REVISTA MATEMÁTICA de la
Universidad Complutense de Madrid
Volumen 10, número 1: 1997

On minimal non $CC-$groups.

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Abstract

In this work it is shown that a locally graded minimal non $CC-$group G has an epimorphic image which is a minimal non $FC-$group and there is no element in G whose centralizer is nilpotent-by-Chernikov. Furthermore Theorem 3 shows that in a locally nilpotent $p-$group which is a minimal non $FC-$group, the hypercentral and hypocentral lengths of proper subgroups are bounded.

1 Introduction

Let G be a group. As is well-known, G is called an $FC-$group ($CC-$group) if for all $x \in G$,

$$[G : C_G(x)] < \infty \, (G/C_G(x^G) \text{ is Chernikov }).$$

G is called a **minimal non** $FC-$**group** if every proper subgroup of G is an $FC-$group but G itself does not have this property. A minimal non $CC-$group is defined similarly.

Belyaev in [2] showed that if G is a perfect locally finite minimal non $FC-$group then either $G/Z(G)$ is simple or G is a $p-$group (p is always a prime number). Recently Kuzucuoğlu and Phillips in [6] have shown that, in fact, G must be a $p-$group. More recently F. Leinen and O. Puglisi in [7] have shown that a perfect locally nilpotent

A.M.S. Classification: 20F
Servicio Publicaciones Univ. Complutense. Madrid, 1997.

$p-$group which is a minimal non $FC-$group can be embedded in the McLain group $M(\mathbb{Q}, GF(p))$. However it is still an open question whether or not a perfect locally nilpotent minimal non $FC-$group can exist.

Otal and Peña in [9] extended some of the properties of a minimal non $FC-$group to a locally graded minimal non $CC-$group. Later the same authors and B. Hartley in [5] have shown that such a group is locally nilpotent $p-$group for some prime p. (A group is called **locally graded** if every nontrivial finitely generated subgroup has a proper subgroup of finite index).

In this work it is shown that a locally nilpotent $p-$group which is a minimal non $CC-$group contains a proper epimorphic image which is a minimal non $FC-$group in which every proper normal subgroup is nilpotent of finite exponent and there is no element in such a group whose centralizer is nilpotent-by-Chernikov. Also Theorem 3 shows that in a perfect locally nilpotent $p-$group which is a minimal non $FC-$group, the hypercentral length and hypocentral length of proper subgroups are bounded.

The main results of this work are stated below.

Theorem 1. *Let G be a locally nilpotent $p-$group in which every proper subgroup is a $CC-$group. Then every proper subgroup of G/G^0 is an $FC-$group. Furthermore if G is perfect then it contains a proper normal subgroup K such that $Z(G)G^0 \leq K$ and every proper subgroup of G/K is nilpotent of finite exponent.*

(For a group X, X^0 denotes the unique maximal radicable abelian subgroup of X whenever it exists).

Corollary 1. *Let G be a locally nilpotent $p-$group which is a minimal non $CC-$group. Then G contains a proper normal subgroup K such that G/K is a minimal non $FC-$group and every proper normal subgroup of G/K is nilpotent of finite exponent.*

Proof. By the Corollary on p. 1232 of [9], G is perfect. Therefore the assertion follows from Theorem 1.

Theorem 2. *Let G be a locally nilpotent $p-$group which is a minimal non $CC-$group. Then $C_G(x)$ is not nilpotent-by-Chernikov (an$NC-$group) for any $x \in G$.*

Theorem 3. *Let G be a perfect locally nilpotent $p-$group which is a* *minimal non $FC-$group. Then for every proper subgroup X of G the following holds.*

(i) $Z_\omega(X) = X$.

(ii) $K_\omega(X) \leq Z(G)$.

(As usual for each ordinal α, $K_\alpha(X)$ and $Z_\alpha(X)$ denote respectively the α th term of the lower and upper central series of X).

2 Proof of the Theorems

Lemma 2.1. *Let A be a periodic $CC-$group and B be a normal abelian subgroup of A such that A/B is radicable abelian. Then A is abelian. If in addition B is radicable abelian then so is A.*

Proof. The first part of the lemma follows from Lemma 2 of [3] and the second part is trivial.

Note that if in the following Lemma H is a $CC-$group, then the conclusion follows from Lemma 1 of [3], but this result is not needed in the proof.

Lemma 2.2. *Let H be a locally nilpotent $p-$group in which every proper subgroup is a $CC-$group. Then H^0 exists and $(H/H^0)^0 = 1$.*

Proof. By (1.1)(3) of [9] every proper subgroup of H is hypercentral, which implies that every subgroup of H is ascendant in H. By Zorn's Lemma, H contains maximal radicable abelian subgroups. Let P and Q be two maximal radicable abelian subgroups. Then $[P, Q] = 1$ by Lemmas 3.2 and 3.4 of [8], since P and Q are ascendant in G. Hence PQ is abelian and so $P = Q$ by the choice of P and Q, which implies that H has a unique maximal radicable abelian subgroup; that is, H^0 exists.

Next let $T/H^0 = (H/H^0)^0$. Then T is radicable abelian by Lemma 2.1 and so $T = H^0$ by the maximality of H^0, which was to be shown.

Lemma 2.3. *Let H be a locally nilpotent $p-$group in which every proper subgroup is a $CC-$group. Then every proper subgroup of H/H^0 is an $FC-$group.*

Proof. Since $(H/H^0)^0 = 1$ by Lemma 2.2, we may suppose without loss of generality that $H^0 = 1$. Let K be any proper subgroup of H. Then K/K^0 is an FC–group by Lemma 1 of [3]. Moreover since H contains a unique maximal radicable abelian subgroup by Lemma 2.2, it follows that $K^0 = 1$ and so K is an FC–group which completes the proof of the Lemma.

Lemma 2.4. *Let H be a perfect locally nilpotent p–group which is a minimal non FC–group. Then H contains a proper normal subgroup K such that $Z(H) \leq K$, and every proper normal subgroup of H/K is nilpotent of finite exponent.*

Proof. By hypothesis $H = H'$, so without loss of generality we may syppose that $Z(H) = 1$. Let $1 \neq a \in H$ and put $C = C_H(a)$. Then $C \neq H$. Let N be a proper normal subgroup of H and put $D = C \cap N$. Then $[N : D]$ is finite, since $N < a >$ is an FC–group. Hence if

$$L = \bigcap_{x \in N} D^x$$

then L is normal in N and N/L is finite. Next let

$$Y = \bigcap_{h \in H} L^h.$$

Then Y is normal in H and N/Y is nilpotent of finite exponent, since it is embedded into the unrestricted direct product

$$\prod_{h \in H} (N/L^h)$$

where $N/L^h \cong N/L$ for all $h \in H$.

Finally let

$$K = \bigcap_{h \in H} C^h.$$

Then $K \neq H$ since $C \neq H$. Also $Y \leq K$ since $Y \leq L \leq D \leq C$. Therefore NK/K is nilpotent since N/Y is nilpotent. Since N is any proper normal subgroup of H, K is a desired subgroup of H.

Proof of Theorem 1. By Lemma 2.3 every proper subgroup of G/G^0 is an FC–group. Now suppose also that G is perfect. Then G/G^0 is also perfect and so it is a minimal non $FC-$group. Therefore by Lemma 2.4 it contains a proper normal subgroup K/G^0 such that $Z(G/G^0) \leq K/G^0$ and every proper normal subgroup of G/K is nilpotent of finite exponent. Obviously $Z(G)G^0 \leq K$. This complete the proof of the theorem.

Proof of Theorem 2. By the Corollary and by (1.1)(3) of [9] G is perfect and every proper subgroup of G is hypercentral. Assume that $C = C_G(a)$ is nilpotent-by-Chernikov ($NC-$group for short) for some $a \in G$. Then $C \neq G$ since G is perfect by hypothesis. First we show that every proper subgroup of G is an $NC-$group. So let X be a proper subgroup of G. Clearly G can be expressed as a union of an ascending chain of proper normal subgroups since it is perfect and locally nilpotent. Hence it follows that $a^G \neq G$. Then also $a^G X \neq G$ since G is perfect but a^G and X both are hypercentral. Put $L = a^G X$. Since L is a $CC-$group, $L/C_L(a^L)$ is Chernikov. Let $R = C_L(a^L)$. Since $R \leq C$, R has a normal nilpotent subgroup K such that R/K is Chernikov. By Lemma 4.7 (i) of [4] we may suppose that K is normal in L since R is normal in L. Also L/K is Chernikov since L/R and R/K are Chernikov. Hence it follows that L is an $NC-$group and then also X has the same property. Consequently it follows that every proper subgroup of G is both a $CC-$group and $NC-$group. But then G is an $NC-$group by the Corollary to Theorem B of [1], which is a contradiction since $G = G'$. This completes the proof of the theorem.

Proof of Theorem 3. (i) Let X be a proper subgroup of G. By hypothesis X is an $FC-$group. Therefore applying Theorem 4.38 of [10] yields that

$$Z_n(X) \leq X \leq Z_\omega(X)$$

For all $n \geq 1$, since X is an $FC-$group. Hence it follows that $X = Z_\omega(X)$.

To show the second assertion first suppose that $Z(G) = 1$. By Lemma 2.21 of [10]

$$[K_m(X),\, Z_m(X)] = 1$$

for all $m \geq 1$. Hence since $K_\omega(X) \leq K_m(X)$ it follows that

$$[K_\omega(X),\, Z_m(X)] = 1$$

for all $m \geq 1$. But since

$$X = \bigcup_{m=1}^{\infty} Z_m(X)$$

by the first part of the proof it follows that $K_\omega(X) \leq Z(X)$.

Next applying Lemma 6 of [2] repeatedly we can write G as

$$G = \bigcup_{i=1}^{\infty} X_i$$

where $X < X_i < X_{i+1}$ for all $i \geq 1$ since G is countably infinite by the Corollary on p.1232 of [9]. Also it follows from the first part of the proof that

$$K_\omega(X_i) \leq Z(X_i)$$

for all $i \geq 1$. But since $K_\omega(X) \leq K_\omega(X_i)$ it follows that

$$K_\omega(X) \leq Z(X_i)$$

for all $i \geq 1$ which yields that

$$K_\omega(X) \leq Z(G) = 1.$$

Now in the general case put $\bar{G} = G/Z(G)$. Then $Z(\bar{G}) = 1$ by hypothesis. Therefore $\overline{K_\omega(X)} = K_\omega(\bar{X}) = 1$ and hence $K_\omega(X) \leq Z(G)$ by the preceding paragraph. This completes the proof of the theorem.

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Recibido: 7 de Febrero de 1996