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Convergence to equilibrium for a hyperbolic/elliptic system modelling the viscoelastic-gravitational deformation of a layered Earth

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Abstract

In this communication we prove the stabilization, as t goes to infinity, of a model (which is an adaptation of the one possed by A. E. H. Love in 1911, see [8]) for the study of the displacements due to internal sources of strain in layered linear viscoelastic-gravitational continua. The existence and uniqueness of weak solutions has been obtained recently in [1] and [2]. Here we prove that, under some additional conditions on the data, the difference of the respective solutions converges to zero, as t goes to infinity, in a suitable functional space. Our proof uses a reformulation of the hyperbolic/elliptic system in terms of a nonlocal hyperbolic system leading to which we apply the La Salle invariance principle for a Lyapounov function involving the nonlocal terms.

1 Introduction. Physical model

The volcanic eruptions are the outcome of significant physical and geological processes (see [6] and [7]). In order to interpret geodetic anomalies (displacements, gravity changes, etc.) which may be tied to volcanic activity, mathematical models allowing the resolution of the inverse problem which consists of obtaining a volcanic intrusion's properties from surface observation, are necessary. The techniques needed for calculation of displacements and gravity change due to internal sources have been developed (see [9] and [10]).

The deformation model we are going to work with consist of an Earth model composed by several viscoelastic-gravitational layers overlying an viscoelastic-gravitational domain.

We consider a spatial domain of the type as shown in Figure 1. The elastic constants and the density of the n-th layer are denoted by λ_n, μ_n and ρ_n. Each layer has thickness d_n. We construct a cylindrical coordinate system with the origin at the surface and with the z axis pointing down into the medium. The lower boundary of the n-th layer is designated by z_n and the depth to the half space by z_p.

Let us define spatial domain in the following way: we will assume p layers "overstrike", that we will denote as Ω_i $\forall i = 1, \ldots, p$, and which union determines global domain Ω, $\Omega = \bigcup_{i=1}^{p} \Omega_i$. Each layer is given through common horizontal set: a open $\omega \subset \mathbb{R}^2$ and so $\Omega_1 := \omega \times (d_1, d_1 + d_2)$, $\Omega_2 := \omega \times (d_1 + d_2, d_1 + d_2 + d_3)$, etc., that is

$$\Omega_i := \omega \times \left(\sum_{j=1}^{i-1} d_j, \sum_{j=1}^{i} d_j \right) \subset \mathbb{R}^3, \text{when} \quad i = 1, \ldots, p-1, \tag{1}$$

and $\Omega_p := \omega \times (H, H + d_r)$,when $H := \sum_{j=1}^{i-1} d_j$ and d_r can be equal to $+\infty$. On each layer $\Omega_i, i = 1, \ldots, p$, the following system of equations holds:

$$\begin{cases} \rho^i \mathbf{u}_{tt}^i (t, \mathbf{x}) - \gamma^i \Delta \mathbf{u}_t^i (t, \mathbf{x}) - \Delta \mathbf{u}^i (t, \mathbf{x}) - \dfrac{1}{1 - 2\nu^i} \nabla \left(div \mathbf{u}^i (t, \mathbf{x}) \right) - \dfrac{\rho^i g}{\mu^i} \nabla \left(\mathbf{u}^i (t, \mathbf{x}) \cdot \mathbf{e}_z \right) + \\ \dfrac{\rho^i g}{\mu^i} \mathbf{e}_z div \mathbf{u}^i (t, \mathbf{x}) = \dfrac{\rho^i}{\mu^i} \nabla \phi^i (t, \mathbf{x}) + \mathbf{f}_u^i (t, \mathbf{x}) \text{ , in } (0, T) \times \Omega_i, \\ -\Delta \phi^i (t, \mathbf{x}) = 4\pi \rho^i G div \mathbf{u}^i (t, \mathbf{x}) + f_\phi^i (t, \mathbf{x}) \text{ , in } (0, T) \times \Omega_i \text{ ,} \end{cases} \tag{2}$$

where $\gamma^i \Delta \mathbf{u}_t^i$ is the term introduced due to the viscoelasticity of each layer, $\mathbf{u}$ denotes the displacement, ϕ the gravitational perturbed potential, ν the Poisson's ratio, ρ the unperturbed density of the medium, g the externally imposed gravitational acceleration, μ is the rigidity and $\mathbf{e}_z$ is the unit vector pointing in the positive z-direction (down into the medium).

Let us describe the boundary of our domain to establish the boundary conditions of the problem. We distinguish, for each layer comprised between the first and the $(p-1)$-th, side, upper and bottom boundary by means of the following notation (see Figure1):

$$\begin{cases} \partial_+ \Omega_i = \omega \times \left\{ \sum_{j=1}^{i-1} d_j \right\}, & \text{top boundary,} \\ \partial_- \Omega_i = \omega \times \left\{ \sum_{j=1}^{i} d_j \right\}, & \text{bottom boundary,} \\ \partial_l \Omega_i = \partial \omega \times \left[\sum_{j=1}^{i-1} d_j, \sum_{j=1}^{i} d_j \right], & \text{side lateral boundary.} \end{cases} \tag{3}$$

Then $\partial \Omega_i = \partial_+ \Omega_i \cup \partial_- \Omega_i \cup \partial_l \Omega_i \qquad \forall i = 1, \ldots, p-1$. For the last layer, that is, the p-th one we have $\partial_+ \Omega_p = \omega \times \{H\}, \partial_- \Omega_p = \omega \times \{H + d_p\}$. To set of partial differential equations we must add the boundary and transmission conditions as follows:

$$\mathbf{u}^i (t, \mathbf{x}) = \mathbf{0}, \mathbf{x} \in \partial_l \Omega_i, t \in (0, T), \tag{4}$$

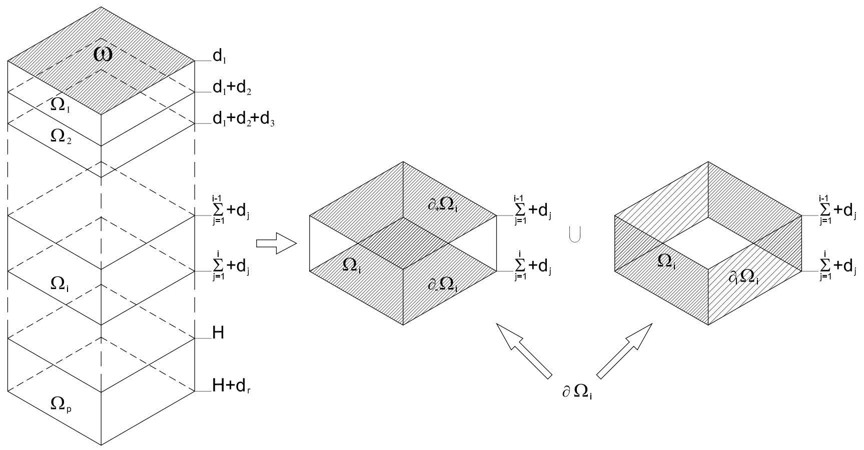

Figure 1: Layered Earth model. Illustration of the coordinate system and variation of the layer properties with depth.

$$\frac{\partial \mathbf{u}^1(t,\mathbf{x})}{\partial z} = \mathbf{0}, \mathbf{x} \in \partial_+\Omega_1, t \in (0,T), \tag{5}$$

$$\mathbf{u}^p(t,\mathbf{x}) = \mathbf{0}, \mathbf{x} \in \partial_-\Omega_p, t \in (0,T), \tag{6}$$

$$\mathbf{u}^i(t,\mathbf{x}) = \mathbf{u}^{i+1}(t,\mathbf{x}), \mathbf{x} \in \partial_-\Omega_i, t \in (0,T), \tag{7}$$

$$\frac{\partial \mathbf{u}^i(t,\mathbf{x})}{\partial z} = \frac{\partial \mathbf{u}^{i+1}(t,\mathbf{x})}{\partial z}, \mathbf{x} \in \partial_-\Omega_i, t \in (0,T). \tag{8}$$

In general, we must impose only that the first derivatives of $\mathbf{u}$ are continuous on the boundaries of the layers, that is, on the boundary between layers. This corresponds to the "transmission conditions".

With respect to the gravitational perturbed potential we will assume that the boundary conditions:

$$\phi(t,\mathbf{x}) = 0, \mathbf{x} \in \partial_l\Omega_i, t \in (0,T), \tag{9}$$

$$\phi^1(t,\mathbf{x}) = \phi_0(\mathbf{x}), \mathbf{x} \in \partial_+\Omega_1, t \in (0,T), \tag{10}$$

$$\phi^p(t,\mathbf{x}) = 0, \mathbf{x} \in \partial_-\Omega_p, t \in (0,T). \tag{11}$$

As before, we shall require some transmission conditions:

$$\phi^i(t,\mathbf{x}) = \phi^{i+1}(t,\mathbf{x}), \mathbf{x} \in \partial_-\Omega_i, t \in (0,T), \tag{12}$$

$$\frac{\partial \phi^i(t,\mathbf{x})}{\partial z} = \frac{\partial \phi^{i+1}(t,\mathbf{x})}{\partial z}, \mathbf{x} \in \partial_-\Omega_i, t \in (0,T). \tag{13}$$

We must add also the initial conditions:

$$\mathbf{u}(0,\mathbf{x}) = \mathbf{u}_0(\mathbf{x}), \mathbf{u}_t(0,\mathbf{x}) = \mathbf{v}_0(\mathbf{x}) \text{ on } \Omega. \tag{14}$$

The above mentioned conditions does not guarantee, in general, the existence of a classical solution of the problem, so we must introduce a notion of weak solution.

Firstly, we define the spaces of test functions (or *energy spaces*):

$$
\begin{aligned}
&V_u{:=}\{(\mathbf{u}^1,\phi^1),\ldots,(\mathbf{u}^p,\phi^p)\in\prod_{i=1}^{p}H^1(\Omega_i)^3\times H^1(\Omega_i)\ \text{ such that } \mathbf{u}^i \text{ verifies (4) to (8)}\},\\
&V_\phi{:=}\{((\mathbf{u}^1,\phi^1),\ldots,(\mathbf{u}^p,\phi^p))\in\prod_{i=1}^{p}H^1(\Omega_i)^3\times H^1(\Omega_i)\ \text{ such that } \phi^i\\
&\text{verifies (9), (11), (12), (13) and } \phi^i\equiv 0 \text{ on } \partial_+\Omega_1\}.
\end{aligned}
\tag{15}
$$

With respect to the boundary datum, ϕ_0, we shall identify it with its extension to the interior of the domain Ω_1, so we assume that there exists a function $\widehat{\phi}_0(t,\mathbf{x})$ such that

$$
\widehat{\phi}_0\in L^2(0,T:H^1(\Omega_1)),\ \widehat{\phi}_0(t,\mathbf{x})=\phi_0(t,\mathbf{x})\ \text{ in } (0,T)\times\partial_+\Omega_1 \tag{16}
$$

and

$$
\widehat{\phi}_0(\mathbf{x})=0 \text{ in } (0,T)\times(\partial_-\Omega_1\cup\partial_l\Omega_1). \tag{17}
$$

We shall assume the following regularity on the data:

$$
\phi_0\in L^2(0,T:\prod_{i=1}^{p}H^1(\Omega_i)) \text{ by satisfying (16) and (17).} \tag{18}
$$

$$
\mathbf{f}_u\in L^2(0,T:\prod_{i=1}^{p}H^{-1}(\Omega_i)^3), \tag{19}
$$

$$
f_\phi\in L^2(0,T:\prod_{i=1}^{p}H^{-1}(\Omega_i)). \tag{20}
$$

$$
\mathbf{u}_0\in V_u\cap\prod_{i=1}^{p}H^2(\Omega_i), \tag{21}
$$

$$
\mathbf{v}_0\in V_u. \tag{22}
$$

Before showing the main result of the problem, we shall give the definition of weak solution.

Definition 1 $(\mathbf{u},\phi)$ *is a weak solution of the problem (2) with the boundary conditions (4)-(13) and (14) if* $(\mathbf{u},\phi-\phi_0)\in H^1(0,T:V)$, $\rho\mathbf{u}_{tt}\in L^2(0,T:V_u')$ *and for any test function* $(\mathbf{w},\theta)\in H^1(0,T:V)$, $\mathbf{w}\in H^2(0,T:V_u')$ *the following identities hold:*

$$
\begin{aligned}
&\int_0^T[\langle\rho\mathbf{u}_{tt}^i(t,\cdot),\mathbf{w}(t,\cdot)\rangle+\sum_{i=1}^{p}\int_{\mathbf{\Omega_i}}\{\frac{1}{1-2\nu^i}div\mathbf{u}^i(t,\mathbf{x})\ div\mathbf{w}^i(t,\mathbf{x})-\frac{\rho^i g}{\mu^i}\nabla\left(\mathbf{u}^i(t,\mathbf{x})\cdot\mathbf{e}_z\right)\cdot\mathbf{w}^i(t,\mathbf{x})\\
&+\frac{\rho^i g}{\mu^i}\mathbf{e}_z div\mathbf{u}^i(t,\mathbf{x})\ \mathbf{w}^i(t,\mathbf{x})+\nabla\mathbf{u}^i(t,\mathbf{x}):\nabla\mathbf{w}^i(t,\mathbf{x})+\nabla\mathbf{u}_t^i(t,\mathbf{x}):\nabla\mathbf{w}^i(t,\mathbf{x})\ d\mathbf{x}]\}dt\\
&=-\int_0^T\left[\sum_{i=1}^{p}\frac{\rho^i}{\mu^i}\int_{\Omega_i}\nabla\phi^i(t,\mathbf{x})\cdot\mathbf{w}^i(t,\mathbf{x})\ d\mathbf{x}+\left\langle\mathbf{f}_u^i(t,\cdot),\mathbf{w}^i(t,\cdot)\right\rangle_{V_u'\times V_u}\right]\\
&\quad\textit{and}\\
&\sum_{i=1}^{p}\int_0^T\int_{\Omega_i}\nabla\phi^i(t,\mathbf{x})\cdot\nabla\theta^i(t,\mathbf{x})\ d\mathbf{x}dt=\int_0^T[\sum_{i=1}^{p}-4\pi\rho^i G\int_{\Omega_i}div\mathbf{u}^i(t,\mathbf{x})\ \theta^i(t,\mathbf{x})\ d\mathbf{x}\\
&+\left\langle f_u(t,\cdot)\ \mathbf{w}^i(t,\cdot)\right\rangle]dt.
\end{aligned}
\tag{23}
$$

Theorem 1 *[2] Assume the regularity (18)-(22) on the data* $\mathbf{f}_u$, f_ϕ, ϕ_0 $\mathbf{u}_0$ *and* $\mathbf{v}_0$. *Then there exists a unique weak solution* $\{\mathbf{u},\phi\}$ *of the problem (2).*

2 Stabilization for $t \to +\infty$.

In this section we are going to prove the convergence of solutions of the hyperbolic/elliptic problem to solutions of the stationary problem $\{\mathbf{u}_\infty(\mathbf{x}), \phi_\infty(\mathbf{x})\}$ when $t \to +\infty$. We suppose that

$$\mathbf{f}_u(t,.) \to \mathbf{f}_{u,\infty}(.) \text{ in } \prod_{i=1}^{p} H^{-1}(\Omega_i) \text{ when } t \to +\infty, \tag{24}$$

$$f_\phi(t,.) \to f_{\phi,\infty}(.) \text{ in } \prod_{i=1}^{p} H^{-1}(\Omega_i) \text{ when } t \to +\infty, \tag{25}$$

$$\widehat{\phi}_0(t,.) \to \widehat{\phi}_{0,\infty}(.) \text{ in } H^1(\Omega_1) \text{ when } t \to +\infty, \tag{26}$$

and we shall work with $\{\mathbf{u}^*, \phi^*\}$, the vectorial difference function

$$\mathbf{u}^*(t,\mathbf{x}) = \mathbf{u}(t,\mathbf{x}) - \mathbf{u}_\infty(\mathbf{x}), \ \phi^*(t,\mathbf{x}) = \phi(t,\mathbf{x}) - \phi_\infty(\mathbf{x}). \tag{27}$$

In order to simplify our study we do not take into account convective terms and assume that, in fact,

$$\mathbf{f}_u(t,.) = \mathbf{f}_{u,\infty}(.), \ f_\phi(t,.) = f_{\phi,\infty}(.) \text{ and } \widehat{\phi}_0(t,.) \to \widehat{\phi}_{0,\infty}(.), \tag{28}$$

(Some arguments allowing to avoid the previous simplifications, can be found in [5]).

Theorem 2 *Under above mentioned hypothesis as well as* $\mathbf{v}_0^i, \mathbf{u}_0^i, \mathbf{u}_\infty^i \in H^2(\Omega_i), i = 1, \ldots, p$, *we have that* $\mathbf{u}^*(t,\mathbf{x}) \longrightarrow 0$ *in* V_u *and* $\phi^*(t,\mathbf{x}) \longrightarrow 0$ *in* V_ϕ *when* $t \to +\infty$.

We shall use the *Lyapunov's method* as stated in [4] for abstract dynamical systems. We recall that if (Z,d) is a complete metric space and $\{S_t\}_{t\geq 0}$ is the dynamical system on Z, given $z \in Z$ the omega-limit set is defined as
$\omega(z) := \{y \in Z, \exists t_n \longrightarrow \infty, S_{t_n} z \longrightarrow \textit{and} \text{ when } n \longrightarrow \infty\}$ and that

$$\omega(z) := \bigcap_{s>0} \overline{\bigcup_{t\geq s} \{S_{t_n} z\}}. \tag{29}$$

Moreover we know that if $\bigcup_{t\geq s} \{S_{t_n} z\}$ is relatively compact in Z then $S_{t_n}(\omega(z)) = \omega(z) \neq \varnothing$ and we have:

Theorem 3 *([4]) [La Salle invariance principle]Let E be a Lyapounov function for* $\{S_t\}_{t\geq 0}$, *(i.e. such that* $E(S_t z) \leq E(z)$ $\forall t \geq 0$ *and* $\forall z \in Z$*), and let* $z \in Z$ *be such that* $\bigcup_{t\geq s} \{S_{t_n} z\}$ *is relatively compact in* Z*, then: (i)* $\lim_{t\to\infty} E(S_t z) = L$ *exists, (ii)* $E(y) = L, \forall y \in \omega(z)$.

Moreover, we find the following result:

Theorem 4 *([4]). Let E be a strict Lyapounov function for* $\{S_t\}_{t\geq 0}$, *(i.e. such that if* $E(S_t z) = E(z)$ $\forall t \geq 0$ *is verified then z is an equilibrium point for* $\{S_t\}_{t\geq 0}$*). Let* $z \in Z$ *be such that* $\bigcup_{t\geq 0} \{S_t z\}$ *is relatively compact in Z. Let $\mathcal{E}$ be the equilibrium points set of* $\{S_t\}_{t\geq 0}$. *Then (i) $\mathcal{E}$ is a closed subset non empty of Z, (ii)* $d(S_t z, \mathcal{E}) \to 0$ *when* $t \to \infty$ *(i.e.* $\omega(z) \subset \mathcal{E}$*).*

Proof of Theorem 2. In order to apply above mentioned abstract results, we shall work with the difference functions $\mathbf{u}^*(t,\mathbf{x})$ and $\phi^*(t,\mathbf{x})$ and, in fact, for the sake of simplicity in the notation we leave the asterisk. So we obtain

$$\begin{cases} \quad \rho^i \mathbf{u}^i_{tt}(t,\mathbf{x}) - \gamma^i \Delta \mathbf{u}^i_t(t,\mathbf{x}) - \Delta \mathbf{u}^i(t,\mathbf{x}) - \dfrac{1}{1-2\nu^i}\nabla\left(div\mathbf{u}^i(t,\mathbf{x})\right) \\ = \dfrac{\rho^i}{\mu^i}\nabla\phi^i(t,\mathbf{x}) \text{ en } (0,+\infty)\times\Omega_i, \\ -\Delta\phi^i(t,\mathbf{x}) = 4\pi\rho G div\mathbf{u}^i(t,\mathbf{x}) \text{ en } (0,+\infty)\times\Omega_i, \\ +\text{Boundary and initial conditions} + \text{initial conditions.} \end{cases} \tag{30}$$

To construct a dynamical system associate to the hyperbolic/elliptic system we take the inverse Laplacian operator (with the specified boundary conditions) on the second equation. So we have $\phi^i(t,\mathbf{x}) = (-\Delta)^{-1}\left(4\pi\rho^i G div\mathbf{u}^i(t,\mathbf{x})\right)$. Replacing this term in the first equation, we obtain a non local equation that only involve displacements:

$$\begin{cases} \rho^i \mathbf{u}^i_{tt}(t,\mathbf{x}) - \gamma \Delta \mathbf{u}^i_t(t,\mathbf{x}) - \Delta \mathbf{u}^i(t,\mathbf{x}) - \dfrac{1}{1-2\nu^i}\nabla\left(div\mathbf{u}^i(t,\mathbf{x})\right) \\ = \dfrac{\rho^i}{\mu^i}\nabla\left((-\Delta)^{-1}\left(4\pi\rho^i G div\mathbf{u}^i(t,\mathbf{x})\right)\right) \text{ , in } (0,+\infty)\times\Omega_i, \\ \mathbf{u}^i(0,\mathbf{x}) = \mathbf{u}^i_0(\mathbf{x}) - \mathbf{u}^i_\infty(\mathbf{x}) \text{ , in } \Omega_i, \\ \mathbf{u}^i_t(0,\mathbf{x}) = \mathbf{v}^i_0(\mathbf{x}) \text{ , in } \Omega_i, \\ \mathbf{u}^i(t,\mathbf{x}) = 0 \text{ , in } \partial_l\Omega_i, \\ \left.\begin{array}{rl} \mathbf{u}^i(t,\mathbf{x}) = & \mathbf{u}^{i+1}(t,\mathbf{x}), \\ \dfrac{\partial\mathbf{u}^i(t,\mathbf{x})}{\partial z} = & \dfrac{\partial\mathbf{u}^{i+1}(t,\mathbf{x})}{\partial z}, \end{array}\right. \text{ , in } (0,+\infty)\times\partial_-\Omega_i \text{ for } i=1,\ldots,p-1, \\ \mathbf{u}^1(t,\mathbf{x}) = 0 \text{ , in } (0,+\infty)\times\partial_+\Omega_1, \\ \mathbf{u}^p(t,\mathbf{x}) = 0 \text{ , in } (0,+\infty)\times\partial_-\Omega_p. \end{cases} \tag{31}$$

We take

$$z := \begin{pmatrix} \mathbf{u}_0 - \mathbf{u}_\infty \\ \mathbf{v}_0 \end{pmatrix} \tag{32}$$

and the dynamic system is given by

$$S_t z := \begin{pmatrix} \mathbf{u}(t,.) \\ \mathbf{u}_t(t,.) \end{pmatrix} \tag{33}$$

with $\mathbf{u}(t,.)$ solution of the above non local problem (31). So, we take as the space of the states $Z := V_u \times V_u$. We define the Lyapunov function E by

$$E\left(\begin{pmatrix} \mathbf{u} \\ \mathbf{u}_t \end{pmatrix}\right) := \sum_{i=1}^{p}[\frac{4\pi\left(\rho^i\right)^2 G}{2}[\int_{\Omega_i}\left(\left|\mathbf{u}^i_t(t,\mathbf{x})\right|^2 + 4\pi\rho^i G\gamma^i\left|\nabla\mathbf{u}^i_t(t,\mathbf{x})\right|^2\right)d\mathbf{x} +$$

$$+2\pi\rho^i G\int_{\Omega_i}(\left|\nabla\mathbf{u}^i(t,\mathbf{x})\right|^2 + \frac{4\pi\rho^i G}{1-2\nu^i}div\mathbf{u}^i(t,\mathbf{x})^2) + \int_\Omega\frac{\rho^i}{\mu^i}\left(\left|\nabla(-\Delta)^{-1}\left(4\pi\rho^i G div\mathbf{u}^i(t,\mathbf{x})\right)\right|^2\right)d\mathbf{x}]. \tag{34}$$

We reach this function by the following steps. Firstly, we multiply the first equation of the system (30) by the term $4\pi\rho^i G\mathbf{u}^i_t$. Then, we integrate over space. Next, we differentiate in t to obtain an expression. Now, we multiply the second equation of the system (30) by $\dfrac{\rho^i}{\mu^i}\phi^i$ and we integrate over space. We add this result to the previous expression after integrating over time in $[0,T]$. Finally, bearing in mind the boundary conditions, we integrate again over time.

The main hypothesis we want to verify is that the function E is a strict Lyapounov function for $\{S_t\}_{t\geq 0}$. But, this coincide exactly with the argument used to prove the uniqueness of solutions of the hyperbolic problem ([2]: see also the DEA report [3]). Finally, to complete the proof we shall check that $\bigcup\limits_{t\geq 0}\{S_t z\}$ is relatively compact in Z. We have that

$$\begin{aligned}
&(\frac{4\pi\left(\rho^i\right)^2 G}{2})\sup\nolimits_{t\in[0,\infty)}\int_{\Omega_i}\left|\mathbf{u}_t^i\left(t,\mathbf{x}\right)\right|^2+\frac{4\pi\left(\rho^i\right)^2 G}{2}4\pi\rho^i G\gamma^i\sup\nolimits_{t\in[0,\infty)}[\int_{\Omega_i}\left|\nabla\mathbf{u}_t^i\left(t,\mathbf{x}\right)\right|^2 d\mathbf{x}\\
&+(2\pi\rho^i G)\sup\nolimits_{t\in[0,\infty)}\int_{\Omega_i}(\left|\nabla\mathbf{u}^i\left(t,\mathbf{x}\right)\right|^2+(\frac{4\pi\rho^i G}{1-2\nu^i})\sup\nolimits_{t\in[0,\infty)}\int_{\Omega_i}div\mathbf{u}^i\left(t,\mathbf{x}\right)^2\\
\leq&\int_{\Omega_i}\frac{\rho^i}{\mu^i}\left|\nabla\phi^i(0,\mathbf{x})\right|^2 d\mathbf{x}+\frac{4\pi\left(\rho^i\right)^2 G}{2}\int_{\Omega_i}\left|\mathbf{v}_0^i\left(\mathbf{x}\right)\right|^2+\frac{4\pi\left(\rho^i\right)^2 G}{2}4\pi\rho^i G\gamma^i[\int_{\Omega_i}\left|\nabla\mathbf{v}_0^i\left(\mathbf{x}\right)\right|^2\\
&+(2\pi\rho^i G)\int_{\Omega_i}(\left|\nabla(\mathbf{u}_0^i\left(\mathbf{x}\right)-\mathbf{u}_\infty^i\left(\mathbf{x}\right))\right|^2+\frac{2\pi\rho^i G}{1-2\nu^i}\int_{\Omega_i}div(\mathbf{u}_0^i\left(\mathbf{x}\right)-\mathbf{u}_\infty^i\left(\mathbf{x}\right))^2.
\end{aligned}\tag{35}$$

Now, we shall prove that $\{\mathbf{u}_t\left(t,.\right)\}$ is relatively compact in V_u. We differentiate in t the system

$$\begin{cases}
\rho^i\mathbf{u}_{ttt}^i\left(t,\mathbf{x}\right)-\gamma^i\Delta\mathbf{u}_{tt}^i\left(t,\mathbf{x}\right)-\Delta\mathbf{u}_t^i\left(t,\mathbf{x}\right)-\dfrac{1}{1-2\nu^i}\nabla div\mathbf{u}_t^i\left(t,\mathbf{x}\right)\\
=\dfrac{\rho^i}{\mu^i}\nabla\phi_t^i\left(t,\mathbf{x}\right)\ \text{, in}\ (0,\infty)\times\Omega_i,\\
\mathbf{u}_t^i\left(0,\mathbf{x}\right)=\mathbf{v}_0^i\left(x\right)\ \text{, in}\ \Omega_i,\\
\mathbf{u}_{tt}^i\left(0,\mathbf{x}\right)=\gamma\Delta\mathbf{v}_0^i\left(\mathbf{x}\right)+\Delta\mathbf{u}_0^i\left(\mathbf{x}\right)-\Delta\mathbf{u}_\infty^i\left(\mathbf{x}\right)+\dfrac{1}{1-2\nu^i}\nabla div\mathbf{u}_0^i\left(\mathbf{x}\right)\text{, in}\ \Omega_i\,.
\end{cases}\tag{36}$$

The assumed regularity $\mathbf{v}_0^i$, $\mathbf{u}_0^i$, $\mathbf{u}_\infty^i\in H^2(\Omega_i)$, $i=1,\ldots,p$ allows us to conclude that $\mathbf{u}_{tt}\in L^\infty\left(0,\infty:H^1(\Omega)\right)$, and the proof follows.

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