Multiplicative functionals on function algebras

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ABSTRACT. Let $X$ be a completely regular Hausdorff space and $C(X)$ the algebra of all continuous $\mathbb{K}$-valued functions on $X$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$). If $A \subseteq C(X)$ is a subalgebra, in [4] can be found conditions on $A$ under which each character of $A$, i.e., each non-zero $\mathbb{K}$-linear multiplicative functional $\phi: A \rightarrow \mathbb{K}$, is given by a point evaluation at some point of $X$.

In this paper we present a «Michael» type theorem for the particular case in which $X$ is a real Banach space. As consequence it is showed that if $E$ is a separable Banach space or $E$ is the topological dual space of a separable Banach space and $A$ is the algebra of all real analytic or the algebra of all real $C^m$-functions, $m = 0, 1, \ldots, \infty$, on $E$, then every character $\phi$ of $A$ is a point evaluation at some point of $E$.

Let $E$ be a real Banach space with topological dual $E'$ and let $C(E)$ be the algebra of all continuous $\mathbb{R}$-valued functions on $E$. Let $l^1(\mathbb{N}) = \{ \alpha = (\alpha_n) \in \mathbb{R}^\mathbb{N} : \sum_{n=1}^{\infty} |\alpha_n| < \infty \}$.

**Theorem 1.** Assume that there exists $(\phi_n)_{n=1}^{\infty} \in E'$, $\|\phi_n\| \leq 1$ for every $n \in \mathbb{N}$, such that $(\phi_n)$ separates points of $E$. Let $A \subseteq C(E)$ be a subalgebra with $1 \in A$. Assume:

(i) If $f \in A$, $f(x) \neq 0$ for all $x \in E$, then $1/f \in A$.

(ii) $E' \subseteq A$ and for every $\alpha = (\alpha_n) \in l^1(\mathbb{N})$, the function $\sum_{n=1}^{\infty} \alpha_n \cdot \phi_n^2$ belongs to $A$.

Then every character $\phi: A \rightarrow \mathbb{R}$, such that $\phi(\phi_n) = \phi_n(a)$ for every $n \in \mathbb{N}$ and some $a \in E$, is the point evaluation at $a$.

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Proof. Let $\alpha=(\alpha_n)\in l^1(\mathbb{N})$ with $\alpha_n>0$ for all $n\in\mathbb{N}$. Condition (ii) implies that the functions:

$$f(x) = \sum_{n=1}^{\infty} \alpha_n \phi_n^2(x-a) \quad \text{and} \quad g(x) = \sum_{n=1}^{\infty} \frac{\alpha_n}{n} \phi_n^2(x-a)$$

belong to $A$.

For each $N \in \mathbb{N}$, let $x \in E$ such that $\phi(f) = f(x)$, $\phi(g) = g(x)$ and $\phi(\phi_i) = \phi_i(x)$, $i = 1, ..., N$ (a such $x$ exists after condition (i)). For this $x \in E$, we have

$$\phi(f) = \sum_{N+1}^{\infty} \alpha_n \phi_n^2(x-a) \quad ; \quad \phi(g) = \sum_{N+1}^{\infty} \frac{\alpha_n}{n} \phi_n^2(x-a)$$

Therefore $0 \leq N \phi(g) \leq \phi(f)$ and it follows that $\phi(g) = 0$.

If $h \in A$ is given, let $y \in E$ such that $\phi(h) = h(y)$ and $\phi(g) = g(y)$. Since $\phi(g) = g(y) = 0$, it follows that $\phi_n(y) = \phi_n(a)$ for all $n \in \mathbb{N}$, i.e., $y = a$ and $\phi(h) = h(a)$.

Remark 1. The hypothesis on the real Banach space $E$ in Theorem 1 is equivalent to say that $E'$ is $\sigma(E'; E)$-separable. Therefore it holds when $E$ is a separable Banach space and when $E$ is the topological dual space of a separable Banach space.

Consequences

Let $A(E)$ be, respectively $C^m(E)$ ($m = 0, 1, ..., \infty$), the subalgebra of $C(E)$ of all real analytic functions (see [2]), respectively of all $C^m$-functions in the Fréchet sense, on $E$.

Corollary 1. If $E$ is finite dimensional and $A = A(E)$ or $A = C^m(E)$, then every character $\phi: A \to \mathbb{R}$ is a point evaluation at some point of $E$.

Proof. This follows from Theorem 1 if we consider $(\phi_n)$ as the canonical projections.

Proposition 1. For every character $\phi: A(E) \to \mathbb{R}$, the restriction $\phi|_E$ is $\sigma(E'; E)$-sequentially continuous.

Proof. Assume that $(x'_n) \subset E'$ converges to zero for the $\sigma(E'; E)$-topology. If $\phi(x'_n) \neq 0$, there are $\alpha > 0$ and $(x'_{np})$, subsequence of $(x'_n)$, such that

$$\left[ \phi \left( \frac{x'_{np}}{\sqrt{\alpha}} \right) \right]^2 > 1$$

for every $p \in \mathbb{N}$. Since $(x'_{np}) \to 0$ ($p \to \infty$) for the $\sigma(E'; E)$-topology,
the function

\[ f(x) = \sum_{p=1}^{\infty} \left[ \frac{x'_{np}(x)}{\sqrt{\alpha}} \right]^{2p} \]

is well defined and \( f \in A(E) \). (See ([2], Th. 6)). For each \( N \in \mathbb{N} \),

\[ \phi(f) \geq \phi \left[ \sum_{p=1}^{N} \left[ \frac{x'_{np}}{\sqrt{\alpha}} \right]^{2p} \right] = \sum_{p=1}^{N} \left[ \phi \left[ \frac{x'_{np}}{\sqrt{\alpha}} \right] \right]^{2p} \]

Therefore \( \sum_{p=1}^{\infty} \left[ \phi \left[ \frac{x'_{np}}{\sqrt{\alpha}} \right] \right]^{2p} < \infty \) and then \( \left[ \phi \left[ \frac{x'_{np}}{\sqrt{\alpha}} \right] \right]^{2p} \to 0 \) \((p \to \infty)\), which is a contradiction because \( \left[ \phi \left[ \frac{x'_{np}}{\sqrt{\alpha}} \right] \right] > 1 \) for all \( p \in \mathbb{N} \).

**Corollary 2.** Let \( E \) be a separable Banach space and \( \phi : A(E) \to \mathbb{R} \) a character. Then \( \phi_E \) is a point evaluation at some point of \( E \).

**Proof.** This is immediate from Prop. 1, since by ([5], Ch. IV; Th. 6.2 and Corollary 3) for \( \phi_E \) to be \( \sigma(E'; E) \)-continuous it suffices to show that \( \phi_E \) is \( \sigma(E'; E) \)-sequentially continuous.

**Corollary 3.** Let \( E \) be a separable Banach space and \( \phi : A(E) \to \mathbb{R} \) a character. Then \( \phi \) is a point evaluation at some point of \( E \).

**Proof.** This is immediate from Theorem 1, Remark 1 and Corollary 2.

Let \( F \) be a separable Banach space and \( (y_n)_{n=1}^{\infty} \) a dense subset in \( \{y \in F : \|y\| \leq 1\} \). Let \( E = F' \). Let \( \phi_n : E \to \mathbb{R} \) be defined as \( \phi_n(x) = x(y_n) \). Then \( \phi_n \in E' \), \( \|\phi_n\| \leq 1 \) and \( (\phi_n)_{n=1}^{\infty} \) separates points of \( E \). The mapping \( y \to \phi_y \), defined as \( \phi_y(x) = x(y) \), allow us identify \( F \) with a subspace of \( E' = F'' \). Thus, if \( \phi : A(E) \to \mathbb{R} \) is a character, Prop. 1 implies that \( \phi_F \) is \( \|\| \)-continuous, therefore \( \phi_F \in F' = E \). Then, it follows that there exists \( a \in E \) such that \( \phi(\phi_n) = \phi_n(a) \) for all \( n \in \mathbb{N} \). Now the following Corollary is clear after Theorem 1.

**Corollary 4.** Let \( E \) be a topological dual space of a separable Banach space and \( \phi : A(E) \to \mathbb{R} \) a character. Then \( \phi \) is a point evaluation at some point of \( E \).

**Corollary 5.** Assume that \( E \) is a separable Banach space or \( E \) is the topological dual space of a separable Banach space. Then every character \( \phi : C^m(E) \to \mathbb{R}, m = 0, 1, ..., \infty \), is a point evaluation at some point of \( E \).
**Proof.** \(\phi_{(A,E)}\) is a point evaluation by Corollary 3 and Corollary 4. Thus, \(\phi\) satisfies conditions of Theorem 1 with \(A = C^m(E)\).

**Remark 2.** The Corollary 5, for the particular case \(E\) a separable Banach space and \(m = \infty\), can be found in [1]. Also, for \(E\) with \(C^m\)-partitions of unity and \(m < \infty\), see [3].

**References**


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