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A variance-expected compliance approach for topology optimization

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Abstract
In this paper we focus on the adaptation to topology optimization of a previous variance-expected compliance applied to truss design. The principal objective of such a model is to find robust structures for a given main load and its perturbations. In particular we are interested in avoiding high compliance values in cases of important perturbations. In the first part, we recall the variance-expected formulation and main results in the case of truss structures. Then, we extend this model to topology optimization. Finally, we study the interest of this model on a 2D benchmark test.

Keywords: Topology optimization; Structural optimization; variance-expected compliance model; Stochastic programming

1 Introduction

We consider an elastic homogeneous body $\Omega \subseteq \mathbb{R}^d$. We impose some support conditions in $\Gamma_u \subseteq \partial\Omega$, where the displacements of the body are not allowed. We apply some external load forces to this body. Our purpose is to find the optimal distribution of material in Ω when the external load force has an stochastic behavior.

Assuming linear response of material, the displacements can be computed by solving the following system of linear partial differential equations:

$$-\operatorname{div}(K\,e(u)) = f \quad \text{in } \Omega, \tag{1}$$
$$u = 0 \quad \text{on } \Gamma_u, \tag{2}$$
$$(K\,e(u))\cdot n = 0 \quad \text{on } \partial\Omega \setminus \Gamma_u \cup \Gamma_t, \tag{3}$$
$$(K\,e(u))\cdot n = g \quad \text{in } \Gamma_t. \tag{4}$$

In this system $\Gamma_u \cap \Gamma_t = \varnothing$, f corresponds to an external load, g is a surface force applied to Γ_t; $u\colon \Omega \to \mathbb{R}^d$ is the vector of displacements, $e(u) = \frac{1}{2}(\nabla u + \nabla u^T)$ denotes the strain tensor and $K = (K_{i,j,k,l})$ is the elasticity tensor (see e.g. [13]). It is well known that under suitable conditions on the data, the previous elasticity problem has a unique weak solution (see e.g. [9]).

For the numerical test presented in Section 4, we assume that the elasticity tensor depends on a parameter $\lambda \in \Lambda$ measuring the amount of material in each point (see [14] and [5] Chapter 1). More precisely $K = \lambda(x)^p K^0$, with $p > 1$ and $0 < \overline{\lambda} \leq \int_\Omega \lambda(x)\,\mathrm{d}x \leq 1$ (see e.g. [14]). We also assume that the material is isotropic, i.e. $K^0_{i,j,k,l} = 2\mu_1\delta_{i,k}\delta_{j,l} + \mu_2\delta_{i,j}\delta_{k,l}$ where $\delta_{i,j}$ denotes the Kronecker symbol and $\mu_1, \mu_2 > 0$ the Lamé constants of the material.

For simplicity, and without lost of generality, we assume that $g \equiv 0$. Following [10] and [5] Chapter 1, we define the functionals

$$A[\lambda](u,v) = \int_\Omega K_{i,j,k,l}e_{i,j}(u)e_{i,j}(v)\,\mathrm{d}x \tag{5}$$

$$l(u) = \int_\Omega f\cdot u\,\mathrm{d}x \tag{6}$$

The well known minimum compliance design problem can be stated as following

$$\min_{\lambda\in\Lambda}\ l(u(\lambda)) \tag{7}$$

$$s.t.\ A[\lambda](u(\lambda),v) = l(v), \quad \text{for all } v\in\hat{H} \tag{8}$$

where $\hat{H} = \{u \in [H^1(\Omega)]^d \mid u_i|_{\Gamma_u} = 0,\ i = 1,\dots,d\}$* corresponds to the space of admissible displacements and $u(\lambda)$ denotes the unique weak solution of (1)-(4). Using finite element discretization of $N\times N$ elements and the same mesh for the displacements and materials (using a constant value λ_i, $i = 1,\dots,N\times N$ for each element of the mesh), the discretized form of (7)-(8) is similar to the truss design problem studied in the Section 2 and it is inherently a large scale problem (see problem (14)).

In an analogous way to previous stochastic results for truss optimization (see [3, 7] and Section 2 of this paper for a brief summary) we assume that the external load force is randomly perturbed by $\xi(\omega)$ (with $\mathbb{E}(\xi) = 0$). The stochastic topology design problem can be stated as

$$\min\{\mathbb{E}_\xi[\Psi(\xi,\lambda)] \mid \lambda\in\Lambda\}, \tag{9}$$

where the functional Ψ is defined by

$$\Psi(\xi,\lambda) = \left\{ \int_\Omega (f+\xi(\omega))\cdot u\,\mathrm{d}x \;\middle|\; u\in\hat{H} \text{ satisfies: } A[\lambda](u,v) = \int_\Omega (f+\xi(\omega))\cdot v\,\mathrm{d}x \text{ for all } v \text{ in } \hat{H} \right\} \tag{10}$$

and $\mathbb{E}_\xi(\cdot)$ stand for the expected value of the corresponding random function.

In this work we will show that (9) can be rewritten as a multiload problem and then efficiently solved. Nevertheless, the stochastic solution of (9) may give unsatisfactory results for some loading scenarios (some loads may produce high compliance values). One way to lead this problem is to consider an stochastic multiobjective problem

$$\min\{\alpha\mathbb{E}_\xi[\Psi(\xi,\lambda)] + \beta\,\mathrm{Var}[\Psi(\xi,\lambda)] \mid \lambda\in\Lambda\} \tag{11}$$

where Ψ is defined by (10), $\alpha,\beta \geq 0$, $\alpha+\beta > 0$ and $\mathbb{E}_\xi(\cdot), \mathrm{Var}_\xi(\cdot)$ stand, respectively, for the expected value and the variance of the corresponding random function.

We use previous results obtained for trusses to give an explicit expression of the discretized problem (11), we show that a good choice of α,β allows us to obtain a material distribution with low risk level.

This work is organized as follows: first we introduce the stochastic setting for trusses and recall the main results. Then we describe the model for the continuous case. Finally, we give preliminary numerical examples in order to see the interest of this formulation.

2 Variance-expected compliance approach for truss optimization

The full description of the work presented in this Section can be found in [7, 8].

Trusses are mechanical structures consisting of an ensemble of slender bars, connecting some pairs of nodal points in $\mathbb{R}^d$ with either $d = 2$ or $d = 3$. The bars are supposed to be made of a linearly elastic, isotropic and homogeneous material. Long bars overlapping small ones are not allowed. They are designed to support some external nodal loads taking into account certain mechanical properties of the bar material.

*We recall that $H^1(\Omega)$ denotes the Sobolev space of all $v\in L^2(\Omega)$ with $\partial_{x_i}v\in L^2(\Omega)$.

Set $n = d \cdot N - s$, the number of *degrees of freedom* of a ground truss structure consisting of $N \geq 2$ nodes, where $s \geq 0$ is the number of fixed nodal coordinate directions (i.e., coordinates corresponding to support conditions are removed) and either $d = 2$ for planar trusses or $d = 3$ for three-dimensional ones. Therefore the nodal displacements can be described entirely by the n *global reduced coordinates* of the structure. Let $m \geq n$ be fixed, corresponding to the number of potential bars in the truss structure (long bars overlapping small bars are not allowed, thus $m \leq N(N-1)/2$), and denote by $\lambda_i \geq 0$ the volume (normalized) of the i-th bar with $i \in \{1, \dots, m\}$. External loads are applied only at nodal points and are described in global reduced coordinates by a vector $f \in \mathbb{R}^n$. Under the assumption that each bar is subjected only to axial tension or compression (thus neglecting large deflections, bending effects and gravity), the mechanical response of the truss is described by the *elastic equilibrium equation* (see [1])

$$K(\lambda)u = f, \tag{12}$$

where $u \in \mathbb{R}^n$ is the nodal displacements vector in global reduced coordinates and $K(\lambda)$ is the *stiffness matrix* of the truss, which has the form

$$K(\lambda) = \sum_{i=1}^{m} \lambda_i K_i. \tag{13}$$

Here $\lambda_i \geq 0$ is the volume of the i-th bar and $K_i \in \mathbb{R}^{n \times n}$ is a positive semi-definite, symmetric matrix, which corresponds to the specific stiffness matrix (see [1]) of the i-th bar in global reduced coordinates.

The problem of finding the minimum *compliance* truss for a normalized volume constraint of material is given by (see [2, 5])

$$\min_{\lambda \in \Delta_m} \left\{ \frac{1}{2} f^T u \mid K(\lambda)u = f, u \in \mathbb{R}^n \right\}, \tag{14}$$

where $\Delta_m = \{\lambda \in \mathbb{R}^m \mid \lambda \geq 0,\ \sum_{i=1}^m \lambda_i = 1\}$ and $\frac{1}{2} f^T u$ is the compliance value of the structure. This problem is known as *single load model.* As the value of the objective function in (14) does not depend on the choice of the equilibrium displacement vector $u \in \mathbb{R}^n$ satisfying (12), problem (14) is well defined.

Taking into account the particular structure of the matrices K_i in (13), it is possible to show (see [2]) that the *single load* model (14) is equivalent to a linear programming problem, and therefore might be efficiently solved. Nevertheless, numerical results using this model show that optimal solutions may be unstable with respect to the mechanical equilibrium, even under small perturbations in the principal load [1, 4]. In fact, there are several examples showing some optimal structures that, under small perturbations, give infinite compliance.

In order to handle this inconvenient, we may consider the *Multiload Model* (see [2]) given by

$$\min_{\lambda \in \Delta_m} \left\{ \frac{1}{2} \sum_{j=1}^{k} \gamma_j (f^j)^T u^j \mid K(\lambda)u^j = f^j,\ j = 1, ..., k \right\}, \tag{15}$$

where $\gamma_j > 0$ corresponds to the influence of the scenario j into the model. In this formulation a weighted average of the compliances, associated with k different loads scenarios, is minimized. The multiload model can be transformed into an equivalent convex quadratic minimization problem and might be also efficiently solved [6]. However, in order to model a structure submitted to a randomly perturbed load using this multi-load approach, we should take into account all the possible perturbations scenarios (or at least, numerically, a number of scenarios high enough).

Under suitable assumptions, we may consider an alternative methodology:

Let $\Psi \colon \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R} \cup \{+\infty\}$defined as

$$\Psi(\xi, \lambda) = \begin{cases} \frac{1}{2}(f + \xi)^T u & \text{if } \lambda \in \Delta_m \text{ and } \exists\ u \in \mathbb{R}^n \text{such that } \ K(\lambda)u = f + \xi. \\ \\ +\infty & \text{otherwise.} \end{cases} \tag{16}$$

Function Ψ is proper (i.e. $\Psi \not\equiv +\infty$), lower semi-continuous and convex (see [3]). Therefore, for each $\lambda \in \Delta_m$, the function

$$\Psi(\cdot, \lambda) \colon \big(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n)\big) \to \big(\mathbb{R} \cup \{+\infty\}, \overline{\mathcal{B}}(\mathbb{R})\big)$$

is measurable. Here $\mathcal{B}(\mathbb{R}^n)$ and $\overline{\mathcal{B}}(\mathbb{R})$ stand for the Borel σ-algebra of $\mathbb{R}^n$ and $\mathbb{R} \cup \{+\infty\}$ respectively.

Next, let us assume that ξ is a random variable corresponding to an uncertain perturbation of f. More precisely, let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space and consider a measurable function

$$\begin{array}{rccc} \xi: & (\Omega, \mathcal{A}) & \to & (\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n)) \\ & \omega & \mapsto & \xi(\omega). \end{array}$$

According to this setting we study the following stochastic minimization problem:

$$\min_{\lambda \in \Delta_m} \{\alpha \mathbb{E}_\xi[\Psi(\xi, \lambda)] + \beta \operatorname{Var}_\xi[\Psi(\xi, \lambda)]\}, \tag{17}$$

where Ψ is defined by (16), $\alpha, \beta \geq 0$, $\alpha + \beta > 0$ and $\mathbb{E}_\xi(\cdot), \operatorname{Var}_\xi(\cdot)$ stand, respectively, for the expected value and the variance of the corresponding random function.

Theorem 1 *Let $\xi \in \mathbb{R}^n$ be a continuous random vector, generated in a subspace of dimension $k \in \mathbb{N}$, such that the distribution of ξ is a n-multivariate normal, $\mathbb{E}(\xi) = 0$ and its covariance matrix is given by $\operatorname{Cov}(\xi) = PP^T$, where $P \in \mathbb{R}^{n \times k}$ (i.e. $\xi \sim \mathcal{N}_n(0, PP^T)$). Then, problem (17) is equivalent to*

$$\min_{\lambda \in \Delta_m} \alpha(\tfrac{1}{2} f^T u + \tfrac{1}{2} \operatorname{Tr}(P^T U)) + \beta(\tfrac{1}{2} \operatorname{Tr}(P^T U)^2 + f^T U U^T f), \tag{18}$$

$$K(\lambda) u = f, \tag{19}$$

$$K(\lambda) U = P. \tag{20}$$

The proof of this theorem can be found in [3].

Similarly to problem (14) the value of the objective function (18) is independent of the choice of $u \in \mathbb{R}^n$ and $U \in \mathbb{R}^{n \times k}$ satisfying (19) and (20) respectively. Therefore, the previous problem is well defined and it can be regarded as a mathematical problem with only λ as a design variable.

We point out that, considering $\beta = 0$ and denoting by $f_0 = f$ and f_j the jth-column of matrix P, $j = 1, \ldots, k$; then (18)-(20) may be rewritten as a multi-load-type problem with $k+1$ scenarios (see (15))

$$\min_{\lambda \in \Delta_m} \left\{ \frac{1}{2} \sum_{j=1}^{k} (f^j)^T u^j \mid K(\lambda) u^j = f^j,\ j = 1, ..., k \right\}.$$

In this case the loading scenario f_j can be interpreted as a load perturbation of the main force f_0 (see [3]). Thus, in order to construct robust structures in this continuous model, it is not necessary to consider explicitly all the loading scenarios but a good representation of them, according to the covariance matrix $P^T P$.

From some numerical experiments presented in [7], we see that formulation (17) with a reasonable balance between α and β can be helpful to generate structures more robust with respect to perturbations of the main load.

3 Variance-expected compliance approach for topology optimization

In this Section, we study the stochastic topology design problem presented in the introduction. We show that this problem can be transformed into a multiload problem in which the loading scenarios are related to the variance of random load applied to the body Ω.

We recall that the stochastic topology optimization problem corresponds to

$$\min \{\mathbb{E}_\xi[\Psi(\xi, \lambda)] \mid \lambda \in \Lambda\},$$

where Λ is the set of feasible material distribution and Ψ is defined by

$$\Psi(\xi, \lambda) = \left\{ \int_\Omega (f + \xi(\omega)) \cdot u \, \mathrm{d}x \;\middle|\; \text{where } u \in \hat{H} \text{ satisfies: } A[\lambda](u, v) = \int_\Omega (f + \xi(\omega)) \cdot v \, \mathrm{d}x \text{ for all } v \text{ in } \hat{H} \right\}$$

where $\hat{H} = \{u \in [H^1(\Omega)]^d \mid u_i|_{\Gamma_D} = 0,\ i = 1, \ldots, d\}$.

In the following we will consider $\{P_i\}_{i=1}^{\infty}$ functions of the Hilbert space $L^2(\Omega)^d$, corresponding to directions of perturbation of the main force f.

Lemma 1 *Let $\xi\colon \Omega\times\mathcal{B}\to \mathbb{R}^d$ be a random load, which in terms of the directions of perturbation $\{P_i\}_{i=1}^{\infty}$ is written as $\xi=\sum_1^{\infty}\varepsilon_i P_i$ where $(\varepsilon_i)_{i=1}^{\infty}$ are random variables with $\mathbb{E}(\varepsilon_i)=0$ and $\mathbb{E}(\varepsilon_i\varepsilon_j)=\alpha_{i,j}$ for $i,j=1,\dots,\infty$. Let G be a linear functional. Then*

$$\mathbb{E}\left(\int_\Omega \xi\cdot G(\xi)\,\mathrm{d}x\right)=\sum_{i=1}^{+\infty}\sum_{j=1}^{+\infty}\alpha_{i,j}\int_\Omega P_i\cdot G(P_j)\ \mathrm{d}x.$$

Proof: using $\xi=\sum_{i=1}^{+\infty}\varepsilon_i P_i$ and Fubini's Theorem we obtain

$$\int_\Omega\int_{\mathbb{R}^d}\xi\cdot G(\xi)\,\mathrm{d}\mathbb{P}\ \mathrm{d}x=\int_\Omega\int_{\mathbb{R}^d}\sum_{i=1}^{+\infty}\varepsilon_i P_i\cdot G(\sum_{j=1}^{+\infty}\varepsilon_j P_j)\,\mathrm{d}\mathbb{P}\ \mathrm{d}x=\int_\Omega\int_{\mathbb{R}^d}\sum_{i=1}^{+\infty}\sum_{j=1}^{+\infty}\varepsilon_i\varepsilon_j P_i\cdot G(P_j)\,\mathrm{d}\mathbb{P}\ \mathrm{d}x$$
$$=\int_\Omega\sum_{i=1}^{+\infty}\sum_{j=1}^{+\infty}\left(\int_{\mathbb{R}^d}\varepsilon_i\cdot\varepsilon_j\,\mathrm{d}\mathbb{P}\right)P_i\cdot G(P_j)\ \mathrm{d}x.$$

□

The following theorem states the close relation between the multiload truss model and the corresponding model for the continuous case.

Theorem 2 *Let us consider $\xi\colon\Omega\times\mathcal{B}\to\mathbb{R}^d$ be a random load, which in terms of the directions $\{P_i\}_{i=1}^{\infty}$ is written as $\xi=\sum_1^{\infty}\varepsilon_i P_i$ where $(\varepsilon_i)_{i=1}^{\infty}$ are independent random variables, $\mathbb{E}(\varepsilon_i\varepsilon_j)=0$ for $i\neq j$, with $\mathbb{E}(\varepsilon_i)=0$ and $\mathrm{Var}(\varepsilon_i)=\sigma_i^2$. Then the stochastic problem defined in* (9) *can be rewritten as the multiload problem:*

$$\min\int_\Omega f\cdot u\,\mathrm{d}x+\sum_{i=1}^{+\infty}\int_\Omega\sigma_i P_i\cdot U_i\,\mathrm{d}x \tag{21}$$
$$A(\lambda)[u,v]=\int_\Omega f\cdot v\ \mathrm{d}x,\ \forall v\in\hat{H} \tag{22}$$
$$A(\lambda)[U_i,v]=\int_\Omega\sigma_i P_i\cdot v\ \mathrm{d}x,\ \forall v\in\hat{H},\ i\in\mathbb{N} \tag{23}$$
$$\lambda\in\Lambda;u\in\hat{H},U_i\in\hat{H}\ \textit{for}\ i\in\mathbb{N}. \tag{24}$$

Proof: Let $\lambda\in\Lambda$ be a feasible material distribution and let us consider $\xi\colon\mathbb{R}^d\times\mathcal{B}\to\mathbb{R}$ be a random load with $\mathbb{E}(\xi)=0$. We define the inverse functional

$$G(f+\xi(\omega))=u(\omega)$$

where $u(\omega)$ is the unique weak solution of the system (1)-(4) changing f by $f+\xi(\omega)$ (recall that we assume $g\equiv 0$). The expected value of the compliance is given by

$$\mathbb{E}(\Psi(\xi,\lambda))=\int_{\mathbb{R}^d}\int_\Omega(f+\xi(\omega))\cdot u(\omega)\,\mathrm{d}x\,\mathrm{d}\mathbb{P}(\omega).$$

Using Fubini's theorem and linearity of inverse operator G we have

$$\begin{aligned}\mathbb{E}(\Psi(\xi,\lambda))&=\int_\Omega\int_{\mathbb{R}^d}(f+\xi)\cdot G(f+\xi)\,\mathrm{d}\mathbb{P}\,\mathrm{d}x\\&=\int_\Omega\int_{\mathbb{R}^d}f\cdot G(f)\,\mathrm{d}\mathbb{P}\ \mathrm{d}x+\int_\Omega\int_{\mathbb{R}^d}\xi\cdot G(f)\,\mathrm{d}\mathbb{P}\ \mathrm{d}x+\int_\Omega\int_{\mathbb{R}^d}f\cdot G(\xi)\,d\mathbb{P}\ \mathrm{d}x+\int_\Omega\int_{\mathbb{R}^d}\xi\cdot G(\xi)\,d\mathbb{P}\ \mathrm{d}x\\&=\int_\Omega\mathbb{P}(\mathbb{R}^d)f\cdot G(f)\ \mathrm{d}x+\int_\Omega\mathbb{E}(\xi)G(f)\ \mathrm{d}x+\int_\Omega G^*(f)\cdot\mathbb{E}(\xi)\ \mathrm{d}x+\int_\Omega\int_{\mathbb{R}^d}\xi\cdot G(\xi)\,\mathrm{d}\mathbb{P}\ \mathrm{d}x\\&=\int_\Omega f\cdot G(f)\ \mathrm{d}x+\int_\Omega\int_{\mathbb{R}^d}\xi\cdot G(\xi)\,\mathrm{d}\mathbb{P}\ \mathrm{d}x,\end{aligned}$$

where G^* denotes the adjoint operator of G.

Using Lemma 1 and the fact that $\mathbb{E}(\varepsilon_i\varepsilon_j)=0$ for $i\neq j$, we obtain

$$\int_\Omega \int_{\mathbb{R}^d} \xi \cdot G(\xi)\, \mathrm{d}\mathbb{P}\ \mathrm{d}x = \int_\Omega \sum_{i=1}^{+\infty} \sigma_i^2 P_i \cdot G(P_i)\ \mathrm{d}x.$$

Denoting by U_i the unique weak solution of

$$A(\lambda)[U_i, v] = \int_\Omega \sigma_i P_i \cdot v\, \mathrm{d}x \text{ for all } v \in \hat{H},$$

we finally get

$$\mathbb{E}(\Psi(\xi, \lambda)) = \int_\Omega f \cdot u\ \mathrm{d}x + \int_\Omega \sum_{i=1}^{+\infty} \sigma P_i \cdot U_i\ \mathrm{d}x.$$

□

In order to avoid scenarios with too large values of the compliance we can consider the variance of the compliance among our objectives to minimize. We introduce the stochastic multiobjective minimization problem:

$$\min_{\lambda \in \Lambda} \{\alpha \mathbb{E}_\xi[\Psi(\xi, \lambda)] + \beta\, \mathrm{Var}_\xi[\Psi(\xi, \lambda)]\}, \tag{25}$$

where $\alpha > 0, \beta \geq 0$ and $\mathrm{Var}_\xi[\cdot]$ stands for the variance of a random variable.

For the numerical test presented in Section 4, we have used $\xi(\omega, x) = \varepsilon(\omega) P(x)$, for this simplest case we can compute explicitly the variance of Ψ in problem (25). We have the following result:

Lemma 2 *Using the notation of the previous theorem, let us consider $\xi = \varepsilon P$ be a random perturbation of f. The vector $P \in L^2(\Omega)^d$ and ε is a random variable with moments $\mathbb{E}(\varepsilon^i) = \mu_i$ (of course we assume $\mu_1 = 0$). Then*

$$\begin{aligned}
\mathrm{Var}\left(\int_\Omega (f + \xi) \cdot G(f + \xi)\, \mathrm{d}x\right) &= \mu_2 \left[\int_\Omega P \cdot G(f)\, \mathrm{d}x + \int_\Omega f \cdot G(P)\, \mathrm{d}x\right]^2 \\
&+ 2\mu_3 \int_\Omega P \cdot G(P)\, \mathrm{d}x \left[\int_\Omega P \cdot G(f)\, \mathrm{d}x + \int_\Omega f \cdot G(P)\, \mathrm{d}x\right] \\
&+ (\mu_4 - \mu_2) \left(\int_\Omega P \cdot G(P)\, \mathrm{d}x\right)^2
\end{aligned} \tag{26}$$

The proof is left to the reader.

Assuming that $\varepsilon \sim \mathcal{N}(0, \sigma^2)$ (in this case $\mu_1 = 0, \mu_2 = \sigma^2, \mu_3 = 0$ and $\mu_4 = 3\sigma^4$) then we get a problem similar to (18)-(20) obtained for trusses.

4 Numerical example

4.1 Problem description

In order to perform a first numerical study to see the interest of formulation (17) and the choice of the parameters (α, β), we consider a 2-D benchmark design problem. A geometrical representation of this benchmark problem is given by Figure 1. We are interested in designing a 'bridge' considering various fixed support areas. The upper part of the bridge (black part in Figure 1) is also fixed with a density equal to 1 and a main load $f = (0, -1)$ is applied homogeneously all over this part. Since we are interested in generating structures that are stable to perturbations of the design force f, we consider a random load $\xi = \xi_1 V_1$, with ξ_1 of law $\mathcal{N}(0, 1)$ and $V_1 = (1, 0)$, applied at the same region than f.

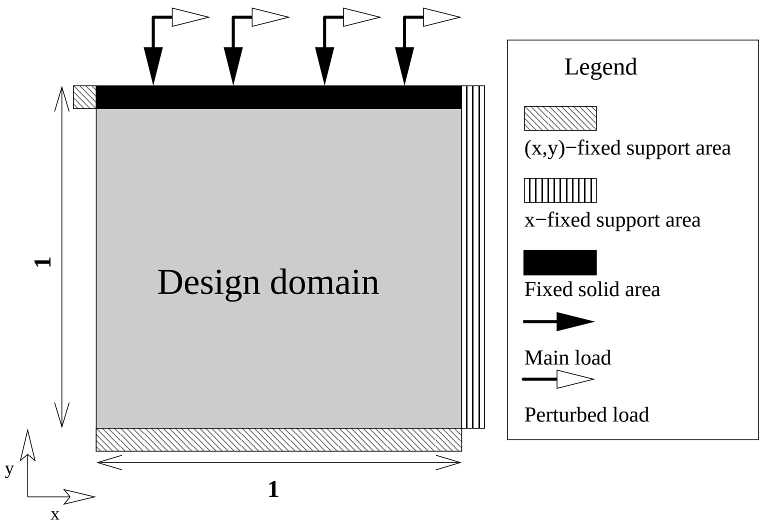

Figure 1: Geometrical representation of the design problem described in Section 4.1: design domain (gray), support areas (dashed), fixed solid area (black) and loads (arrow) of the considered numerical example.

We solve the corresponding problem (17), considering a finite element approach similar to the one proposed in [14] with a grid of 30×30, for various set of values of (α, β). More precisely, (α, β) is chosen in the set $\Sigma = \{(1, 0), (1, 0.5), (1, 1), (0.5, 1), (0, 1)\}$. In addition to those problems, we also solve the same design problem considering the perturbation load as the main load and omitting f, with $(\alpha, \beta) = (1, 0)$. In order to solve (17) for the considered design problems, which seem to be non-convex problems with several local minima [7], we use the *Global Optimization Platform* (*GOP*) software with the steepest descent method as the core algorithm and where the initial condition is generated using the secant method. A complete description and validation of this algorithm can be found in [12, 11]. The obtained solutions are denoted by $\{\lambda_{(\alpha,\beta)}\}_{(\alpha,\beta)\in\Sigma}$ for the full design problem and $\lambda_{\text{Pert.}}$ for the only perturbation load case.

In order to have a qualitative comparison of those structures, we analyze their robustness when they are submitted to random loads and their shape evolution. More precisely, for each solution λ, we consider the random variable $\Phi_\lambda = \Psi(\xi, \lambda)$, where Ψ is defined by (16), and we approximate its density function ρ_{Φ_λ} using a Monte-Carlo approach [11] that generates $M \in \mathbb{N}$ possible *scenarios* (i.e. values of ξ). Then, we compute a particular risk measure of Φ_λ (i.e. mappings $\varpi : L^\infty(\Omega, \mathcal{A}, \mathbb{P}) \to \mathbb{R}$, where $(\Omega, \mathcal{A}, \mathbb{P})$ is a probability space). Risk measures are used in various areas, such as financial analysis [11], in order to study the value of the worst scenarios (in our case, the random loads generating the highest structure compliances). Here we focus on a popular one called the *Coherent-Value at Risk* (C-VaR), defined as:

$$\text{C-VaR}_\nu(\chi) = \frac{1}{\nu}\int_0^\nu \inf\Big\{z \in \mathbb{R} \text{ s.t. } \int_0^z 100\rho_\chi(x)\mathrm{d}x > (100 - y)\Big\}\mathrm{d}y,$$

where ν is a percentile, $\chi \in L^\infty(\Omega, \mathcal{A}, \mathbb{P})$ and ρ_χ is the density function of χ. C-VaR_ν corresponds to the average value of the worst ν % case scenarios of χ (i.e. the ν % highest values of χ). In our case we take $\chi = \Phi_\lambda$ and $\nu = 1\%$. A complete presentation or risk measures and C-VaR can be found in [11].

4.2 Results

All results are reported in Table 1 and the different shape configurations are presented in Figure 2.

As we can observe in Table 1, the solution $\lambda_{(1,0)}$ is less stable to perturbations of the main load, considering the 1 % worst case scenarios (C-VaR_1), than the solutions $\lambda_{(1,0.5)}$, $\lambda_{(1,1)}$ $\lambda_{(0.5,1)}$. Although the value of the expected compliance increases as the value of β increases, those three strucures represent a good alternative to $\lambda_{(1,0)}$. In fact the increase of the expected compliance is reasonable (between 2 and 6%) in comparison to the C-VaR_1 diminution (between 9 and 12%). However, considering only a variance minimization problem generates the structure with worst characteristics.

Figures 2 shows that the shape of the structure changes with the evolution of the coefficients α and β. In fact, it seems that there is a mass transfer phenomenon when β goes from 0 to 1: From the right

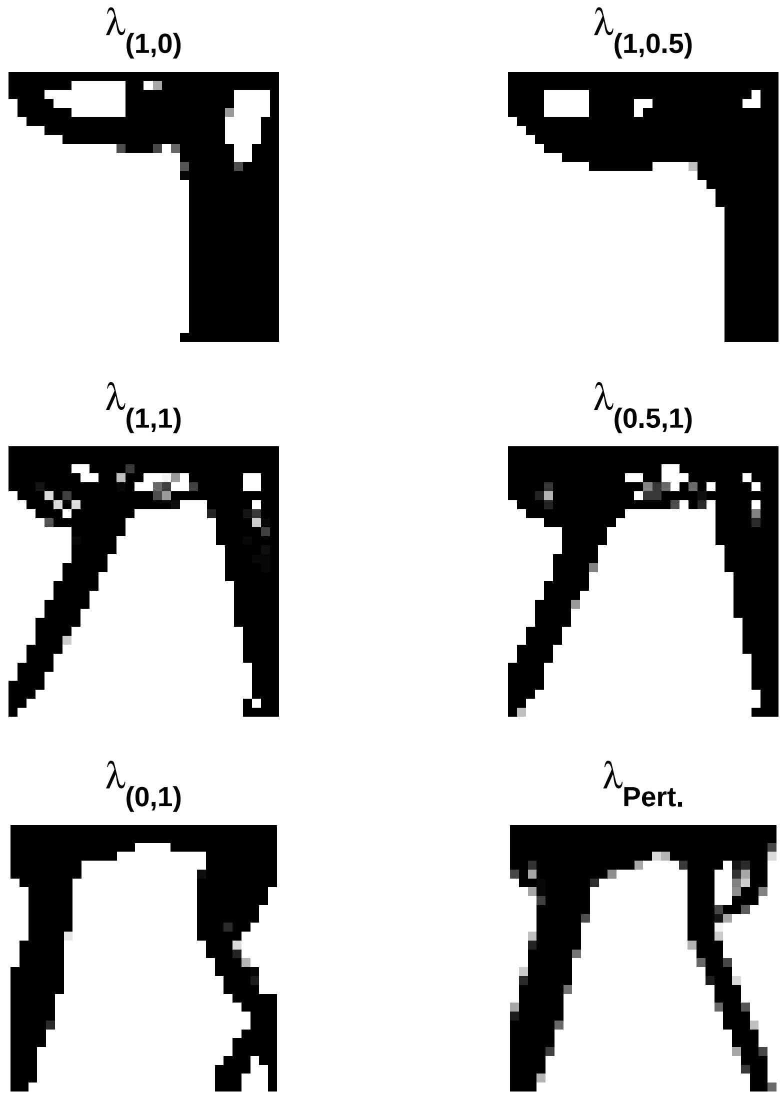

Figure 2: Shape of the solutions $\{\lambda_{(\alpha,\beta)}\}_{(\alpha,\beta)\in\Sigma}$ and $\lambda_{\text{Pert.}}$.

Table 1: Results obtained considering $\{\lambda_{(\alpha,\beta)}\}_{(\alpha,\beta)\in\Sigma}$ and $\lambda_{\text{Pert.}}$: Expected compliance (**EC**), Variance (**Vari**) and Coherent Value at Risk (**C-VaR$_1$**) of the solutions. The value in parenthesis represents the difference in percentage between the values of the considered solution and $\lambda_{(1,0)}$.

Solution	$\lambda_{(1,0)}$	$\lambda_{(1,0.5)}$	$\lambda_{(1,1)}$	$\lambda_{(0.5,1)}$	$\lambda_{(0,1)}$	$\lambda_{\text{Pert.}}$
EC	2.66	2.71 (+2%)	2.76 (+4%)	2.82 (+6%)	4.62 (+73 %)	5.23 (+97%)
Vari	4.42	4.16 (-6%)	3.04 (-30%)	2.85 (-35%)	2.56 (-42%)	2.72 (-38%)
C-VaR$_1$	13.2	12.0 (-9%)	11.9 (-10%)	11.6 (-12%)	13.5 (+1%)	13.8 (+2%)

initial pylon to the creation of the left inclined pylon. Furthermore, the shape $\lambda_{(0,1)}$ is close to the $\lambda_{\text{Pert.}}$ ones. This shape evolution is intuitive as the right pylon provides a good resistance to the vertical load f while the inclined pylons support inclined loads resulting of the perturbation of f.

As in the truss optimization case, we deduce that considering formulation (17) for topology optimization with an adequate balance between α and β helps to generate structures more robust to perturbations of the main load.

5 Conclusions

We have adapted a variance-expected compliance formulation to topology optimization. This method has been tested numerically on a 2-D benchmark test case. This new formulation, combined with an adequate balance between the compliance variance and the expected compliance, allows to generate structures that are more stable, in the sense of the worst scenarios, to perturbations of the main load.

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