldots, N_C$, such that

$$\mathbf{curl}(\nu_i(|\mathbf{curl}\ \mathbf{A}|)\mathbf{curl}\ \mathbf{A}) = \frac{I_i(t)}{\mathrm{meas}(\Omega_i)}\, \mathbf{e}_z \quad \text{in } \Omega_i,\ i = 1, \ldots, N, \tag{20}$$

$$\mathbf{curl}(\nu_{N+1}(|\mathbf{curl}\ \mathbf{A}|)\mathbf{curl}\ \mathbf{A}) = \mathbf{0} \quad \text{in } \Omega_{N+1}, \tag{21}$$

$$\mathbf{A} = \mathbf{0} \quad \text{on } \partial\Omega, \tag{22}$$

$$\frac{d}{dt}\int_{\Omega_i}\sigma_i A_z(x,y,t)\,\mathrm{d}x\mathrm{d}y + I_i(t) = -C_i(t)\alpha_i^{-1}, \quad i=1,\ldots,N_C, \tag{23}$$

$$I_i(0) = I_i^0, \quad i=1,\ldots,N_C. \tag{24}$$

We notice that, in (20), the currents for $i = N_C+1,\ldots,N$ are given, but the rest of them, i.e., those for $i=1,\ldots,N_C$ are unknown. In order to compute the latter we have added Eqs. (23) and (24) to the system.

From the computational point of view, it is convenient to eliminate the unknown currents $I_i(t),\ i=1,\ldots,N_C$ from the system. For this purpose, we first obtain $I_i(t)$ from (23) and then replace it in (20) for $i=1,\ldots,N_C$. Then Problem 2.1 can be rewritten as:

Problem 2.2. Given functions $C_i(t),\ i=1,\ldots,N_C, I_i(t),\ i=N_C+1,\ldots,N$, and initial currents $I_i^0,\ i=1,\ldots,N_C$, find a field $\mathbf{A} = A_z(x,y,t)\mathbf{e}_z$ such that

$$\frac{1}{\mathrm{meas}(\Omega_i)}\frac{d}{dt}\int_{\Omega_i}\sigma_i A_z(x,y,t)\,\mathrm{d}x\mathrm{d}y\,\mathbf{e}_z + \mathbf{curl}(\nu_i(|\mathbf{curl}\,\mathbf{A}|)\mathbf{curl}\,\mathbf{A})$$
$$= -\frac{C_i(t)\alpha_i^{-1}}{\mathrm{meas}(\Omega_i)}\,\mathbf{e}_z \quad \text{in } \Omega_i,\ i=1,\ldots,N_C, \tag{25}$$

$$\mathbf{curl}(\nu_i(|\mathbf{curl}\,\mathbf{A}|)\mathbf{curl}\,\mathbf{A}) = \frac{I_i(t)}{\mathrm{meas}(\Omega_i)}\,\mathbf{e}_z \quad \text{in } \Omega_i,\ i=N_C+1,\ldots,N, \tag{26}$$

$$\mathbf{curl}(\nu_{N+1}(|\mathbf{curl}\,\mathbf{A}|)\mathbf{curl}\,\mathbf{A}) = \mathbf{0} \quad \text{in } \Omega_{N+1}, \tag{27}$$

$$\mathbf{A} = \mathbf{0}, \quad \text{on } \partial\Omega, \tag{28}$$

$$I_i(0) = I_i^0, \quad i=1,\ldots,N_C. \tag{29}$$

Remark 2.1. Notice that when the currents are supported on a surface S (i.e., on a curve in 2D), they have to be represented by a distribution rather than a function. In fact, if $\mathbf{J}_S = J_{S_z}\mathbf{e}_z$ denotes the surface current density (A/m), then it is well-known that the tangential component of the magnetic field is discontinuous across S, more precisely

$$[\mathbf{H}\times\mathbf{n}] = [\nu(|\mathbf{curl}\,\mathbf{A}|)\mathbf{curl}\,\mathbf{A}\times\mathbf{n}] = \mathbf{J}_S \quad \text{on } S, \tag{30}$$

where $\mathbf{n}$ is a unit normal vector and $[\cdot]$ denotes the jump across S.

2.3. A particular case: couples of conductors with opposite currents

Let us consider the particular case where there exist two indices i_1 and i_2, $1 \le i_1, i_2 \le N_C$, such that $I_{i_1}(t) = -I_{i_2}(t) = I(t)$. In this case, the number of unknown currents in system (25)–(29) is $N_C - 1$ and accordingly, we cannot prescribe each potential drop $C_{i_j}(t), j=1,2$ arbitrarily, but only the difference of potential drops: $V(t) := C_{i_1}(t) - C_{i_2}(t)$ (in fact, it is this difference of potential drops the magnitude that is physically known).

In this case, Eq. (23), for $i = i_1, i_2$, yields

$$\frac{d}{dt}\int_{\Omega_{i_1}}\sigma_{i_1} A_z(x,y,t)\,\mathrm{d}x\mathrm{d}y + I(t) = -C_{i_1}(t)\alpha_{i_1}^{-1}, \tag{31}$$

$$\frac{d}{dt}\int_{\Omega_{i_2}}\sigma_{i_2} A_z(x,y,t)\,\mathrm{d}x\mathrm{d}y - I(t) = -C_{i_2}(t)\alpha_{i_2}^{-1}. \tag{32}$$

By subtracting Eqs. (31) and (32) after multiplication by $\alpha_{i_j},\ j=1,2$, respectively, we get

$$\alpha_{i_1}\frac{d}{dt}\int_{\Omega_{i_1}}\sigma_{i_1} A_z(x,y,t)\,\mathrm{d}x\mathrm{d}y - \alpha_{i_2}\frac{d}{dt}\int_{\Omega_{i_2}}\sigma_{i_2} A_z(x,y,t)\,\mathrm{d}x\mathrm{d}y$$
$$+ (\alpha_{i_1}+\alpha_{i_2})I(t) = -C_{i_1}(t) + C_{i_2}(t) = -V(t). \tag{33}$$

This equation replaces the two ones in (23), for $i = i_1, i_2$ and allows us to eliminate the currents in Eqs. (20), for $i = i_1,\ i_2$. Thus, the corresponding equations in (25), i.e., those for $i = i_1,\ i_2$, become,

$$\frac{1}{(\alpha_{i_1}+\alpha_{i_2})\mathrm{meas}(\Omega_{i_1})}\left(\alpha_{i_1}\frac{d}{dt}\int_{\Omega_{i_1}}\sigma_{i_1} A_z(x,y,t)\,\mathrm{d}x\mathrm{d}y - \alpha_{i_2}\frac{d}{dt}\int_{\Omega_{i_2}}\sigma_{i_2} A_z(x,y,t)\,\mathrm{d}x\mathrm{d}y\right)\mathbf{e}_z$$
$$+\mathbf{curl}(\nu_{i_1}(|\mathbf{curl}\,\mathbf{A}|)\mathbf{curl}\,\mathbf{A}) = -\frac{1}{\mathrm{meas}(\Omega_{i_1})(\alpha_{i_1}+\alpha_{i_2})}V(t)\,\mathbf{e}_z \quad \text{in } \Omega_{i_1}, \tag{34}$$

$$\frac{1}{(\alpha_{i_1}+\alpha_{i_2})\mathrm{meas}(\Omega_{i_2})}\left(-\alpha_{i_1}\frac{d}{dt}\int_{\Omega_{i_1}}\sigma_{i_1}A_z(x,y,t)\,\mathrm{d}x\mathrm{d}y+\alpha_{i_2}\frac{d}{dt}\int_{\Omega_{i_2}}\sigma_{i_2}A_z(x,y,t)\,\mathrm{d}x\mathrm{d}y\right)\mathbf{e}_z$$
$$+\,\mathbf{curl}(\nu_{i_2}(|\mathbf{curl\,A}|)\mathbf{curl\,A})=\frac{1}{\mathrm{meas}(\Omega_{i_2})(\alpha_{i_1}+\alpha_{i_2})}V(t)\,\mathbf{e}_z\quad\text{in }\Omega_{i_2}. \tag{35}$$

If σ_i is constant in Ω_i, from (19) we have

$$\alpha_i\int_{\Omega_i}\sigma_iA_z(x,y,t)\,\mathrm{d}x\mathrm{d}y=\frac{1}{\mathrm{meas}(\Omega_i)}\int_{\Omega_i}A_z(x,y,t)\,\mathrm{d}x\mathrm{d}y. \tag{36}$$

In that case, Eqs. (34) and (35) can be written as

$$\frac{1}{(\alpha_{i_1}+\alpha_{i_2})}\left(\frac{1}{\mathrm{meas}(\Omega_{i_1})}\frac{d}{dt}\int_{\Omega_{i_1}}A_z(x,y,t)\,\mathrm{d}x\mathrm{d}y-\frac{1}{\mathrm{meas}(\Omega_{i_2})}\frac{d}{dt}\int_{\Omega_{i_2}}A_z(x,y,t)\,\mathrm{d}x\mathrm{d}y\right)\mathbf{e}_z$$
$$+\,\mathrm{meas}(\Omega_{i_1})\mathbf{curl}(\nu_{i_1}(|\mathbf{curl\,A}|)\mathbf{curl\,A})=-\frac{1}{(\alpha_{i_1}+\alpha_{i_2})}V(t)\,\mathbf{e}_z\quad\text{in }\Omega_{i_1}, \tag{37}$$

$$\frac{1}{(\alpha_{i_1}+\alpha_{i_2})}\left(-\frac{1}{\mathrm{meas}(\Omega_{i_1})}\frac{d}{dt}\int_{\Omega_{i_1}}A_z(x,y,t)\,\mathrm{d}x\mathrm{d}y+\frac{1}{\mathrm{meas}(\Omega_{i_2})}\frac{d}{dt}\int_{\Omega_{i_2}}A_z(x,y,t)\,\mathrm{d}x\mathrm{d}y\right)\mathbf{e}_z$$
$$+\,\mathrm{meas}(\Omega_{i_2})\mathbf{curl}(\nu_{i_2}(|\mathbf{curl\,A}|)\mathbf{curl\,A})=\frac{1}{(\alpha_{i_1}+\alpha_{i_2})}V(t)\,\mathbf{e}_z\quad\text{in }\Omega_{i_2}. \tag{38}$$

Remark 2.2. Notice that, after solving the problem, we know the vector potential field $\mathbf{A}=A_z\mathbf{e}_z$ from which the current density and then the current across Ω_{i_j}, $j=1,2$ can be computed. Next, the potential drop in each conductor, $C_{i_j}(t)$, can be obtained from Eqs. (31) and (32).

3. Numerical solution

The numerical solution of the above problems is done by using the implicit Euler scheme for time discretization combined with standard continuous piecewise linear finite elements on triangular meshes for space discretization. After full discretization, Problem 2.2 is solved at each time of a mesh of the time interval (see below). Let us notice that, if $N_C=0$, Eqs. (25) and (29) are not needed and then, for each time $t\in[0,T]$, the problem is similar to a nonlinear standard magnetostatic problem. On the contrary, if $N_C\neq 0$ the problem is not standard and some currents have to be computed. For this purpose, in the next section we introduce a time discretization of the problem.

3.1. Time discretization

Let $0=t_0<t_1<\cdots<t_M=T$ be a partition of the time interval of simulation $[0,T]$. We propose the following implicit Euler-like scheme:

- $m=0$ (initial time).
  By using the (given) initial currents I_i^0, $i=1,\dots,N_C$ and $I_i(0)$, $i=N_C+1,\dots,N$, we solve the nonlinear magnetostatic problem (20)–(22) to compute $\mathbf{A}^0$.
- $m\geq 1$.
  An approximation of the magnetic vector potential at time t_{m-1} is known from the previous time step, namely, $\mathbf{A}^{m-1}$. Then $\mathbf{A}^m=A_z^m\mathbf{e}_z$ is the solution of the problem,

$$\frac{1}{t_m-t_{m-1}}\frac{1}{\mathrm{meas}(\Omega_i)}\int_{\Omega_i}\sigma_iA_z^m(x,y)\,\mathrm{d}x\mathrm{d}y\,\mathbf{e}_z+\mathbf{curl}(\nu_i(|\mathbf{curl\,A}^m|)\mathbf{curl\,A}^m)$$
$$=\frac{1}{\mathrm{meas}(\Omega_i)}\left(-C_i(t_m)\alpha_i^{-1}+\frac{1}{t_m-t_{m-1}}\int_{\Omega_i}\sigma_iA_z^{m-1}(x,y)\,\mathrm{d}x\mathrm{d}y\right)\mathbf{e}_z\quad\text{in }\Omega_i,\ i=1,\dots,N_C, \tag{39}$$

$$\mathbf{curl}(\nu_i(|\mathbf{curl\,A}^m|)\mathbf{curl\,A}^m)=\frac{I_i(t_m)}{\mathrm{meas}(\Omega_i)}\mathbf{e}_z\quad\text{in }\Omega_i,\ i=N_C+1,\dots,N, \tag{40}$$

$$\mathbf{curl}(\nu_{N+1}(|\mathbf{curl\,A}^m|)\mathbf{curl\,A}^m)=\mathbf{0}\quad\text{in }\Omega_{N+1}, \tag{41}$$

$$\mathbf{A}^m=\mathbf{0}\quad\text{on }\partial\Omega. \tag{42}$$

3.2. An iterative algorithm

We notice that, at each time step, problem (39)–(42) is nonlinear so we propose an iterative algorithm for solution, known as Bermúdez–Moreno algorithm, that has been introduced in [16] in a different abstract context.

Let us denote by $\mathcal{H}_i$ the nonlinear operator from $\mathbb{R}^2$ into itself giving the magnetic field from the magnetic induction in domain Ω_i, that is $\mathbf{H} = \mathcal{H}_i(\mathbf{B}) := \nu_i(|\mathbf{B}|)\mathbf{B}$. We recall that ν_i, the magnetic reluctivity of Ω_i, is the inverse of the magnetic permeability μ_i. Actually we have:

$$\mathbf{B} = \mathcal{B}_i(\mathbf{H}) := \mu_i(|\mathbf{H}|)\mathbf{H}$$

and $\mathcal{H}_i = \mathcal{B}_i^{-1}$.

Let us assume that $\mathcal{H}_i$ can be extended to a maximal monotone operator (see for instance, [17]). Let us associate a positive real number ω_i with each domain Ω_i, $i = 1, \dots, N+1$. For a given field $\mathbf{A}$ let us introduce a multiplier $\mathbf{P}_i$ defined in Ω_i by

$$\mathbf{P}_i := -\omega_i \mathbf{curl}\ \mathbf{A} + \mathcal{H}_i(\mathbf{curl}\ \mathbf{A}). \tag{43}$$

Then, it is possible to prove (see [16]) that (43) is equivalent to

$$\mathbf{P}_i = \mathcal{H}^{\omega_i}_{i,\lambda_i}(\mathbf{curl}\ \mathbf{A} + \lambda_i \mathbf{P}_i), \tag{44}$$

for all $0 < \lambda_i < 1/\omega_i$. In (44), $\mathcal{H}^{\omega_i}_{i,\lambda_i}$ denotes the so-called Yosida regularization of operator $\mathcal{H}_i - \omega_i \mathcal{I}$ given by

$$\mathcal{H}^{\omega_i}_{i,\lambda_i}(\mathbf{B}) := \frac{\mathbf{B} - \mathcal{J}^{\omega_i}_{i,\lambda_i}(\mathbf{B})}{\lambda_i}, \tag{45}$$

where $\mathcal{J}^{\omega_i}_{i,\lambda_i}$ denotes the resolvent operator defined by

$$\mathbf{H} = \mathcal{J}^{\omega_i}_{i,\lambda_i}(\mathbf{B}) \quad \text{if and only if } \mathbf{H} + \lambda_i(\mathcal{H}_i - \omega_i \mathcal{I})(\mathbf{H}) = \mathbf{B},$$

and $\mathcal{I}$ is the identity operator.

Then, equality (44) suggests the use of the following algorithm to solve the nonlinear problem (39)–(42):

- *Initial iteration* ($s = 1$):

$$\mathbf{P}^0_{i,[1]} = \mathbf{0}, \quad i = 1, \dots, N+1, \tag{46}$$

$$\mathbf{P}^m_{i,[1]} = \mathbf{P}^{m-1}_i, \quad i = 1, \dots, N+1, \text{ for } m \geq 1. \tag{47}$$

- *Iteration* $s > 1$: $\mathbf{P}^m_{i,[s]}$, $i = 1, \dots, N+1$ are known. Then compute, successively,

1. $\mathbf{A}^m_{[s]}$ as the solution of the (linear) problem,

$$\begin{aligned} &\frac{1}{t_m - t_{m-1}} \frac{1}{\mathrm{meas}(\Omega_i)} \int_{\Omega_i} \sigma_i A^m_{z,[s]}(x,y)\, \mathrm{d}x\mathrm{d}y\, \mathbf{e}_z + \omega_i\, \mathbf{curl}(\mathbf{curl}\ \mathbf{A}^m_{[s]}) \\ &\quad = \frac{1}{\mathrm{meas}(\Omega_i)} \left(-C_i(t_m)\alpha_i^{-1} + \frac{1}{t_m - t_{m-1}} \int_{\Omega_i} \sigma_i A^{m-1}_z(x,y)\, \mathrm{d}x\mathrm{d}y \right) \mathbf{e}_z \\ &\qquad - \mathbf{curl}\ \mathbf{P}^m_{i,[s]}, \quad \text{in } \Omega_i,\ i = 1, \dots, N_C, \end{aligned} \tag{48}$$

$$\omega_i\, \mathbf{curl}(\mathbf{curl}\ \mathbf{A}^m_{[s]}) = \frac{I_i(t_m)}{\mathrm{meas}(\Omega_i)}\, \mathbf{e}_z - \mathbf{curl}\ \mathbf{P}^m_{i,[s]} \quad \text{in } \Omega_i,\ i = N_C + 1, \dots, \mathrm{N}, \tag{49}$$

$$\omega_0\, \mathbf{curl}(\mathbf{curl}\ \mathbf{A}^m_{[s]}) = -\mathbf{curl}\ \mathbf{P}^m_{0,[s]} \quad \text{in } \Omega_{N+1}, \tag{50}$$

$$\mathbf{A}^m_{[s]} = \mathbf{0}, \quad \text{on } \partial\Omega. \tag{51}$$

2. $\mathbf{P}^m_{i,[s+1]} = \mathcal{H}^{\omega_i}_{i,\lambda_i}(\mathbf{curl}\ \mathbf{A}^m_{[s]} + \lambda_i \mathbf{P}^m_{i,[s]})$, $i = 1, \dots, N+1$.

Remark 3.1. In the case of a magnetically linear material we do not need to introduce the corresponding multiplier $\mathbf{P}$. More precisely, parameter ω_i should be replaced with the magnetic reluctivity in that subdomain and the term involving $\mathbf{P}$ on the right-hand side suppressed.

Remark 3.2. We notice that the linear partial differential operators involved in partial differential equations (48)–(50) are independent of both time step m and iteration $[s]$ as long as the time step is constant. Moreover, when they are discretized by finite elements, the corresponding coefficient matrix is symmetric and positive definite so it can be assembled and factorized only once, out of the two loops corresponding to time steps and iterations.

Remark 3.3. The iterative algorithm proposed above is actually a fixed point method. Moreover, if ω is interpreted as a fixed magnetic reluctivity, this algorithm is similar to the well-known polarization method [18] in that both are based on the splitting (43). However, the updating of the multiplier is different: in [18] it is done by using (43), while in the present paper it is based on some properties of maximal monotone operators and its Yosida regularization. The convergence of the proposed algorithm has been proved in [16] for $\lambda\omega \leq 1/2$ and its performance depends on the choice of these parameters; we refer the reader to [19] for the choice of optimal parameters in some cases. In a similar way, the convergence of the polarization methods also depends on the value of ω (see [18,20]).

4. Computing periodic solutions

If $N_C = 0$, the nonlinear boundary-value problem (20)–(24) has a periodic solution when the given currents $I_1(t), \dots, I_N(t)$ are periodic functions of period T. However, the problem of computing periodic solutions is more involved when there are conductors for which we know the potential drops $C_i(t)$ instead of the currents, i.e., if $N_C \neq 0$. In this case we will assume that the given potential drops are periodic with the same period T and null average, that is,

$$\int_0^T C_i(t)\mathrm{d}t = 0, \quad i = 1, \dots, N_C.$$

We will also assume that the given currents $I_i(t),\ i = N_C + 1, \dots, N$ are periodic functions with common period T.

In order to compute a periodic solution, we could take any initial conditions $\vec{I}_0 = (I_1^0, \dots, I_{N_C}^0)$ and integrate the algebraic-differential system of equations until convergence to a periodic solution. However, this procedure can be very costly from the computational point of view if the "initial currents" are far from the ones corresponding to the periodic solution we are looking for. In what follows we propose a method to determine these "initial currents", in such a way that the periodic solution can be obtained by integrating the problem along one single period.

For $t \in [0, T]$, let us denote by $\mathbf{F}_t = (F_{t,1}, \dots, F_{t,N_C})$ the mapping from $\mathbb{R}^{N_C}$ into itself such that, to the vector of currents $\vec{I} = (I_1, \dots, I_{N_C}) \in \mathbb{R}^{N_C}$ associates the numbers

$$F_{t,i}(\vec{I}) = \alpha_i \int_{\Omega_i} \sigma_i A_z(x, y, t)\, \mathrm{d}x\mathrm{d}y, \quad i = 1, \dots, N_C.$$

We notice that computing $\mathbf{F}_t(\vec{I})$ requires to solve a nonlinear magnetostatic problem at each time t, in order to determine field $A_z(x, y, t)$. By using this mapping, Eqs. (23) can be rewritten as

$$\frac{dF_{t,i}(\vec{I}(t))}{dt} + \alpha_i I_i(t) = -C_i(t), \quad i = 1, \dots, N_C. \tag{52}$$

Moreover, since the equations to compute $F_{t,i}(\vec{I}(t))$ do not involve any time derivative (they are a magnetostatic problem for each time t), if all currents $\{I_1(t), \dots, I_N(t)\}$ are periodic then functions $F_{t,i}(\vec{I}(t))$ are also periodic. By integrating (52) along a period, we get

$$F_{T,i}(\vec{I}(T)) - F_{0,i}(\vec{I}_0) + \alpha_i \int_0^T I_i(t)\mathrm{d}t = -\int_0^T C_i(t)\mathrm{d}t, \quad i = 1, \dots, N_C, \tag{53}$$

and then, as C_i is assumed to have null average, we deduce

$$\int_0^T I_i(t)\mathrm{d}t = 0, \quad i = 1, \dots, N_C. \tag{54}$$

In fact (54) is a necessary and sufficient condition for $F_{t,i}(\vec{I}(t))$ to be a periodic function. Let us integrate (52) from 0 to t:

$$F_{t,i}(\vec{I}(t)) - F_{0,i}(\vec{I}_0) + \alpha_i \int_0^t I_i(s)\mathrm{d}s = -\int_0^t C_i(s)\mathrm{d}s, \quad i = 1, \dots, N_C, \tag{55}$$

and then again from 0 to T:

$$\int_0^T F_{t,i}(\vec{I}(t))\mathrm{d}t - F_{0,i}(\vec{I}_0)T + \alpha_i \int_0^T \left(\int_0^t I_i(s)\mathrm{d}s\right)\mathrm{d}t = -\int_0^T \left(\int_0^t C_i(s)\mathrm{d}s\right)\mathrm{d}t, \quad i = 1, \dots, N_C. \tag{56}$$

From these equations it follows that

$$\int_0^T F_{t,i}(\vec{I}(t))\mathrm{d}t - F_{0,i}(\vec{I}_0)T + \alpha_i \int_0^T (T-s) I_i(s)\mathrm{d}s = -\int_0^T (T-s) C_i(s)\mathrm{d}s, \quad i = 1, \dots, N_C. \tag{57}$$

These equations allow us to compute the initial currents $\vec{I}_0$ leading to a periodic solution from the initial time. This computation can be done by using iterative methods which require solving problem (20)–(24) in $[0, T]$ at each iteration.

In what follows we propose a much simpler alternative method by approximating equations (52) in a way to be precised below. Notice that these equations can be written as

$$\sum_{j=1}^{N_C}(D\mathbf{F}_t(\vec{I}))_{ij}\frac{dI_j(t)}{dt}+\alpha_i I_i(t)=-C_i(t),\quad i=1,\dots,N_C, \tag{58}$$

where $(D\mathbf{F}_t(\vec{I}))_{ij}$ denotes the ij-th element of the Jacobian matrix $D\mathbf{F}_t(\vec{I})$ of $\mathbf{F}_t$ at point $\vec{I}$. Let us assume the following hypothesis:

$$\frac{\alpha_i T}{\min\limits_{t,\vec{I}}(|D\mathbf{F}_t(\vec{I})|)_{ii}}\ll 1,\quad i=1,\dots,N_C.$$

In this case, the term involving α_i can be neglected in (58) and hence in Eqs. (57), which become

$$\int_0^T F_{t,i}(\vec{I}(t))\mathrm{d}t-F_{0,i}(\vec{I}_0)T=-\int_0^T(T-s)C_i(s)\mathrm{d}s,\quad i=1,\dots,N_C. \tag{59}$$

Solving this system of equations to compute $\vec{I}_0$ has the same difficulty as (57). However, we want that property (54) be still satisfied by the approximate solution so let us suppose for a moment that all the materials are magnetically linear. Then, even in the case where permanent magnets are present, mapping $\mathbf{F}_t$ is affine. In fact, it can be written as follows,

$$\mathbf{F}_t(\vec{I})=\mathbf{G}\vec{I}+\mathbf{F}_t(\mathbf{0}),$$

for some matrix $\mathbf{G}$ of order N_C independent of t. Then (59) yields

$$\int_0^T(\mathbf{G}\vec{I}+\mathbf{F}_t(\mathbf{0}))\mathrm{d}t-\mathbf{F}_0(\vec{I}_0)T=-\int_0^T(T-s)\vec{C}(s)\mathrm{d}s, \tag{60}$$

and, since we want to keep property (54),

$$\int_0^T\mathbf{F}_t(\mathbf{0})\,\mathrm{d}t-\mathbf{F}_0(\vec{I}_0)T=-\int_0^T(T-s)\vec{C}(s)\mathrm{d}s. \tag{61}$$

Hence, in order to compute an initial condition leading to a periodic solution from the initial time, we have to solve the system of equations:

$$\mathbf{F}_0(\vec{I}_0)=\frac{1}{T}\left(\int_0^T\mathbf{F}_t(\mathbf{0})\mathrm{d}t+\int_0^T(T-s)\vec{C}(s)\mathrm{d}s\right). \tag{62}$$

Notice that, in order to solve (62), it is first necessary to compute the term $\mathbf{F}_t(\mathbf{0})$ by solving a magnetostatic problem for each value of $t\in[0,T]$. Once this term has been computed, the nonlinear system (62), unlike (57), only involves the magnetostatic problem for time $t=0$.

Remark 4.1. We notice that if there are no permanent magnets and as far as the given currents $I_{N_C+1}(t),\dots,I_N(t)$ have null average, we have

$$\int_0^T\mathbf{F}_t(\mathbf{0})\mathrm{d}t=\mathbf{0},$$

and hence (62) simplifies to

$$\mathbf{F}_0(\vec{I}_0)=\frac{1}{T}\left(\int_0^T(T-s)\vec{C}(s)\mathrm{d}s\right). \tag{63}$$

Moreover, if $N=N_C$ and there are permanent magnets, $\mathbf{F}_t(\mathbf{0})$ is constant in time, that is, $\mathbf{F}_t(\mathbf{0})=\mathbf{F}_0(\mathbf{0})$ for all t. Then,

$$\frac{1}{T}\left(\int_0^T\mathbf{F}_t(\mathbf{0})\mathrm{d}t\right)=\mathbf{F}_0(\mathbf{0})$$

and Eq. (62) becomes

$$\mathbf{G}\vec{I}_0=\frac{1}{T}\int_0^T(T-s)\vec{C}(s)\mathrm{d}s, \tag{64}$$

which means that we could compute equivalent initial currents without including the magnets in the procedure.

In the nonlinear case, may be Eq. (62) does not hold exactly. However, the numerical experiments show that its solution is still a good approximation of the initial currents leading to a periodic solution.

Remark 4.2. Let us adapt the previous method to the particular case considered in Section 2.3. Let us write (62) for the couple of indices i_1 and i_2 as follows:

$$F_{0,i_1}(\vec{I}^0) = \frac{1}{T}\left(\int_0^T F_{t,i_1}(\mathbf{0})\mathrm{d}t + \int_0^T (T-s)C_{i_1}(s)\mathrm{d}s\right), \tag{65}$$

$$F_{0,i_2}(\vec{I}^0) = \frac{1}{T}\left(\int_0^T F_{t,i_2}(\mathbf{0})\mathrm{d}t + \int_0^T (T-s)C_{i_2}(s)\mathrm{d}s\right). \tag{66}$$

Since we do not know either $C_{i_1}(t)$ or $C_{i_2}(t)$ but the difference $V(t)$, we subtract these equations to get

$$F_{0,i_1}(\vec{I}^0) - F_{0,i_2}(\vec{I}^0) = \frac{1}{T}\left(\int_0^T (F_{t,i_1}(\mathbf{0}) - F_{t,i_2}(\mathbf{0}))\mathrm{d}t + \int_0^T (T-s)V(s)\mathrm{d}s\right). \tag{67}$$

This equation will replace (65) and (66). Accordingly, the number of unknowns is also reduced because $I^0_{i_1} = -I^0_{i_2}$.

Eq. (62) is a nonlinear system for $\vec{I}^0$ that can be solved by numerical algorithms; in particular, by the Newton's method. For this purpose it is necessary to compute the partial derivatives of functions $F_{0,i}$. This task will be done below.

Firstly, let us introduce the vector $\vec{g} \in \mathbb{R}^{N_C}$ whose components are

$$g_i := \frac{1}{T}\left(\int_0^T F_{t,i}(\mathbf{0})\mathrm{d}t + \int_0^T (T-s)C_i(s)\mathrm{d}s\right), \quad i = 1,\ldots,N_{\mathrm{C}}. \tag{68}$$

The Newton's method constructs a sequence, $\{\vec{I}_{[n]}\}$, converging to $\vec{I}^0$, namely,

$$\vec{I}_{[n+1]} = \vec{I}_{[n]} - D\mathbf{F}_0(\vec{I}_{[n]})^{-1}(\mathbf{F}_0(\vec{I}_{[n]}) - \vec{g}), \tag{69}$$

where $\mathbf{F}_0(\vec{I}) := (F_{0,1}(\vec{I}),\ldots,F_{0,N_{\mathrm{C}}}(\vec{I}))$ and $D\mathbf{F}_0(\vec{I})$ denotes the Jacobian matrix of $\mathbf{F}_0$ at point $\vec{I}$.

Let $\varDelta\vec{I}_{[n]}$ be the solution of the linear system of order N_{C},

$$D\mathbf{F}_0(\vec{I}_{[n]})\varDelta\vec{I}_{[n]} = -\mathbf{F}_0(\vec{I}_{[n]}) + \vec{g}, \tag{70}$$

then we can write (69) as

$$\vec{I}_{[n+1]} = \vec{I}_{[n]} + \varDelta\vec{I}_{[n]}.$$

Thus, each Newton iteration amounts to solve a linear system whose matrix is the Jacobian matrix $D\mathbf{F}_0(\vec{I}_{[n]})$.

In order to build this matrix, let us consider the following weak magnetostatic problem:

Given $\vec{I} \in \mathbb{R}^{N_{\mathrm{C}}}$, find $A_z \in \mathrm{H}^1_0(\Omega)$ such that

$$\sum_{i=1}^{N+1}\int_{\Omega_i} \nu_i(|\mathbf{curl}\,\mathbf{A}|)\mathbf{curl}\,\mathbf{A}\cdot\mathbf{curl}\,\mathbf{Z}\,\mathrm{d}x\mathrm{d}y = \langle f, Z\rangle + \langle B\vec{I}, Z\rangle \quad \forall Z \in \mathrm{H}^1_0(\Omega), \tag{71}$$

with $\mathbf{A} := A_z\mathbf{e}_z$, $\mathbf{Z} := Z\mathbf{e}_z$ and

$$\langle f, Z\rangle := \sum_{i=N_{\mathrm{C}}+1}^{N}\int_{\Omega_i}\frac{I_i(0)}{\mathrm{meas}(\Omega_i)}Z\,\mathrm{d}x\mathrm{d}y,$$

$$\langle B\vec{I}, Z\rangle := \sum_{i=1}^{N_{\mathrm{C}}}\int_{\Omega_i}\frac{I_i}{\mathrm{meas}(\Omega_i)}Z\,\mathrm{d}x\mathrm{d}y.$$

Let us emphasize that $I_i(0)$, $i = N_{\mathrm{C}}+1,\ldots,N$ are known. We only have to compute the initial currents for conductors where the potential drops are prescribed.

Let us consider the nonlinear operator Ψ from $\mathbb{R}^{N_{\mathrm{C}}}$ into $\mathrm{H}^1_0(\Omega)$ and the linear operator L from $\mathrm{H}^1_0(\Omega)$ into $\mathbb{R}^{N_{\mathrm{C}}}$ defined by,

$$\Psi(\vec{I}) := A_z,$$

$$(L(A_z))_i := \alpha_i\int_{\Omega_i}\sigma_i A_z(x,y,0)\,\mathrm{d}x\mathrm{d}y, \quad i = 1,\ldots,N_{\mathrm{C}}.$$

It is clear that $\mathbf{F}_0$ can be expressed as composition of functions Ψ and L defined above

$$\mathbf{F}_0(\vec{I}) = (L\circ\Psi)(\vec{I}).$$

As a consequence, by using the chain rule and the linearity of operator L, we deduce

$$D\mathbf{F}_0(\vec{I}) = L \circ D\Psi(\vec{I}). \tag{72}$$

Hence, computing $D\mathbf{F}_0(\vec{I})$ amounts to compute $D\Psi(\vec{I})$.

In order to avoid the non-differentiability of the norm at the null vector, we use the trick

$$\nu(|\mathbf{curl}\ \mathbf{A}|) = \frac{1}{2}\,\varphi(|\mathbf{curl}\ \mathbf{A}|^2)$$

where

$$\varphi(x) := 2\,\nu\left(\sqrt{x}\right). \tag{73}$$

We assume that φ is differentiable at any non-negative value. Then, we can rewrite (71) as

$$\sum_{i=1}^{N+1} \int_{\Omega_i} \frac{1}{2}\,\varphi_i(|\mathbf{curl}\ \mathbf{A}|^2)\mathbf{curl}\ \mathbf{A} \cdot \mathbf{curl}\ \mathbf{Z}\, \mathrm{d}x\mathrm{d}y = \langle f, Z\rangle + \langle B\vec{I}, Z\rangle \quad \forall Z \in \mathrm{H}_0^1(\Omega).$$

By applying the implicit function theorem to the previous equation, we can deduce that, for any $\delta\vec{I} \in \mathbb{R}^{N_C}$, the vector field $\delta\mathbf{A} := D\Psi(\vec{I})(\delta\vec{I})$ is the solution of the following weak linear problem:

$$\begin{aligned}\sum_{i=1}^{N+1} \int_{\Omega_i} & (\varphi_i'(|\mathbf{curl}\ \mathbf{A}|^2)(\mathbf{curl}\ \mathbf{A} \otimes \mathbf{curl}\ \mathbf{A})\mathbf{curl}\ \delta\mathbf{A} \cdot \mathbf{curl}\ \mathbf{Z} \\ &+ \frac{1}{2}\,\varphi_i(|\mathbf{curl}\ \mathbf{A}|^2)\mathbf{curl}\ \delta\mathbf{A} \cdot \mathbf{curl}\ \mathbf{Z})\, \mathrm{d}x\mathrm{d}y = \langle B\delta\vec{I}, Z\rangle \quad \forall Z \in \mathrm{H}_0^1(\Omega).\end{aligned} \tag{74}$$

Let us recall that the tensor product of two vectors $\mathbf{a}$ and $\mathbf{b}$ of the same dimension is the endomorphism $\mathbf{a} \otimes \mathbf{b}$ defined by

$$(\mathbf{a} \otimes \mathbf{b})\mathbf{c} = \mathbf{c} \cdot \mathbf{b}\, \mathbf{a}$$

for any vector $\mathbf{c}$.

Therefore, by using (72),

$$(D\mathbf{F}_0(\vec{I})(\delta\vec{I}))_i = \alpha_i \int_{\Omega_i} \sigma_i \delta A_z(x, y, 0)\, \mathrm{d}x\mathrm{d}y, \quad i = 1, \ldots, N_C, \tag{75}$$

$\delta\mathbf{A}$ being the solution of (74).

The symmetric bilinear form, $a(\cdot, \cdot)$, associated to problem (74) is given by:

$$\begin{aligned} a(u, v) := \sum_{i=1}^{N+1} \int_{\Omega_i} & (\varphi_i'(|\mathbf{curl}\ \mathbf{A}|^2)(\mathbf{curl}\ \mathbf{A} \otimes \mathbf{curl}\ \mathbf{A})\mathbf{curl}\ \mathbf{u} \cdot \mathbf{curl}\ \mathbf{v} \\ &+ \frac{1}{2}\,\varphi_i(|\mathbf{curl}\ \mathbf{A}|^2)\mathbf{curl}\ \mathbf{u} \cdot \mathbf{curl}\ \mathbf{v})\, \mathrm{d}x\mathrm{d}y \end{aligned} \tag{76}$$

on $\mathrm{H}^1(\Omega) \times \mathrm{H}^1(\Omega)$, where $\mathbf{u} = u\mathbf{e}_z$ and $\mathbf{v} = v\mathbf{e}_z$. Let us introduce the tensor fields

$$S_{\mathbf{A},i} := \varphi_i'(|\mathbf{curl}\ \mathbf{A}|^2)(\mathbf{curl}\ \mathbf{A} \otimes \mathbf{curl}\ \mathbf{A}) + \frac{1}{2}\,\varphi_i(|\mathbf{curl}\ \mathbf{A}|^2)\mathbf{I}, \quad i = 1, \ldots, N_C,$$

where $\mathbf{I}$ represents the identity matrix. Then $\delta\mathbf{A}$ is the solution of the linear magnetostatic problem:

$$\int_{\Omega} S_{\mathbf{A}}\, \mathbf{curl}\ \delta\mathbf{A} \cdot \mathbf{curl}\ \mathbf{Z}\, \mathrm{d}x\mathrm{d}y = \langle B\delta\vec{I}, Z\rangle \quad \forall Z \in \mathrm{H}_0^1(\Omega), \tag{77}$$

where $S_{\mathbf{A}}$ is the tensor defined by $S_{\mathbf{A}}|_{\Omega_i} := S_{\mathbf{A},i}$, $i = 1, \ldots, N + 1$.

Since $\mathbf{curl}\ \mathbf{A}$ is constant in every triangle of the mesh, the same is true for $S_{\mathbf{A}}$. Moreover, in the case where material Ω_i is linear, we have $S_{\mathbf{A},i} = \nu_i\, \mathbf{I}$.

Then, in order to obtain the order N_C matrix $D\mathbf{F}_0(\vec{I})$, it will be enough to solve (77) for $\delta\vec{I} = \mathbf{e}_j$, $j = 1, \ldots, N_C$, where $\mathbf{e}_j$ is the j-th vector of the canonical basis in $\mathbb{R}^{N_C}$. Indeed, for this choice, vector $L(\delta\mathbf{A}_j)$ is just the j-th column of the Jacobian matrix $D\mathbf{F}_0(\vec{I})$.

Thus, in every iteration of the Newton's method we have to solve N_C linear magnetostatic problems with the same "reluctivity" matrix $S_{\mathbf{A}^n}$, where $\mathbf{A}^n$ is the solution of the nonlinear magnetostatic problem corresponding to $\vec{I}_{[n]}$. It is worth mentioning that the coefficient matrix of the finite element approximation of problem (77) does not depend on index j so it can be computed and factorized only once per Newton's iteration.

Finally, in order to solve the linear system (70) we can use Gauss method. We notice that vector $\vec{g}$ in the right-hand side is independent of n so it should be computed only once out of the Newton's iteration loop.

Remark 4.3. We can adapt the above algorithm to the particular case described in Section 2.3. For this purpose let us assume that among the N_C conductors where the currents are unknown, $2N_V$ of them are couples with opposite currents. Let us denote by $\mathcal{U}$ the Boolean matrix of order $N_C \times (N_C - N_V)$ such that to each "reduced current vector" $\mathbb{I}$ in $\mathbb{R}^{N_C - N_V}$ associates the full vector of currents, $\vec{I}$, along the whole N_C conductors i.e., $\vec{I} = \mathcal{U}\mathbb{I}$.

Let $\mathbf{G}_0$ be the mapping from $\mathbb{R}^{N_C - N_V}$ into itself defined by

$$\mathbf{G}_0(\mathbb{I}) := \mathcal{U}^t \mathbf{F}_0(\mathcal{U}\mathbb{I}).$$

Then we have

$$D\mathbf{G}_0(\mathbb{I}) = \mathcal{U}^t D\mathbf{F}_0(\mathcal{U}\mathbb{I})\mathcal{U}.$$

In order to determine the initial currents we have to solve the following nonlinear system (see Remark 4.2):

$$\mathbf{G}_0(\mathbb{I}) = \mathcal{U}^t \vec{g},$$

with the components of $\mathcal{U}^t \vec{g}$ given in (67). In this case the Newton's method becomes

$$\mathbb{I}_{[n+1]} = \mathbb{I}_{[n]} + \Delta\mathbb{I}_{[n]},$$

where $\Delta\mathbb{I}_{[n]}$ is the solution of the linear system:

$$\mathcal{U}^t D\mathbf{F}_0(\mathcal{U}(\mathbb{I}_{[n]}))\,\mathcal{U}\,\Delta\mathbb{I}_{[n]} = -\mathcal{U}^t\,\mathbf{F}_0(\mathcal{U}\mathbb{I}) + \mathcal{U}^t \vec{g}. \tag{78}$$

Remark 4.4. In practical applications, for each subdomain Ω_i with a nonlinear material, we have a B–H table. In other words, we have discrete values for the magnetic reluctivity of each domain Ω_i. From these values, we can also obtain a table of discrete values for φ_i (see (73)). Using this table we build an interpolating cubic spline from which the derivative φ_i' is easily computed.

Remark 4.5. Since we have neglected the term involving the $I_i(t)$ in Eq. (52), the initial condition $\vec{I}^0$ obtained by the above procedure may not yield exactly the periodic solution of (52) we are seeking, from $t = 0$. Anyway, as numerical results show, we get a good approximation that could be improved by computing a few more periods if needed.

5. An example with cylindrical symmetry

In this section we describe an example with cylindrical geometry which has an analytical solution even in the nonlinear case. This feature will be exploited in the next section to validate the above numerical methods.

Let us consider a cylindrical domain composed by a laminated magnetic core, surrounded by an infinitely thin coil (see Fig. 2). We denote by (r, θ, z) the cylindrical coordinate system and by $\mathbf{e}_r$, $\mathbf{e}_\theta$, $\mathbf{e}_z$ the corresponding unit vectors of the local orthonormal basis. We assume that the z axis coincides with the central axis of the cylindrical domain. Moreover, the core and the coil are assumed to be infinite in the z-direction.

Let us suppose axisymmetry of the current sources and also that they are supported on the core–air interphase. Thus, the coil is modeled as a surface conductor in 3D and then by a curve in 2D; namely, let the inner coil be located on the surface S_1, $r = R_1$, and the outer one S_2, placed on $r = R_2$. In order to apply the 2D transient magnetic model described in previous sections we suppose that the surface current density of the sources is given by $\mathbf{J}_S(r, \theta, z, t) = J_{S_z}(r, t)\mathbf{e}_z$, for $r = R_1,\ R_2$, with

$$J_{S_z}(R_1, t) = \frac{n_e I(t)}{2\pi R_1}$$

and

$$J_{S_z}(R_2, t) = -\frac{n_e I(t)}{2\pi R_2},$$

n_e being the number of turns of the coil and $I(t)$ the current in the coil.

In this example, the data of the problem can be either the current $I(t)$ or the difference of potential drops in the two conductors, namely, $V(t) = C_1(t) - C_2(t)$.

In this case, all fields are independent of the azimuthal variable. In particular, the magnetic vector potential is of the form

$$\mathbf{A}(r, \theta, z, t) = A_z(r, t)\mathbf{e}_z. \tag{79}$$

Hence,

$$\mathbf{curl}\ \mathbf{A} = -\frac{\partial A_z}{\partial r}\mathbf{e}_\theta \tag{80}$$

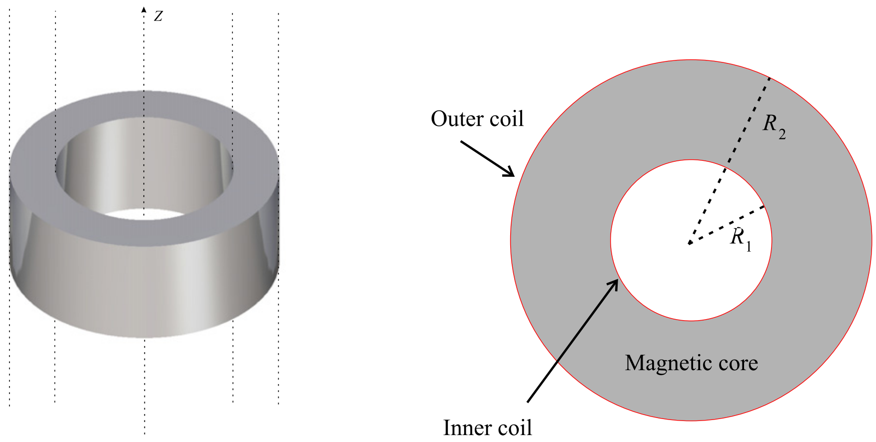

Fig. 2. Magnetic core (left) and sketch of the cross section (right).

and Eqs. (12), (30) become

$$\frac{\partial}{\partial r}\left(\frac{r}{\mu}\frac{\partial A_z}{\partial r}\right) = 0 \quad \text{in the laminated core and in the air,} \tag{81}$$

$$\left[\frac{1}{\mu}\frac{\partial A_z}{\partial r}\right] = J_{S_z} \quad \text{on the surface of the core,} \tag{82}$$

where $[\cdot]$ denotes the jump across the surface of the core.

In this case Problem 2.1 can be easily solved. Notice that we neglect the electrical conductivity of the laminated core along the axial direction and we actually solve a magnetostatic problem at each time.

Indeed, since $\mathbf{B} = \mathbf{curl}\, \mathbf{A}$, from (80) we deduce $\mathbf{B} = B_\theta \mathbf{e}_\theta$ and then

$$A_z(r,t) = A_z(\infty,t) + \int_r^\infty B_\theta(s,t)\,\mathrm{d}s = \int_r^\infty B_\theta(s,t)\,\mathrm{d}s. \tag{83}$$

Moreover, from the constitutive law we have

$$B_\theta(r,t) = \mathcal{B}(H_\theta(r,t)),$$

with $\mathcal{B}(H_\theta) = \mu(|H_\theta|)H_\theta$, and from the Ampére's law the magnetic field H_θ can be obtained as

$$H_\theta(r,t) = \begin{cases} 0, & 0 \le r \le R_1, \\ \dfrac{n_e I(t)}{2\pi r}, & R_1 \le r \le R_2, \\ 0, & r \ge R_2. \end{cases}$$

Let us notice that, in this example, the magnetic field intensity does not depend on the magnetic properties of the core, that is, on the particular function $\mathcal{B}$ relating the magnetic induction to the magnetic field intensity.

Therefore,

$$A_z(r,t) = \begin{cases} \displaystyle\int_{R_1}^{R_2} B_\theta(s,t)\,\mathrm{d}s, & 0 \le r \le R_1, \\ \displaystyle\int_r^{R_2} B_\theta(s,t)\,\mathrm{d}s, & R_1 \le r \le R_2, \\ 0, & r \ge R_2. \end{cases}$$

In the linear case, $\mathcal{B}(H_\theta) = \mu H_\theta$, and we get

$$A_z(r,t) = \begin{cases} \dfrac{\mu n_e I(t)}{2\pi} \log\left(\dfrac{R_2}{R_1}\right), & 0 \le r \le R_1, \\ \dfrac{\mu n_e I(t)}{2\pi} \log\left(\dfrac{R_2}{r}\right), & R_1 \le r \le R_2, \\ 0, & r \ge R_2. \end{cases}$$

Now, let us suppose function $\mathcal{B}$ is given by

$$\mathcal{B}(H_\theta) = \mu_0 H_\theta + \alpha \arctan(\gamma H_\theta), \tag{84}$$

where α and γ are two constants depending on the material.

Then, if $R_1 \leq r \leq R_2$,

$$B_\theta(r,t) = \mathcal{B}(H_\theta(r,t)) = \mathcal{B}\left(\frac{n_e I(t)}{2\pi r}\right) = \mu_0 \frac{n_e I(t)}{2\pi r} + \alpha \arctan\left(\frac{\gamma n_e I(t)}{2\pi r}\right). \tag{85}$$

Let us compute A_z. First, let us denote,

$$\beta(t) := \frac{\gamma n_e I(t)}{2\pi}.$$

Then, we have

$$A_z(r,t) = \mu_0 \frac{n_e I(t)}{2\pi} \log\left(\frac{R_2}{R_1}\right) + \alpha\left(\frac{\beta(t)}{2} \log\left(\frac{\beta(t)^2 + R_2^2}{\beta(t)^2 + R_1^2}\right) + R_2 \arctan\left(\frac{\beta(t)}{R_2}\right) - R_1 \arctan\left(\frac{\beta(t)}{R_1}\right)\right),$$

for $0 \leq r \leq R_1$,

$$A_z(r,t) = \mu_0 \frac{n_e I(t)}{2\pi} \log\left(\frac{R_2}{r}\right) + \alpha\left(\frac{\beta(t)}{2} \log\left(\frac{\beta(t)^2 + R_2^2}{\beta(t)^2 + r^2}\right) + R_2 \arctan\left(\frac{\beta(t)}{R_2}\right) - r \arctan\left(\frac{\beta(t)}{r}\right)\right),$$

for $R_1 \leq r \leq R_2$ and

$$A_z(r,t) = 0, \quad \text{for } r \geq R_2.$$

Then

$$\int_{S_2} \sigma_2^S A_z(r,t)\,\mathrm{d}r = 0,$$

while

$$\begin{aligned}\int_{S_1} \sigma_1^S A_z(r,t)\,\mathrm{d}r &= 2\pi R_1 \sigma_1^S \left\{\mu_0 \frac{n_e I(t)}{2\pi} \log\left(\frac{R_2}{R_1}\right)\right. \\ &\quad \left. +\ \alpha\left(\frac{\beta(t)}{2} \log\left(\frac{\beta(t)^2 + R_2^2}{\beta(t)^2 + R_1^2}\right) + R_2 \arctan\left(\frac{\beta(t)}{R_2}\right) - R_1 \arctan\left(\frac{\beta(t)}{R_1}\right)\right)\right\},\end{aligned}$$

where σ_i^S (Ohm^{-1}) are the *surface* electrical conductivities of coils S_i, $i = 1, 2$. Then,

$$\alpha_i = \frac{1}{2\pi R_i \sigma_i^S}, \quad i = 1, 2,$$

and Eq. (23) yields (we notice that in this case $N = N_C = 2$):

$$\frac{d}{dt} \int_{S_1} \sigma_1^S A_z(x,y,t)\,\mathrm{d}l + n_e I(t) = -C_1(t) 2\pi R_1 \sigma_1^S, \tag{86}$$

$$-n_e I(t) = -C_2(t) 2\pi R_2 \sigma_2^S. \tag{87}$$

If current $I(t)$ is given, Eqs. (86) and (87) allow us to compute the potential drops in each of the two conductors. More specifically, we have,

$$C_1(t) = -\frac{n_e I(t)}{2\pi R_1 \sigma_1^S} - \frac{n_e I'(t)}{2\pi}\left(\mu_0 \log\left(\frac{R_2}{R_1}\right) + \frac{\gamma\,\alpha}{2} \log\left(\frac{\beta^2 + R_2^2}{\beta^2 + R_1^2}\right)\right), \tag{88}$$

$$C_2(t) = \frac{n_e I(t)}{2\pi R_2 \sigma_2^S}. \tag{89}$$

Conversely, by subtracting (89) from (88) we get

$$\frac{n_e I'(t)}{2\pi}\left(\mu_0 \log\left(\frac{R_2}{R_1}\right) + \frac{\gamma\,\alpha}{2} \log\left(\frac{\beta^2 + R_2^2}{\beta^2 + R_1^2}\right)\right) + \left(\frac{1}{R_1 \sigma_1^S} + \frac{1}{R_2 \sigma_2^S}\right) \frac{n_e I(t)}{2\pi} = -V(t). \tag{90}$$

This is a (nonlinear) first order ordinary differential equation that can be integrated (may be, numerically) with the initial condition $I(0) = I^0$, in order to obtain the current $I(t)$ from the difference of potential drops in the two conductors, $V(t)$.

In the linear case, $\mathcal{B}(H_\theta) = \mu H_\theta$ and the above equation becomes

$$\frac{\mu n_e}{2\pi} \log\left(\frac{R_2}{R_1}\right) I'(t) + \left(\frac{1}{R_1 \sigma_1^S} + \frac{1}{R_2 \sigma_2^S}\right) \frac{n_e I(t)}{2\pi} = -V(t). \tag{91}$$

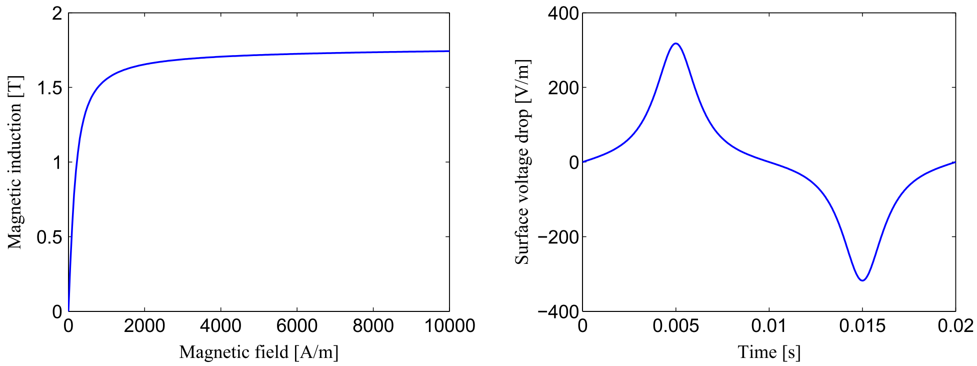

Fig. 3. $\mathcal{B}(H_\theta)$ curve used in the analytical test (left). Voltage drop vs. time for $I(t) = 3000\cos(100\pi t)$ (right).

We recall that its unique solution for $I(0) = I^0$ is given by,

$$I(t) = e^{-\frac{a}{b}t}\left(I^0 - \int_0^t \frac{1}{b}V(s)e^{\frac{a}{b}s}\mathrm{d}s\right), \tag{92}$$

where

$$a = \left(\frac{1}{R_1\sigma_1^S} + \frac{1}{R_2\sigma_2^S}\right)\frac{n_e}{2\pi}$$

and

$$b = \frac{\mu n_e}{2\pi}\log\left(\frac{R_2}{R_1}\right).$$

Remark 5.1. Notice that this example corresponds to the particular case considered in Section 2.3. Eq. (33) becomes,

$$\alpha_1\frac{d}{dt}\int_{S_1}\sigma_1^S A_z(x,y,t)\,\mathrm{d}l + (\alpha_1+\alpha_2)n_e I(t) = -C_1(t) + C_2(t) = -V(t). \tag{93}$$

(Subtract (87) from (86) after multiplying these equations by α_2 and α_1, respectively.)

6. Numerical results

In this section we report some numerical results obtained with a Fortran code implementing the numerical methods described above. First, in order to check the implementation of the numerical code we have solved the analytical example presented in the previous section by providing a surface voltage drop as data. Next, by using the same geometry, we have solved the problem with a PWM voltage drop where the computation of the initial current is crucial to reduce the computational effort. We illustrate this fact by using linear and nonlinear examples. Finally, we present an application in a two-dimensional geometry taken from [14] and compare the computational effort needed to reach the steady-state starting from the current obtained by the method in Section 4 or from the null current.

6.1. Test with known analytical solution

Let us consider the cylindrical device presented in Section 5 the transversal section of which is depicted in Fig. 2. The coil is modeled as a surface conductor as it has been done in the analytical computations; namely, the inner coil is placed at $r = R_1$ and the outer one at $r = R_2$. Surface sources defined on these surfaces (curves in 2D) will be the data for the discrete problem.

Since both conductors carry the same current with opposite sign, we can compute analytically the difference of their respective potential drops for a known current $I(t)$; namely, $V(t) = C_1(t) - C_2(t)$, with C_1 and C_2 given by (88)–(89). We have considered as reference current a cosine function of frequency $f = 50$ Hz, namely, $I(t) = 3000\cos(2\pi f t)$, and solved the discrete problem by providing the corresponding $V(t)$ given by (90) which is represented in Fig. 3-right.

The magnetic core has a nonlinear magnetic behavior given by expression (84) with $\alpha = 3.5/\pi$, and $\gamma = 4999\,\mu_0/\alpha$. Fig. 3-left illustrates this nonlinear function. On the other hand, $\sigma_1^S = \sigma_2^S = \sigma d$ where d is the thickness of the "surface" conductors. The geometrical and physical data that complete the example are given in Table 1.

Firstly, we have computed the initial current by using the procedure developed in Section 4. The tolerance parameter used in the two iterative algorithms involved in the calculus has been 10^{-4} in relative error. The computed initial current is

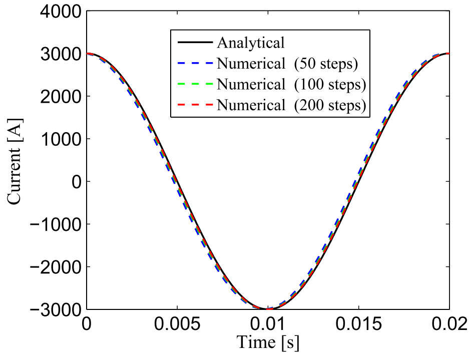

Fig. 4. Approximated current vs. time by using different number of time steps.

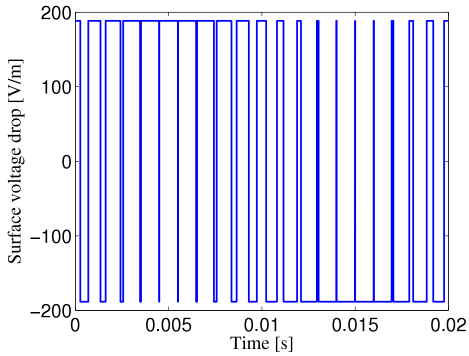

Fig. 5. PWM surface voltage drop.

Table 1
Axisymmetric test. Geometrical data and physical parameters (SI Units).

Description	Parameter	Value
Inner radius of the magnetic core	R_1	1 m
Outer radius of the magnetic core	R_2	1.401 m
Thickness of the coil	d	0.001 m
Electrical conductivity of the coil	σ	5.96 e7 $(\text{Ohm m})^{-1}$
Magnetic permeability of the vacuum	μ_0	$4\pi \times 10^{-7}$ Hm^{-1}
Number of turns of the coil	n_e	1

2999.93, that is, a very good approximation of the exact value which is equal to 3000. Secondly, by using this initial current we have solved the problem in a cycle, that is, in the time interval [0, 0.02] with the difference of potential drops, $V(t)$, as data. Fig. 4 shows the approximate and analytical current versus time by using 50, 100 and 200 steps in a cycle. Notice that the approximation is very good and clearly improves as the time step tends to zero.

6.2. Test with a surface PWM voltage drop

In this section we consider the geometry of the previous test supplied with a pulse-width modulation (PWM) voltage. This kind of source is often used to feed electrical machines. It is a discontinuous function with a great number of discontinuities in each period (see, for instance, [21]). Thus, we should use a very small time step in the numerical method to obtain accurate results. Moreover, since we do not know the initial current corresponding to the steady-state solution, in principle we should solve the problem along a large number of cycles to compute the electromagnetic field. To illustrate this last feature, we start analyzing the case of a linear magnetic core where, from a known value of the initial current, we can compute the exact current $I(t)$ by means of expression (92).

Let us consider the surface voltage depicted in Fig. 5 whose period is equal to 0.02 and oscillates between 188.46 V/m and −188.46 V/m. If the relative magnetic permeability of the core is equal to 500, Fig. 6-left shows the current vs. time obtained starting with an initial value $I(0) = 0$. Notice that the curve seems to reach the steady-state after about 9000 periods. However, if the initial current is taken to be 18 320 A, which is the value obtained with the methodology described above, the current shown in Fig. 6-right is periodic from the beginning. More precisely, Fig. 7 shows the current obtained by the step-by-step method in the 2400th period and the one obtained in the first period with the suitable initial current.

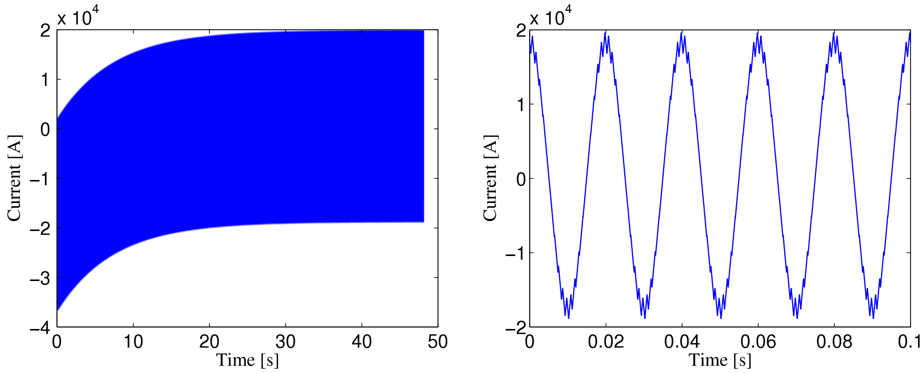

Fig. 6. Current [A] vs. time [s] with different starting points: $I(0) = 0$ (left) and $I(0)$ approximated with the methodology proposed in this paper (right). Linear magnetic core and PWM voltage supply.

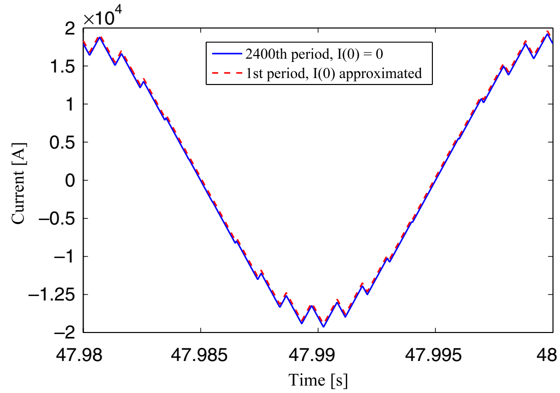

Fig. 7. Comparison of coil current by using a null initial current and the computed value. Linear magnetic core and PWM voltage supply.

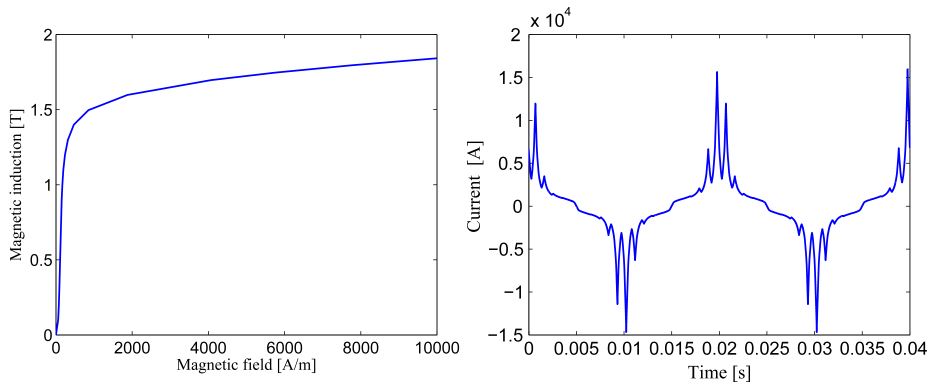

Fig. 8. H–B curve used in the case of PWM excitation (left). Approximated current vs. time under PWM excitation (right).

To end this section, we exploit the procedure to compute the initial current in a nonlinear case and show the obtained current under PWM excitation. In particular, we suppose that the magnetic core has the nonlinear behavior depicted in Fig. 8-left which corresponds to an industrial laminated material. The initial current approximated by the above methodology is equal to 6689.27 A and starting from this value we have computed the current for two cycles with the finite element method. In each of them we have used 8000 time steps and the current obtained is shown in Fig. 8-right. The steady state is reached in the first cycle which is an important advantage because the number of time steps needed in each cycle is very large. We emphasize that this is the reason of having included the previous linear example, instead of a nonlinear one, in order to show the importance of determining a good initial current in the case of PWM signals. Starting from $I(0) = 0$ as some commercial packages do, would be extremely expensive from the computational point of view.

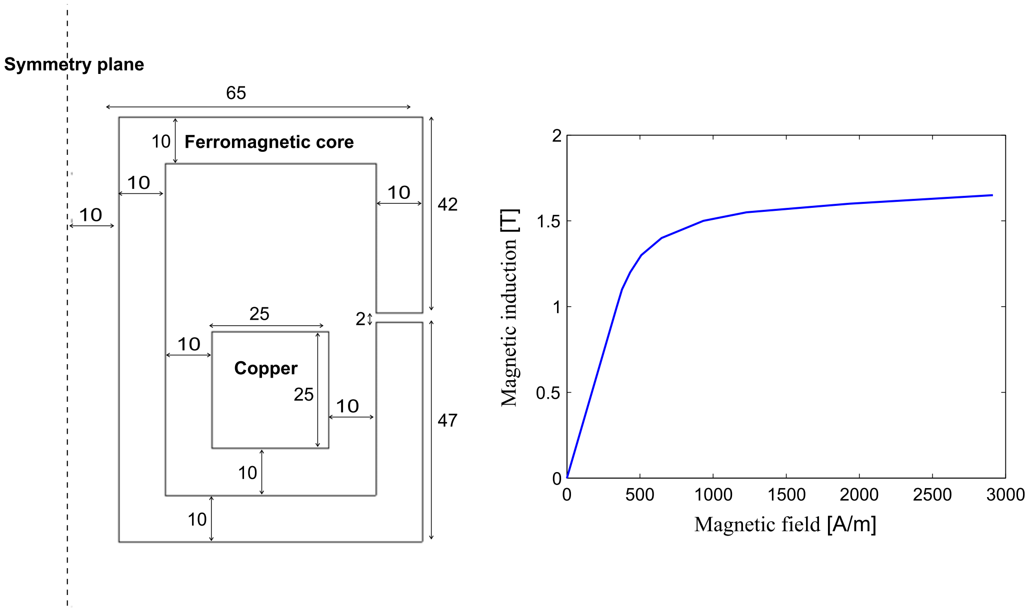

Fig. 9. Cross section of the 2D domain (left); lengths are given in millimeter. H–B curve of the ferromagnetic core (right).

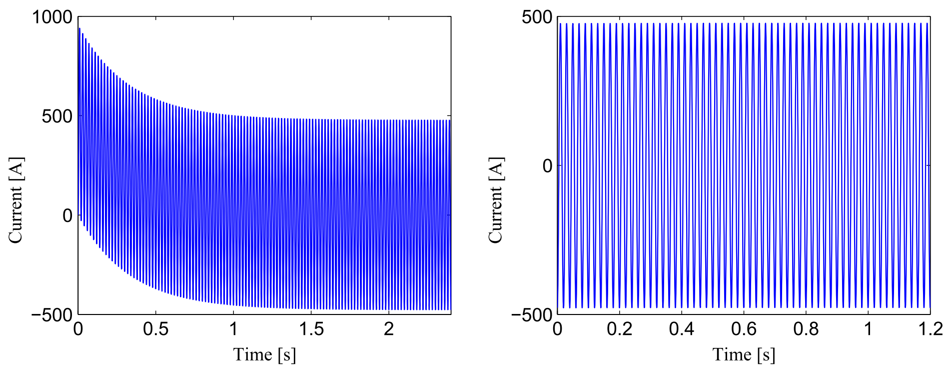

Fig. 10. Current [A] vs. time [s] with different starting points: $I(0) = 0$ (left) and $I(0)$ obtained with the methodology proposed in this paper (right).

6.3. A two-dimensional nonlinear application

In this section we present the numerical results obtained in a two-dimensional domain which is composed by a copper coil and a ferromagnetic screen. The geometry is depicted in Fig. 9 and has been taken from [14] where the authors solve a nonlinear transient eddy current model by a novel method which also accelerates the transient process to the steady-state. Here, we will use the same data but assuming that the ferromagnetic screen is not conducting and the copper section is a stranded coil; notice that these assumptions are due to the fact our model does not include eddy currents. The H–B curve of the ferromagnetic screen is shown in Fig. 9 and detailed in numerical format in [14]; the electrical conductivity of the coil is equal to 5.7 e7 $(\text{Ohm m})^{-1}$.

The source for the coil is a potential drop per unit length in the z-direction given by $C(t) = 1.4\sin(100\pi t)$ V/m. Firstly, we have solved the nonlinear problem starting with null initial current and advanced in time until reaching the steady-state. More precisely, we have solved 120 periods by using 50 time-steps per period and a stopping criterion of 0.01% for the relative error in the nonlinear iterative algorithm. The evolution of the current in the coil is shown in Fig. 10-left.

On the other hand, we have computed the initial current by using the methodology described above. Starting from this current, we have solved 60 periods and we can see in Fig. 10-right that the steady state is practically reached in the first period. More precisely, the relative error in the Euclidean norm between the vector of currents in the first period and in the 60th period is less than 0.5%. Thus, once the right initial current is known one could solve only along one period to obtain a good solution, which represents a very important saving in computational effort. Fig. 11 shows the current obtained by the step-by-step method in the 120th period and the one obtained in the first period with the suitable initial current. The difference is very small, actually less than 0.5% measured in the Euclidean norm.

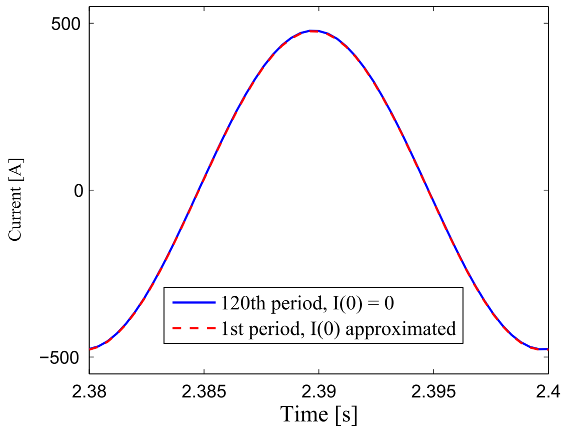

Fig. 11. Comparison of coil current by using a null initial current and the computed value.

Table 2
Average number of iterations vs. C_m and ω.

C_m	1.4	3	5	6.5
$w_a = 5094.66$	13	97	101	91
$w_b = 757.48$	14	15	42	73
$w_c = 9961.00$	181	194	198	124

Finally, we present some results focused on the convergence of the numerical algorithm proposed in Section 3.2. Although a detailed analysis of the convergence is out of the scope of this work, this algorithm seems to be an interesting alternative for the numerical solution of nonlinear magnetic problems. In particular, we have solved the previous problem for different levels of saturation by using different values of parameter ω. These values have been computed from the discrete data conforming the H–B curve by following different procedures. More precisely, if $H_i, B_i, i = 1, \ldots, n$ are the discrete data characterizing the nonlinear behavior of the ferromagnetic core, we define:

$$\omega_a = \frac{1}{n}\sum_{i=1}^{n-1}\frac{H_{i+1}-H_i}{B_{i+1}-B_i}, \qquad \omega_b = \frac{\sum_{i=1}^{n} H_i}{\sum_{i=1}^{n} B_i} \quad \text{and} \quad \omega_c = \frac{\nu_{\min}+\nu_{\max}}{2},$$

where $\nu_{\min}$ and $\nu_{\max}$ denote the minimum and maximum slope, respectively, of the H–B curve. The last value ω_c guarantees convergence for the polarization method presented in [18]. In all cases, the value of λ has been taken equal to $1/(2\omega)$.

On the other hand, we have modified the amplitude C_m of the voltage drop per unit length in $C(t) = C_m \sin(100\pi t)$, in order to consider different levels of saturation of the material. Table 2 shows the average number of iterations needed in a cycle by using a tolerance parameter of 10^{-4} in relative error. Notice that, in each case, we have solved the problem only in one cycle because the initial current has been computed by using the methodology proposed in the paper.

Notice also that the number of iterations clearly depends on the value of ω. In particular, ω_b seems to be a suitable choice when using a fixed parameter. In a forthcoming paper we will analyze the convergence when ω is no longer constant but a function of (x, y), which could improve the performance of the method [19].

7. Conclusions

We have introduced a numerical method to compute nonlinear transient magnetic fields in two-dimensional domains under different source conditions. If the source is given in terms of voltage drops, the transient problem also requires the initial currents as data. The paper proposes an efficient methodology to compute the initial currents that correspond to the steady-state solution thus allowing for an important saving in computational effort. In particular, the approach is very useful to work with PWM signals which require many time steps per cycle, or with large complex geometries. The performance of the methodology is shown by means of numerical experiments.

Acknowledgments

The authors express their gratitude to Dr J. Poza and Dr G. Almandoz from the Universidad de Mondragón (Spain) and to A. González from ORONA company for useful discussions about PWM signals, providing the data corresponding to these signals and the picture of the laminated core presented in Fig. 1. This collaboration was supported by ORONA company under CENIT contract Net0Lift.

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