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On certain compact topological spaces.

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Abstract

A compact topological space K is in the class $\mathcal{A}$ if it is homeomorphic to a subspace H of $[0,1]^I$, for some set of indexes I, such that, if L is the subset of H consisting of all $\{x_i : i \in I\}$ with $x_i = 0$ except for a countable number of i's, then L is dense in H. In this paper we show that the class $\mathcal{A}$ of compact spaces is not stable under continuous maps. This solves a problem posed by Deville, Godefroy and Zizler.

If K is a compact space, $C(K)$ denotes the Banach space of all real continuous functions defined on K with the supremum norm. We denote by ω_0 the first infinite ordinal and ω_1 will be the first uncountable ordinal. We shall deal with the interval $[0,\omega_1]$, endowed with the order topology, which is a compact space. If I is a non-void set, by $\Sigma(I)$ we mean the subset of $[0,1]^I$ formed by those elements $\{x_i : i \in I\}$ with $x_i = 0$ except for a countable number of i's. A compact space K is of the class $\mathcal{A}$ if it is homeomorphic to a subspace H of $[0,1]^I$, for some set I, such that $H \cap \Sigma(I)$ is dense in H. When H itself is contained in $\Sigma(I)$, then K is said to be a Corson compact. The spaces $[0,\omega_1]$ and $[0,1]^I$, for an uncountable set I, are clearly compact spaces of the class $\mathcal{A}$ which are not Corson. Every Corson compact space K is angelic, i. e., since K is compact, this equals saying that: For every $A \subset K$ and x

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in the closure of A, then there is a sequence in A converging to x. Projectional resolutions of the identity operator are constructed for $C(K)$ and some of its subspaces, when K belong to the class $\mathcal{A}$, in [2] and [3].

In [1], the members of the class $\mathcal{A}$ are called Valdivia compacts and the following question is asked (Problem VII.2): *Is the class of Valdivia compact sets stable under continuous maps?* We shall give now a negative answer to this question.

Theorem. *There exists a compact space K satisfying the following conditions:*

1. *K is a continuous image of $[0, \omega_1]$.*
2. *K does not belong to the class $\mathcal{A}$.*
3. *$C(K)$ is isometric to a hyperplane of $C([0, \omega_1])$.*
4. *$C(K)$ is isomorphic to $C([0, \omega_1])$.*

Proof. Let K be equal to $[0, \omega_1[$ endowed with the following topology τ. If $\alpha \neq \omega_0$, then the neighborhoods of α are those of the order topology in $[0, \omega_1[$. A fundamental system of neighborhoods for ω_0 is given by the sets

$$\{]n, \omega_0] \cup]\alpha, \omega_1[: \alpha \in]\omega_0, \omega_1[, n = 1, 2, \cdots\}.$$

It is not difficult to see that K is compact and also that the map

$$\varphi : [0, \omega_1] \to K,$$

defined by $\varphi(\alpha) = \alpha$, if $\alpha < \omega_1, \varphi(\omega_1) = \omega_0$, is continuous.

Let us suppose that K belongs to the class $\mathcal{A}$. Let I be a non-empty set, H a compact subspace of $[0, 1]^I$ with $H \cap \Sigma(I)$ dense in H and ϕ a homeomorphism from K onto H. We define

$$D := \phi^{-1}(H \cap \Sigma(I)).$$

Hence, D is dense in K and it is also sequentially closed. If $\alpha \in K$, $\alpha \neq \omega_0$, there is a sequence in D converging to α. Consequently, $K \backslash \{\omega_0\} \subset D$. On the other hand, the sequence $(n)_{n=1}^{\infty}$ in K converges to ω_0 and so $D = K$. Thus $H \subset \Sigma(I)$ and K will be a Corson compact space in that case. If we take $A :=]\omega_0, \omega_1[$, ω_0 is in the closure of A and there will be a sequence α_n in A converging to ω_0. This sequence must converge to ω_1 in $[0, \omega_1]$ which is a contradiction. These facts conclude the proofs of 1 and 2. It is not difficult to show that $C(K)$ is isometric with the hyperplane of $C([0, \omega_1])$ formed by all the functions with the same value in ω_0 and ω_1. Finally, the set of functions in $C([0, \omega_1])$ which vanish at 0 is a hyperplane isometric to $C([1, \omega_1])$, then isometric to $C([0, \omega_1])$. Consequently, $C(K)$ is isomorphic to $C([0, \omega_1])$.

q.e.d.

Let us remark that $C(K)$, in the former theorem, is isomorphic to $C([0, \omega_1])$; nevertheless, these spaces are not isometric by Stone's theorem since K is not homeomorphic to $[0, \omega_1]$.

Note. Let us assume that K is a compact space in the class $\mathcal{A}$. Let I be a non-empty set, H a compact subspace of $[0, 1]^I$ with $H \cap \Sigma(I)$ dense in H and ϕ a homeomorphism from K onto H. Let us write

$$D := \phi^{-1}(H \cap \Sigma(I)), \ L := \phi^{-1}(H \backslash \Sigma(I)).$$

If L is a non-empty closed set, the ideas of the proof of our theorem can be used to find a continuous image of K which is not in the class $\mathcal{A}$. Indeed, let us denote by $\mathcal{U}$ the family of neighborhoods of L, and for every $x \in K$, let $\mathcal{U}_x$ represent the family of neighborhoods of x. We take $x_0 \in D$ and we call M the set D endowed with the following topology: If $x \in D$, $x \neq x_0$, the neighborhoods of x are the intersections of members of $\mathcal{U}_x$ with D; the neighborhoods of x_0 are the sets

$$(U \cap D) \cup (V \cap D), \ U \in \mathcal{U}, \ V \in \mathcal{U}_{x_0}.$$

Then, M is a compact space that is not in the class $\mathcal{A}$, but the mapping $\varphi : K \to M$ defined as $\varphi(x) = x_0$, if $x \in L$, $\varphi(x) = x$, if $x \in D$, is onto and continuous.

The following question comes out naturally

Open Question. *Does there exist a compact space of the class $\mathcal{A}$ whose continuous images still remain in this class and such that it is not Corson?*

References

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