

Two geometric constants for operators acting on a separable Banach space

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ABSTRACT. The main result of this paper is the following: A separable Banach space X is reflexive if and only if the infimum of the Gelfand numbers of any bounded linear operator defined on X can be computed by means of just one sequence of nested, closed, finite codimensional subspaces with null intersection.

INTRODUCTION

Let A be an operator from a separable Banach space X into another Banach space Y . For every Markushevich basis of X , (a_n) , we define two numbers $h_{A,(a_n)}$, $H_{A,(a_n)}$, which give some geometrical insight about the space X and also the operator A . In fact reflexivity of X is characterized by stability of $H_{A,(a_n)}$ through changes of the M -basis, for every operator A , as theorem 1 states.

In the framework of reflexive Banach spaces these constants will be denoted simply H_A and h_A and they coincide respectively with the infimum of the Gelfand numbers of A , and with a precise lower bound of the Bernstein numbers of A defined by Zemanek in [11]. So we come to the conclusion stated in theorem 2, that the infimum of the Gelfand numbers can be computed by means of a nested sequence of closed finite codimensional subspaces of null intersection.

In the third part we relate these numbers also with the spectral properties of A . Finally we see that for the particular case of a Hilbert space X , they are exactly the maximum and the minimum of the limit points of the spectrum of $(A^*A)^{1/2}$.

NOTATIONS, DEFINITIONS AND REMARKS

We denote a Banach space by letters such as X, Y . The symbol $[\]$ will stand for closed linear span, and the symbol \rightharpoonup for weak convergence. An M -basis will be the short name of a Markushevich basis, i.e. a fundamental sequence (a_n) in a separable Banach space X , for which there exists another sequence (a_n^*) in the dual X^* , called the conjugate sequence, with $a_n^*(a_n) = \delta_{mn}$ and such that $a_n^*(x) = 0 \ \forall n \in N$, implies $x = 0$. It is a well known result that M -bases always exist in a separable Banach space.

An M -basis (a_n) of X is called shrinking if for every f of X^*

$$\|f|_{[a_n \dots]}\| \xrightarrow{n \rightarrow \infty} 0$$

James has proven that this condition is equivalent to completeness of (a_n^*) in X^* , that is $[(a_n^*)] = X^*$.

The set of all bounded linear operators from X into Y will be called $B(X, Y)$, and $\Phi_+(X, Y)$ will stand for the set of semi-Fredholm $+$ operators, i.e. those bounded linear operators with closed range and finite dimensional kernel.

1. Let X be a separable Banach space, $A \in B(X, Y)$ and let (a_n) be an M -basis of X . We define the two numbers:

$$H_{A, (a_n)} = \inf_n \|A|_{[a_n \dots]}\|$$

and

$$h_{A, (a_n)} = \sup_n m(A|_{[a_n \dots]})$$

where m stands for minimum modulus, i.e.

$$m(A) = \inf_{\|x\|=1} \|Ax\|$$

The first of these numbers leads us to a characterization of reflexivity of X , as can be seen in

Theorem 1. *The following statements are equivalent:*

1. X is reflexive.
2. For every bounded linear operator from X into an arbitrary Banach space Y , $H_{A, (a_n)}$ is independent of the M -basis (a_n) .
3. For every $f \in X^*$, $H_{f, (a_n)}$ is independent of the M -basis (a_n) .

Before giving the proof of the theorem we state three lemmas, which are easy to prove.

Lemma 1. *If X is reflexive and separable, (a_n) an M -basis of X and $x_n \xrightarrow{w} 0$ with $\|x_n\| = 1$, then $x_n \xrightarrow{w} 0$.*

Lemma 2. *Let (x_n) be a normalized sequence in a Banach space X . Then $x_n \xrightarrow{w} 0$ if and only if for every finite codimensional, closed subspace E , there exists (x'_n) in E , with $x'_n \xrightarrow{w} 0$, $\|x'_n\| = 1$ and $\|x_n - x'_n\| \rightarrow 0$.*

It is a well known result, due to Zippin, that a Banach space X with a basis is reflexive, if and only if every basis of X is shrinking. The following lemma is a slight modification of this, not requiring the space to have a basis.

Lemma 3. *A separable Banach space X is reflexive if and only if every M -basis of X is shrinking.*

Proof. If X is reflexive a closed subspace F of X^* which is total on X must be the whole of X^* . Thus, for every M -basis (a_n) , $[(a_n^*)] = X^*$ and equivalently (a_n) is shrinking.

Conversely, if X is non reflexive two cases arise. First, if X^* is non separable, no M -basis can be shrinking. Second, if X^* is separable, by the result of Gaposkin and Kadets, (see [4] p. 120), for every total subspace F of X^* there exists an M -basis (a_n) of X such that $[(a_n^*)] = F$. Thus (a_n) is a non shrinking M -basis.

Proof of theorem 1

1) \rightarrow 2) Let $(a_n), (b_n)$ be two different normalized, M -bases of X . Denote by $\mu_n = \|A_{[a_n, a_{n+1}, \dots]}\|$. Clearly $\mu_n \geq \mu_{n+1} \geq \dots \geq 0$ and analogously $\mu'_n = \|A_{[b_n, \dots]}\|$.

Let (ε_n) be a decreasing sequence of real numbers with $\varepsilon_n \rightarrow 0$. Choose for every n , $x_n \in [a_n, \dots]$ such that $\|x_n\| = 1$ and $|\mu_n - \|Ax_n\|| < \varepsilon_n$. By lemma 1, $x_n \xrightarrow{w} 0$. For a fixed $p \in \mathbb{N}$, take into account the descomposition $X = [b_1, \dots, b_{p-1}] \oplus [b_p, \dots]$ and choose (x'_n) in $[b_p, \dots]$, such that $\|x'_n\| = 1$ and $\|x_n - x'_n\| \rightarrow 0$.

Clearly $\| \|Ax_n\| - \|Ax'_n\| \| \leq \|A(x_n - x'_n)\| \rightarrow 0$ and $\|Ax'_n\| \rightarrow H_{A, (a_n)}$. For every $n \in \mathbb{N}$, $\|Ax'_n\| \leq \|A_{[b_p, \dots]}\| = \mu'_p$ and thus $H_{A, (a_n)} \leq \mu'_p$ ($\forall p \in \mathbb{N}$). Therefore

$$H_{A, (a_n)} \leq \inf_p \mu'_p = H_{A, (b_n)}$$

Changing the roles of (a_n) and (b_n) , the converse inequality is obtained, so $H_{A, (a_n)} = H_{A, (b_n)}$, which we will call H_A .

2 \rightarrow 3 is obvious.

3→1 Assume X non reflexive. By lemma 3 there is a non shrinking M -basis (a_n) in X . Take $g \notin [(a_n^*)]$ and take a biorthogonal system (c_n, c_n^*) with

$$[(c_n)_{n=1}^\infty] = \text{Ker } g \text{ and } (c_n^*) \text{ total on Ker } g. \text{ Choose then } b_1 \in X \text{ such that}$$

$g(b_1) = 1$ and $b_n = c_{n-1}$ for $n \geq 2$. Now (b_n) is an M basis of X with conjugate system (b_n^*) given by $b_1^* = g$ and $b_{n+1}^* = c_n^* - c_n^*(b_1)g, n \geq 2$. Since $g \in [(b_n^*)]$, $H_{g, (b_n)} = 0$. On the other hand $g \notin [(a_n^*)]$, which implies $H_{g, (a_n)} \neq 0$. Thus $H_{g, (b_n)} \neq H_{g, (a_n)}$.

Remark 1. A parallel to the theorem, setting $h_{A, (a_n)}$ instead of $H_{A, (a_n)}$, does not hold. Although 1)→2) could be proved with the same argument, in 3)→1) we would obtain $h_{f, (a_n)} = 0$ for every (a_n) , not depending on reflexivity of X .

Remark 2. Recall the following result of Milman and Tumarkin: «For a separable Banach space X and a sequence (E_n) of closed subspaces of X such that $\text{codim } E_n = n, E_n \supset E_{n+1} \forall n \in \mathbb{N}$, and $\bigcap_n E_n = \{0\}$, there exists a biorthogonal system (a_n, a_n^*) such that $[a_{n+1}, a_{n+2}, \dots] = E_n$ ». (see [4], thm 1.6). According to it, theorem 1 could be stated without reference to an M -basis, only with a sequence of nested subspaces of finite codimension and null intersection.

From now on we abbreviate h_A and H_A for $h_{A, (a_n)}$ and $H_{A, (a_n)}$ respectively, taking for granted that we are dealing with a separable, reflexive, Banach space X .

2. We relate now h_A and H_A with s -numbers and some characteristics given by Zemanek in [11]. First recall that the Gelfand numbers of an operator A are $c_n(A) = \inf\{\|A|_W\|, \text{codim } W < n\}$, and that the Bernstein numbers are $u_n(A) = \sup\{m(A|_W), \text{dim } W \geq n\}$.

The sequences $c_n(A)$ and $u_n(A)$ are decreasing, bounded from below by

$$c(A) = \inf\{\|A|_W\|, \text{codim } W < \infty\}$$

and

$$u(A) = \sup\{m(A|_W), \text{dim } W = \infty\}$$

respectively. Also note that

$$G(A) = \inf\{\|A|_W\|, \text{dim } W = \infty\} \leq c(A)$$

and

$$B(A) = \sup\{m(A|_W), \text{codim } W < \infty\} \leq u(A)$$

In the quoted paper Zemanek obtains nice results concerning these numbers, and Kolmogorof numbers, Mityagin numbers, and their infima; he

compares them with each other and with geometrical concepts such as Semi-Fredholm radius, reduced minimum modulus, etc. In the frame of separable reflexive Banach spaces we relate them with h_A and H_A as follows:

Proposition 1. *Let X be a separable reflexive Banach space and A an operator from X into any Banach space Y . Then $H_A = c(A)$ and $h_A = B(A)$.*

Proof. Fix an M -basis (a_n) and define as before $\mu_n = \|A|_{[a_n, \dots]}\|$. Since $c_n(A) = \inf\{\|A|_W\|, \text{codim } W < n\}$, it follows that $c_n(A) \leq \mu_n, \forall n \in N$. Therefore

$$c(A) = \inf_n c_n(A) \leq \inf_n \mu_n = H_A$$

For the converse inequality, fix a real number $\varepsilon > 0$. Take $K \in N$ such that $c_K(A) < c(A) + \varepsilon$ and take also a subspace W with $\text{codim } W < k$ and $\|A|_W\| < c(A) + \varepsilon$. Any M -basis of W can be extended in order to obtain an M -basis of X (see [4] thm. 1.8). Thus, assume $(a_n)_{n=k}^\infty$ is an M -basis of W and extend it to an M -basis of X , $(a_n)_{n=1}^\infty$. We have $H_A \leq \|A|_{[a_n, \dots]}\| < c(A) + \varepsilon$. This implies $H_A < c(A)$, and so $H_A = c(A)$.

A similar argument proves $h_A = B(A)$.

Now we can rewrite theorem 1 as follows:

Theorem 2. *A separable Banach space X is reflexive if and only if for every decreasing sequence of closed subspaces (E^n) such that*

- a) $n = \text{codim } E^n$.
- b) $\bigcap_n E^n = \{0\}$.

and for every operator A from X into another Banach space Y , $\inf_n \|A|_{E^n}\| = c(A)$, where $c(A)$ is the infimum of the Gelfand numbers of A .

Proof. Just take into account Remark 2, and Proposition 2.

We can say further that when X is separable and reflexive, $B(A)$ can also be computed by means of a nested sequence of closed subspaces satisfying the same conditions as those of theorem 2.

- Open questions.* 1) What can be said when separability of X is dropped?
2) Why does reflexivity establish such a difference?

3. We relate these constants now with spectral theory. The real numbers of the interval $[h_A, H_A]$ can be characterized as follows:

Theorem 3. *Let X be a separable reflexive Banach space, A an operator defined from X into an arbitrary Banach space Y . Then $\lambda \in [h_A, H_A]$ if and only if there exists (x_n) in X , with $\|x_n\| = 1, x_n \xrightarrow{w} 0$, such that $\|Ax_n\| \rightarrow \lambda$.*

Proof. Let $\lambda \in (h_A, H_A)$ and let (a_n) be an M -basis of X . For every $n \in N$, let $v_n = \inf\{\|Ax\|; x \in [a_n, \dots], \|x\| = 1\}$. Then $v_n \leq h_A < \lambda < H_A \leq \mu_n = \|A|_{[a_n, \dots]}\|$. Take now $z_n, y_n \in [a_n, \dots]$, with $\|z_n\| = \|y_n\| = 1$ such that $v_n \leq \|Az_n\| < \lambda$ and $\mu_n \geq \|Ay_n\| > \lambda$. Because the unit sphere of $[a_n, \dots]$ is connected, one can find $x_n \in [a_n, \dots]$, $\|x_n\| = 1$ such that $\|Ax_n\| = \lambda$. Iterating for $n \in N$ we obtain a sequence (x_n) , which by lemma 1 converges weakly to zero, and $\|Ax_n\| = \lambda$.

Slight modifications in this argument give the proof for $\lambda = h_A$ and $\lambda = H_A$.

Conversely, let $x_n \xrightarrow{w} 0$, $\|x_n\| = 1$ and $\|Ax_n\| \rightarrow \lambda$.

By lemma 2, for every fixed $p \in N$ we can choose $(z_n^{(p)})_{n=1}^\infty \subset [a_p, a_{p+1}, \dots]$ with $\|z_n^{(p)}\| = 1 \forall n \in N$, $z_n^{(p)} \xrightarrow{w} 0$ and $\|x_n - z_n^{(p)}\| \xrightarrow{n \rightarrow \infty} 0$ which implies $|\|Az_n^{(p)}\| - \|Ax_n\|| \rightarrow 0$. Thus $\|Az_n^{(p)}\| \rightarrow \lambda$.

Select now a diagonal sequence $(z_{n(p)}^{(p)})_{n=1}^\infty$, with $\|Az_{n(p)}^{(p)}\| \rightarrow \lambda$.

For every $p \in N$, $v_p \leq \|Az_{n(p)}^{(p)}\| \leq \mu_p$. Therefore, $\lambda \in [h_A, H_A]^{p \rightarrow \infty}$.

Equality of h_A and H_A , for an operator satisfying the former conditions, is determined by:

Corollary 1. *The following are equivalent:*

- 1) $h_A = H_A = r$.
- 2) Every normalized sequence (x_n) of X , with $x_n \xrightarrow{w} 0$ verifies $\|Ax_n\| \rightarrow r$.
- 3) For every normalized $(x_n) \subset X$ with $x_n \xrightarrow{w} 0$, $(\|Ax_n\|)$ is convergent.

Corollary 2. *The operator A is compact if and only if $h_A = H_A = 0$.*

Corollary 3. *The operator A is semi-Fredholm+ if and only if $h_A > 0$.*

We omit the proof of these corollaries which are straight-forward, and do the same with the following propositions.

Proposition 2. *Let $A \in B(X, Y)$. The following properties hold:*

- 1) h_A and H_A are stable through compact perturbations, i.e., $h_{A+K} = h_A$ and $H_{A+K} = H_A$, for every compact operator K .
- 2) If $A, T \in B(X, X)$
 $H_{AT} < \|A\|H_T$
 $h_{AT} \geq h_A h_T$
 If $T \in \Phi_+$, $H_{AT} \leq H_A H_T$.
- 3) $H_{A+B} \leq H_A + H_B$ and $h_{A+B} \leq h_A + h_B$.

Proposition 3. *Let $A \in B(X, Y)$. The following properties are equivalent:*

- 1) $\|A|_E\| = \|A\|$ for every closed, linear finite-codimensional subspace E .
- 2) $\|A\| = H_A$.
- 3) There exists a normalized weakly null sequence, (x_n) , such that $\|Ax_n\| \rightarrow \|A\|$.

4. GEOMETRICAL MEANING OF THE CONSTANTS h_A AND H_A FOR A HILBERT SPACE OPERATOR A

Now let X be a separable Hilbert space, A a selfadjoint operator on X , and E the spectral measure of A . Recall that a number λ is a limit point of the spectrum of A , $\sigma(A)$, iff $E(V)$ has infinite rank for every neighborhood V of λ .

Theorem 4. *If A is a selfadjoint operator on X , H_A and h_A are respectively, the maximum and the minimum of the limit points of the spectrum of A .*

Proof. Suppose λ is a limit point. According to Weyl's result (s. [3], p. 266) there exists (x_n) in X , $\|x_n\| = 1$, $x_n \rightharpoonup 0$ and $\|(A - \lambda I)x_n\| \rightarrow 0$. In particular, $\|Ax_n\| \rightarrow \lambda$ and therefore $h_A \leq \lambda \leq H_A$, although obviously not all the points of $[h_A, H_A]$ are limit points. Taking into account that addition of a self adjoint compact operator does not change the limit points on the spectrum, and by the results of Weyl and Von Neumann (s. [3], th. 2.1 p. 253), without loss of generality, we assume A is a diagonal operator (obviously referred to an orthonormal basis of eigenvectors (e_n)), say

$$A = \begin{bmatrix} \lambda_1 & \dots & 0 & \dots \\ 0 & \dots & \lambda_n & \dots \\ \dots & \dots & \dots & \dots \end{bmatrix} \quad \lambda_i \geq 0, \forall i \in \mathbb{N}$$

Let $\mu_n = \sup_{\substack{x \in [e_n, \dots] \\ \|x\|=1}} \|Ax\|$.

Clearly $\mu_n = \sup_{i > n} \lambda_i$

Choose now $(\lambda_{p_n}) \subset (\lambda_n)$ such that $\mu_n - \lambda_{p_n} \xrightarrow{n \rightarrow \infty} 0$. We have $\lambda_{p_n} \rightarrow H_A$.

The fact $Ae_{p_n} = \lambda_{p_n}e_{p_n}, \forall n \in \mathbb{N}$, implies $\|(A - H_A)e_{p_n}\| \rightarrow 0$, and therefore H_A is a limit point, actually the largest of them. A similar argument proves that h_A is the minimum of the limit points of A .

Remark 3. If A is not selfadjoint then H_A and h_A are not necessarily limit points of $\sigma(A)$. Nevertheless it can be easily proved, by polar descomposition and theorem 2, that h_A and H_A are exactly the maximum and the minimum of the limit points of $(A^*A)^{1/2}$.

Remark 4. Other geometric properties of operators A with $h_A = H_A$, can be

seen in [7] and [8], where such operators are called «asymptotic similarities», because of their geometric behaviour.

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