Polishing Lead Crystal Glass

Università degli Studi di Firenze
University of Oxford
Universidad Complutense de Madrid

June 24, 2008
Introduction
Model 1: constant normal velocity
Model 2: linear velocity
Model 3: exponential velocity
Conclusions

The Team

Agnese Bondi
Francisco López
Luca Meacci
Cristina Pérez
Luis Felipe Rivero
Elena Romero
Summary

1. Introduction

2. Model 1: constant normal velocity
   - The model
   - Numerical results

3. Model 2: linear velocity
   - The model
   - Numerical simulations and analysis

4. Model 3: exponential velocity

5. Conclusions
Introduction

Irish manufacturer produces lead crystal glasses.

They become opaque and rough after the cutting process.

Polishing with immersion in acid.
Polishing process

Acid immersion → Rinsing process → Settle down

The glasses are introduced in inserts like the one we can see in the picture.

Those inserts belong to a basket which is immersed in the acid solution.
Polishing process

Acid immersion → Rinsing process → Settle down

The glasses are introduced in inserts like the one we can see in the picture.

Those inserts belong to a basket which is immersed in the acid solution.
Polishing process

Acid immersion $\rightarrow$ Rinsing process $\rightarrow$ Settle down

The glasses are introduced in inserts like the one we can see in the picture.

Those inserts belong to a basket which is immersed in the acid solution.
Polishing process

Acid immersion → Rinsing process → Settle down

The glasses are introduced in inserts like the one we can see in the picture.

Those inserts belong to a basket which is immersed in the acid solution.
Polishing process

- Reactions.

\[
\begin{align*}
SiO_2 + 4HF & \rightarrow SiF_4 + 2H_2O \\
PbO + H_2SO_4 & \rightarrow PbSO_4 + H_2O \\
K_2O + 2HF & \rightarrow 2KF + H_2O \\
SiF_4 + 2HF & \rightarrow H_2SiF_6
\end{align*}
\]

- Oxid + Acid = Salts.
- Soluble salts disappear in the water.
- Insoluble salts precipitate.
What is the problem?

- How does the process work?
- How long should the glass be immersed?
- Optimising the problem?
General assumptions

- One dimensional problem
- Initial form as the roughness: sinus.
- Homogeneous Neumann conditions.
Model 1: constant normal velocity

- \( s(x, t) \) surface.
- \( F(x, z, t) = z - s(x, t) = 0 \).
- \( n = \frac{\nabla F}{||\nabla F||} = \frac{(-s_x, 1)}{\sqrt{1+s_x^2}} \).
- Material Derivative
  \[ \frac{\partial F}{\partial t} + v_n ||\nabla F|| = 0 \]
  \( v_n \) rate removal surface

\[ s_t = -v \sqrt{1 + s_x^2} \]
Model 1: constant normal velocity

- $s(x, t)$ surface.
- $F(x, z, t) = z - s(x, t) = 0$.
- $n = \frac{\nabla F}{||\nabla F||} = \frac{(-s_x, 1)}{\sqrt{1+s_x^2}}$.
- Material Derivative
  \[ \frac{\partial F}{\partial t} + v_n ||\nabla F|| = 0 \]
  $v_n$ rate removal surface

First Model Equation

\[ s_t = -v \sqrt{1 + s_x^2} \]
Introducione
Model 1: constant normal velocity
Model 2: linear velocity
Model 3: exponential velocity
Conclusions

Charpit Method

Non-dimensionalised equation

\[ s_t = -\sqrt{1 + s_x^2} \]

\[ F(x, t, s, p, q) = q + \sqrt{1 + p^2} = 0, \quad p = s_x, \quad q = s_t \]

Problem

\[
\begin{aligned}
\dot{x} &= F_p \\
\dot{t} &= F_q \\
\dot{s} &= pF_p + qF_q \\
\dot{p} &= -F_x - pF_s \\
\dot{q} &= -F_t - qF_s 
\end{aligned}
\]
Problem and Solution

\[
\begin{align*}
\dot{x} &= \frac{p}{\sqrt{1+p^2}} \\
\dot{t} &= 1 \\
\dot{s} &= -\frac{1}{\sqrt{1+p^2}} \\
\dot{p} &= 0 \\
\dot{q} &= 0
\end{align*}
\]

\[
S(X(\xi, t), t) = S_0(\xi) - \frac{1}{\sqrt{1+S_0'^2}} t
\]

\[
X(\xi, t) = \xi + \frac{S_0'}{\sqrt{1+S_0'^2}} t.
\]
Plotting with Matlab

\[ t \in [0, 3] \]
\[ x \in [-15, 15] \]
Plotting with Matlab

\[ t \in [0, 3], \, x \in [0, 2\pi] \]

\textit{time step} = 1, \textit{space step} = 1
Plotting with COMSOL Multiphysics
Model 2: linear velocity

- Linear relationship between velocity and surface curvature $k$.
- $v = v_0 + v_1 k$.
- $k = -\frac{s_{xx}}{(1+s_x^2)^{3/2}}$.

Second Model Equation

$$s_t = -v_0 \sqrt{1 + s_x^2} + v_1 \frac{s_{xx}}{1 + s_x^2}.$$
Model 2: linear velocity

- Linear relationship between velocity and surface curvature $k$.
  \[ v = v_0 + v_1 \kappa. \]
- $\kappa = -\frac{s_{xx}}{(1+s_x^2)^{3/2}}$.

**Second Model Equation**

\[ s_t = -v_0 \sqrt{1 + s_x^2} + v_1 \frac{s_{xx}}{1 + s_x^2}. \]
Numerical simulations

1. Non-dimensionalization

\[ s_t = - \left( 1 + s_x^2 \right)^{\frac{1}{2}} + \epsilon \frac{s_{xx}}{1 + s_x^2} \]

\[ \epsilon = \frac{v_1}{l v_0} \]

2. Finite elements method (COMSOL).

(video)
## Critical $\epsilon$ value

<table>
<thead>
<tr>
<th>$\epsilon$ values</th>
<th>Results</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0$</td>
<td>Previous model.</td>
</tr>
<tr>
<td>$\epsilon &gt; 0.141$</td>
<td>The surface goes up at the beginning.</td>
</tr>
<tr>
<td>$0 &lt; \epsilon \leq 0.141$</td>
<td>The surface always goes down.</td>
</tr>
</tbody>
</table>

(video)

### Polishing Lead Crystal Glass

Numerical simulations and analysis

Conclusions

The model

Model 1: constant normal velocity
Model 2: linear velocity
Model 3: exponential velocity

Introduction
Finite Difference Method (Matlab)

Numerical discretisation:

\[
S_t = \frac{S_{n+1} - S_n}{\tau}
\]

\[
S_x = \frac{S_n(x + h) - S_n(x - h)}{2h}
\]

\[
S_{xx} = \frac{S_{n+1}(x - h) - 2S_{n+1}(x) + S_{n+1}(x + h)}{h^2}
\]
Solutions

Critical $\epsilon$ value.
About initial conditions

\[ A = \frac{a}{l} \]

where \( a = \text{height} \) and \( l = \text{length} \).

The bigger \( A \) is, the slower velocity goes.

\[ A = 0.5 \quad A = 1 \]

\textbf{Polishing Lead Crystal Glass}
Model 3: exponential velocity

Exponential relationship between normal velocity and surface curvature.

\[ v = v_0 + v_1 k = v_0 \left(1 + \frac{v_1}{v_0} k\right) \approx v_0 \exp \left(\frac{-v_1 s_{xx}}{v_0 (1+s_x^2)^{3/2}}\right) \]
Conclusions

1. Model 1, $\nu$ as a constant. Hamilton-Jacobi non-linear equation:

   $$ s_t = -\nu \sqrt{1 + s_x^2} $$

2. $t^* = \frac{l}{\nu} t_c^*(A)$

3. Model 2, $\nu$ linearly dependent on $k$ ($\nu = \nu_0 + \nu_1 k$). Diffusion equation:

   $$ s_t = - \left( 1 + s_x^2 \right)^{\frac{1}{2}} + \epsilon \frac{s_{xx}}{1 + s_x^2} $$

4. $\epsilon = \frac{\nu_1}{l \nu_0}$ critical value, if it is too large it becomes unphysical.


6. Not as easy finding a proper velocity rate when several acids appear. New research?
Bibliography


Google [http://www.google.es/](http://www.google.es/)


MACSI (Mathematics applications Consortium for Science and Industry) [http://www.macsi.ie](http://www.macsi.ie)

Infante, Juan Antonio, *Numerical Methods Notes*
Introduction
Model 1: constant normal velocity
Model 2: linear velocity
Model 3: exponential velocity

Conclusions

...questions?