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The Artin conjecture for Q-algebras.

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Abstract

We give a simplification, in the case of **Q**-algebras, of the proof of Artin's Conjecture, which says that a regular morphism between noetherian rings is the inductive limit of smooth morphisms of finite type.

Introduction

This article is devoted to the proof of Artin Conjecture for **Q**-algebras which can be formulated as follow :

Artin Conjecture *Let* $A \xrightarrow{\phi} B$ *be a regular morphism between noetherian rings, where we assume that* A *contains* **Q**. *Then, the ring* B *is a filtered inductive limit of smooth* A*-algebras of finite type.*

The Artin Conjecture is also called Generalised Neron desingularization or Appproximation of regular morphisms. The proof of the Artin Conjecture is based on ideas of Popescu [Po] and completed by André [An1]. We will follow this proof. The conjecture had also been proved by Spivakovsky [Sp] using another method, containing some geometrical ideas, such as blowing up.

In these references, the Artin Conjecture is proved without the assumption that A contains **Q**. To deal with this desingularization problem in general characteristic, the André Homology [An2] seems to

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be the appropriate tool. However, it appears to be quite natural, for **Q**-algebras, to give a proof without using this André Homology formalism. This is our main goal : to simplify André 's proof for **Q**-algebras.

The Artin Conjecture is a powerful tool in commutative algebra which admits a lot of applications, some of them are described in [Te].

It is formally equivalent to the following formulation :

Artin Conjecture (2nd form) *Consider a commutative diagram of noetherian rings* $\begin{array}{ccc} A & \xrightarrow{\phi} & B \\ \downarrow & \nearrow & \\ C & & \end{array}$ *, where* ϕ *is regular and* C *of finite type over* A*. Assume that* A *contains* $\mathbf{Q}$*. Then, there exists a factorization* $\begin{array}{ccc} A & \xrightarrow{\phi} & B \\ \downarrow & \nearrow & \uparrow \\ C & \rightarrow & D \end{array}$ *, where* D *is smooth of finite type over* A*.*

We will work with this form of the conjecture, the factorization through D is called a desingularization or smoothing of C.

The section 1 recalls the definitions of smoothness and regularity for a morphism. We also introduce the singular locus of C over A which is the set of all prime ideals of C where C is not smooth over A. Then, the ideal of the singular locus is defined to be the intersection of all these prime ideals. We denote it by $\mathcal{H}_{C/A}$.

In section 2 we show, among other things, that if $\sqrt{\mathcal{H}_{C/A}B} = B$ (i.e. if the inverse image of the ideal of the singular locus of C over A in $\operatorname{Spec} B$ is empty), then we have a smoothing of C.

In section 3 we explain the main lines of the proof. The idea is to make the ideal $\sqrt{\mathcal{H}_{C/A}B}$ increase, until we finally arrive to B. For this, we reduce to the case where $\sqrt{\mathcal{H}_{C/A}B}$ is a prime ideal $\mathfrak{q}$. Set $\mathfrak{p} = \mathfrak{q} \cap A$.

In section 4 we treat the basic case where $\operatorname{ht} \mathfrak{p} = \operatorname{ht} \mathfrak{q} = 0$. In section 5 we explain the reduction to this basic case, by means of a lifting property of the smoothing. This last section contains the most technical part of the proof.

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1 Preliminaries

First, recall some classical notations. We denote by $\operatorname{Spec} B$ the set of all the prime ideals of B. For any $\mathfrak{p} \in \operatorname{Spec} B$, $k(\mathfrak{p})$ is the residue field of B at $\mathfrak{p}$ and $\operatorname{ht} \mathfrak{p}$ is the height of $\mathfrak{p}$ in B. Let $\phi : A \to B$ be a ring homomorphism. Let I be an ideal of A, J an ideal of B and $\mathfrak{q}$ be a prime ideal of B. Then, $IB = \phi(I)B$, $J \cap A = \phi^{-1}(J)$ and $A_{\mathfrak{q}} = A_{\phi^{-1}(\mathfrak{q})}$.

In this section, we introduce the classical algebraic results about smoothness and regularity of morphisms. For these notions we will refer to [M1] and [G1].

1.1 Smoothness and regularity for a ring homomorphism

Definition 1.1. (Formal smoothness). *Let $A \xrightarrow{\phi} B$ be a ring homomorphism and I an ideal of B. The ring B is* formally smooth *over A* for the *I*-adic topology *if for any A-algebra C and any nilpotent ideal N of C, each continous morphism $B \xrightarrow{u} C/N$ (for the discrete topology on C/N and the I-adic topology on B, i.e. $u(I^e) = 0$ for some integer e) can be lifted to a morphism $B \xrightarrow{v} C$. That is, we have a commutative diagram :*

$$\begin{array}{ccc} B & \xrightarrow{u} & C/N \\ \uparrow & \overset{v}{\searrow} & \uparrow \\ A & \longrightarrow & C \end{array}$$

*We also say that B is I-*smooth *over A. If $I = 0$ then we say that B is* smooth *over A (for the discrete topology). If the lifting v is unique, then we say that B is I-*etale *over A.*

Definition 1.2. (Regularity). *Let k be a field and B a noetherian k-algebra. The ring B is* geometrically regular *over k if for any finite field extension $k \to k'$, the ring $B \bigotimes_k k'$ is regular (it is regular at each prime ideal). The morphism $A \xrightarrow{\phi} B$ is* regular *if ϕ is flat and for each prime ideal $\mathfrak{p}$ of A, $B \bigotimes_A k(\mathfrak{p})$ is geometrically regular over $k(\mathfrak{p})$.*

Let $\mathfrak{q} \in \operatorname{Spec} B$ and $\mathfrak{p} = \mathfrak{q} \cap A$. The morphism $A \xrightarrow{\phi} B$ is regular at $\mathfrak{q}$, if $B_{\mathfrak{q}}$ is flat over $A_{\mathfrak{p}}$ and the ring $B_{\mathfrak{q}} \bigotimes_{A_{\mathfrak{p}}} k(\mathfrak{p})$ is geometrically regular over $k(\mathfrak{p})$.

Remark 1.3. The notion of a geometrically regular ring is local. Namely, the ring B is geometrically regular over k if and only if, for any prime ideal $\mathfrak{q}$ of B, $B_\mathfrak{q}$ is geometrically regular over k.

In the same way, the morphism ϕ is regular if and only if ϕ is regular at each prime ideal $\mathfrak{q}$ of B.

When the rings $A_\mathfrak{p}$ and $B_\mathfrak{q}$ are artinian ($\operatorname{ht}\mathfrak{q} = \operatorname{ht}\mathfrak{p} = 0$), it is clear that ϕ is regular at $\mathfrak{q}$ if and only if the morphism $\phi_\mathfrak{q} : A_\mathfrak{p} \to B_\mathfrak{q}$ is regular.
Beware that it is not true in general. Indeed, Nagata showed [Na, Appendix E7.1] the existence of a noetherian normal local $\mathbf{Q}$-algebra R, whose completion $\widehat{R}$ is not a domain. In this case, if the morphism $R \to \widehat{R}$ were regular then $\widehat{R}$ should be normal by [M1, Th 32.2]. Which is not possible since $\widehat{R}$ is not a local domain.
We may also mention nevertheless that this property is true if we assume that the $\mathbf{Q}$-algebras A and B are excellent. Then by [G2, 7.9.8], the morphism $A \to B$ is regular at $\mathfrak{q}$ if and only if $\phi_\mathfrak{q} : A_\mathfrak{p} \to B_\mathfrak{q}$ is regular.

The definition 1.2 has a simple formulation for $\mathbf{Q}$-algebras since we do not have to deal with field extensions :

Proposition 1.4. *Let B be a noetherian algebra over a field k containing $\mathbf{Q}$. Then, B is geometrically regular over k if it is a regular ring.*

The proof of the Proposition is immediate by using the two lemmas [M1, 28.Lemma 1] and [M1, 28.7].

1.2 Relations between smooth and regular morphisms

We will see that the two notions of regular and smooth morphisms coincide under some finiteness assumption.

Proposition 1.5. *Let ϕ be a morphism of noetherian rings $A \xrightarrow{\phi} B$. The morphism ϕ is regular at $\mathfrak{q}$ if and only if the morphism $A \xrightarrow{\phi_\mathfrak{q}} B_\mathfrak{q}$ is formally smooth for the $\mathfrak{q}$-adic topology. Then, the morphism ϕ is regular if and only if the morphism $A \xrightarrow{\phi_\mathfrak{q}} B_\mathfrak{q}$ is $\mathfrak{q}$-smooth for each prime ideal $\mathfrak{q}$ of B.*

The proposition follows from [M1, 28.7] and the difficult theorem [G1, 19.7.1] :

Lemma 1.6.([M1, 28.7]). *Let $(A, \mathfrak{m}, k)$ be a noetherian local ring containing a field L. Then, A is $\mathfrak{m}$-smooth over L if and only if A is geometrically regular over L.*

Theorem 1.7.([G1, 19.7.1]). *Let $(A, \mathfrak{m}, k)$ and $(B, \mathfrak{n}, K)$ be two noetherian local rings and $\phi : A \to B$ be a local homomorphism. Let $B_0 = B \bigotimes_A k = B/\mathfrak{m}B$ and $\mathfrak{n}_0 = \mathfrak{n}/\mathfrak{m}B$. Then, B is $\mathfrak{n}$-smooth over A if and only if B is flat over A and B_0 is $\mathfrak{n}_0$-smooth over k.*

If we assume that B is a finite type A-algebra then we get the following :

Proposition ([G1, 22.6.6]). *Let A be a noetherian ring and B a finite type A-algebra. Then we have the equivalences :*

- *B is smooth over A*
- *For each prime ideal $\mathfrak{q}$ of B, $B_\mathfrak{q}$ is smooth over A*
- *For each prime ideal $\mathfrak{q}$ of B, $B_\mathfrak{q}$ is $\mathfrak{q}$-smooth over A*

We can summarize the previous results into the following proposition :

Proposition 1.8. *Let $\phi : A \to B$ be a morphism of noetherian rings. If B is smooth over A then ϕ is regular.*

Assume that B is a finite type A-algebra. Then B is smooth over A if and only if ϕ is regular.

Smoothness criterion using the module of differentials

Let A be a noetherian ring and B an A-algebra. Consider a presentation $B \simeq F/I$ with F a polynomial ring $F = A[u_i]_{i \in J}$. We recall the second fondamental exact sequence ([M1, Th 25.2]) :

$$I/I^2 \to \Omega^1_{F/A} \bigotimes_F B \to \Omega^1_{B/A} \to 0$$

Denote by $\mathcal{H}_1(A, B)$ the kernel of the first arrow. This is justified by the fact that this B-module does not depend on the choice of F. It is actually the B-module $\mathcal{H}_1(A, B, B)$ in the theory of André Homology of commutative algebras defined in [An2].

We deduce from [G1, 22.6.3] :

Proposition 1.9. *Let $A \xrightarrow{\phi} B$ be a morphism of noetherian rings. The ring B is smooth over A if and only if $\mathcal{H}_1(A, B) = 0$ and $\Omega^1_{B/A}$ is a projective B-module.*

Remark. By using the André Homology, we can also prove that the ring B is regular over A if and only if $\mathcal{H}_1(A, B) = 0$ and $\Omega^1_{B/A}$ is a flat B-module.

Indeed, a projective module is always flat by [M1, Th 2.5]. Conversely, a finitely presented flat module is projective by [M1, Corollary of Th 7.12]. Since B is a finite type A-algebra and B is noetherian, we deduce that $\Omega^1_{B/A}$ is a finitely presented B-module. Hence, we find that if the morphism $A \xrightarrow{\phi} B$ is smooth then it is regular. Furthermore, if B is a finite type A-algebra then ϕ is smooth if and only if it is regular. Which is an improvement of Proposition 1.8.

The Jacobian criterion

When A is a noetherian ring and B is a finite type A-algebra, we have another smoothness criterion, which is the Jacobian criterion (which is again deduced from the smoothness criterion using the module of differentials). Consider the presentation $B = A[u_1, \ldots, u_n]/I$ with $I = (f_1, \ldots, f_m)$. The Jacobian criterion of smoothness [G1, 22.6.3] says that :

The A-algebra B is smooth at the prime ideal $\mathfrak{p}$ of B if and only if there exist some subsets $(g_1, \ldots, g_r)$ of $(f_1, \ldots, f_m)$ and $(v_1, \ldots, v_r)$ of $(u_1, \ldots, u_n)$ with $r \leq n$ such that :

$$\begin{cases} \det(\partial g_i/\partial v_j)_{1\leq i,j\leq r} \notin \mathfrak{p}, \\ B_\mathfrak{p} = ((g) : I)_\mathfrak{p} \text{ where } ((g) : I) = \{\alpha \in A[u_1, \ldots, u_n] \mid \alpha I \subset (g)\}. \end{cases}$$

1.3 The singular locus of an algebra

Let A be a noetherian ring and C be a finite type A-algebra. Consider the presentation $C = A[u_1, \ldots, u_n]/I$ with $I = (f_1, \ldots, f_m)$.

We write Δ_g for the ideal of $A[u]$ generated by the $r \times r$ minors of the Jacobian matrix $(\partial g_i/\partial v_j)_{1\leq i\leq r, 1\leq j\leq n}$ for $g = (g_1, \ldots, g_r) \subset I$. The *singular locus* of the ring C over A is the set of all prime ideals $\mathfrak{p}$ of C such that C is not smooth at $\mathfrak{p}$ over A. We define the *ideal of the singular locus* of C over A to be $\mathcal{H}_{C/A} = \sqrt{\sum_{\{g\}\subset\{f\}} \Delta_g((g):I)C}$. The main advantage of the ideal of the singular locus is that the relative dimension of the jacobian criterion does not appear any more. The Jacobian criterion takes the following form :

Proposition 1.10. *Let A be a noetherian ring and C a finite type A-algebra. Let $\mathfrak{p}$ be a prime ideal of C. The ring $C_\mathfrak{p}$ is smooth over A if and only if $\mathcal{H}_{C/A} \not\subset \mathfrak{p}$. Let $x \in C$, equivalently we have C_x is smooth over A if and only if $x \in \mathcal{H}_{C/A}$.*

The ideal of the singular locus $\mathcal{H}_{C/A}$ does not depend on the choice of the presentation of C over A.

All the classical results on smoothness can be formulated using the ideal of the singular locus. We state below the examples of composition and base change formulas for smoothness.

Proposition 1.11. (Composition). *Let A be a noetherian ring, B and C be two finite type A-algebras. Suppose we have two morphisms $A \to C \to B$. Then $\sqrt{\mathcal{H}_{B/C}\mathcal{H}_{C/A}B} = \sqrt{\mathcal{H}_{B/C}\mathcal{H}_{B/A}}$ or equivalently $\mathcal{H}_{B/C} \cap \sqrt{\mathcal{H}_{C/A}B} = \mathcal{H}_{B/C} \cap \mathcal{H}_{B/A}$.*

Proof. Let $\mathfrak{p}$ be a prime ideal of B. We have the following equivalences :

$$\begin{aligned}
\mathcal{H}_{B/C}\mathcal{H}_{C/A}B \not\subset \mathfrak{p} &\iff \mathcal{H}_{B/C} \not\subset \mathfrak{p} \text{ and } \mathcal{H}_{C/A}B \not\subset \mathfrak{p} \\
&\iff B_\mathfrak{p} \text{ smooth over } C \text{ and } C_\mathfrak{p} \text{ smooth over } A \\
&\iff B_\mathfrak{p} \text{ smooth over } C \text{ and } B_\mathfrak{p} \text{ smooth over } A \\
&\qquad \text{[M2, 33.B Lemma 1]} \\
&\iff \mathcal{H}_{B/C} \not\subset \mathfrak{p} \text{ and } \mathcal{H}_{B/A} \not\subset \mathfrak{p} \\
&\iff \mathcal{H}_{B/C}\mathcal{H}_{B/A} \not\subset \mathfrak{p}
\end{aligned}$$

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Proposition 1.12. (Base change). *Let A be a noetherian ring, B and C be two finite type A-algebras. Let $D = B \bigotimes_A C$. Then we have $\sqrt{\mathcal{H}_{B/A}D} \subset \mathcal{H}_{D/C}$.*

Proof. Let $\mathfrak{q}$ be a prime ideal of D and $\mathfrak{p} = \mathfrak{q} \cap B$. If $B_{\mathfrak{p}}$ is smooth over A, then $B_{\mathfrak{p}} \bigotimes_{A_{\mathfrak{p}}} C_{\mathfrak{q}}$ is smooth over $C_{\mathfrak{q}}$ and so is $D_{\mathfrak{q}}$ which is a localization of $B_{\mathfrak{p}} \bigotimes_{A_{\mathfrak{p}}} C_{\mathfrak{q}}$.

■

We explicitly mention some important consequences we will need in the following .

Lemma 1.13. *Let A be a noetherian ring and B be a finite type A-algebra.*

- *Let I be an ideal of A, $\bar{A} = A/I$ and $\bar{B} = B/IB = B \bigotimes_A A/I$. Then we have $\sqrt{\mathcal{H}_{B/A}\bar{B}} \subset \mathcal{H}_{\bar{B}/\bar{A}}$.*
- *Let $A' = A[u_1, \ldots, u_n]$ and $B' = B[u_1, \ldots, u_n] = B \bigotimes_A A'$. Then we have the equality $\sqrt{\mathcal{H}_{B/A}B'} = \mathcal{H}_{B'/A'}$. (More generaly, this equality is true whenever A' is flat over A by [An2, Prop 15.18] or [An1, I Lemme 30]).*

Proof. We only have to prove the non obvious inclusion of the second equality. The morphism $A \to A'$ is smooth, so is the morphism $B \to B'$ by base change. Thus, we have $\mathcal{H}_{A'/A} = A'$ and $\mathcal{H}_{B'/B} = B'$. We apply Proposition 1.11 to the sequence $A \to B \to B'$ to get $\mathcal{H}_{B'/B} \cap \sqrt{\mathcal{H}_{B/A}B'} = \mathcal{H}_{B'/B} \cap \mathcal{H}_{B'/A}$, and hence $\sqrt{\mathcal{H}_{B/A}B'} = \mathcal{H}_{B'/A}$. Again, we apply Proposition 1.11 to the sequence $A \to A' \to B'$ to get $\mathcal{H}_{B'/A'} \cap \sqrt{\mathcal{H}_{A'/A}B'} = \mathcal{H}_{B'/A'} \cap \mathcal{H}_{B'/A}$ which gives $\mathcal{H}_{B'/A'} \subset \mathcal{H}_{B'/A}$. Combining the two relations, we deduce $\mathcal{H}_{B'/A'} \subset \sqrt{\mathcal{H}_{B/A}B'}$.

■

1.4 Local criterion of flatness

Flatness is a prerequisite for smoothness or regularity, hence we will need the following forms of the local criterion of flatness which are corollaries of [M1, 22.3].

Proposition 1.14. *Let $A \to B \to C$ be local morphisms of noetherian local rings. Assume that B is flat over A, and let k be the residue field* *of A. Then, C is flat over B if and only if C is flat over A and $C \bigotimes_A k$ is flat over $B \bigotimes_A k$.*

Recall that an element x of a ring A is A-regular in A if it is not a zero divisor. Moreover, a sequence $(x_1, \ldots, x_n)$ of elements of A is said to be A-regular if x_1 is A-regular and if for any $i \in \{2 \ldots n\}$, x_i is $A/(x_1, \ldots, x_{i-1})$-regular, and $A/(x_1, \ldots, x_n) \neq 0$. We have :

Propostion 1.15. *Let $(A, \mathfrak{m})$ be a noetherian local ring containing a field k and $(x_1, \ldots, x_n)$ be an A-regular sequence in $\mathfrak{m}$. Then, the subring $k[x_1, \ldots, x_n]$ of A is isomorphic to the free polynomial ring $k[X_1, \ldots, X_n]$ and A is flat over $k[x_1, \ldots, x_n]$.*

2 Elementary smoothing, standardization

Consider the commutative diagram of noetherian rings $\begin{array}{ccc} A & \xrightarrow{\phi} & B \\ \downarrow & \nearrow & \\ C & & \end{array}$, where C is of finite type over A. In this section, we construct a factorization which is smooth at the elements of C whose images in B are in $\sqrt{\mathcal{H}_{C/A}B}$.

Propostion 2.1. *Consider the commutative diagram of noetherian rings $\begin{array}{ccc} A & \xrightarrow{\phi} & B \\ \downarrow & \nearrow & \\ C & & \end{array}$, where C is of finite type over A. Let x be an element of C whose image in B is in $\sqrt{\mathcal{H}_{C/A}B}$. Then, there exists a factorization $\begin{array}{ccc} A & \xrightarrow{\phi} & B \\ \downarrow & \nearrow & \uparrow \\ C & \rightarrow & D \end{array}$, where D is of finite type over A, $\mathcal{H}_{C/A}D \subset \mathcal{H}_{D/C}$ and the image of x in D is an element of $\mathcal{H}_{D/A}$.*

Proof. Let $c = (c_1, \ldots, c_s)$ be a system of generators of $\mathcal{H}_{C/A}$. There exists an integer N such that in B, $x^N = \sum_{1 \leq i \leq s} \phi(c_i) z_i$ with $z_i \in B$. Set

$$D = C[Z_1, \ldots, Z_s]/(x^N - \sum_{1 \leq i \leq s} c_i Z_i)$$

and send Z_i onto z_i to define a morphism $D \rightarrow B$ which factorizes the diagram. Moreover, for any i , $c_i \in \mathcal{H}_{D/C}$ since $\partial/\partial Z_i(x^N -$

$\sum_{1\leq i\leq s} c_i Z_i) = -c_i$ which is invertible in D_{c_i}. Thus, $\mathcal{H}_{C/A}D \subset \mathcal{H}_{D/C}$. Since $x^N = \sum_{1\leq i\leq s} c_i Z_i$ in D, we may apply Proposition 1.11 to the sequence $A \to C \to D$ and conclude that the image of x in D is in $\mathcal{H}_{D/A}$.

■

If we take $x = 1$ in the previous Proposition, we get :

Proposition 2.2. *Consider the commutative diagram of noetherian rings*

$$\begin{array}{ccc} A & \xrightarrow{\phi} & B \\ \downarrow & \nearrow & \\ C & & \end{array}$$

where C is of finite type over A. Assume that $\sqrt{\mathcal{H}_{C/A}B} = B$. Then, there exists a factorization

$$\begin{array}{ccc} A & \xrightarrow{\phi} & B \\ \downarrow & \nearrow & \uparrow \\ C & \rightarrow & D \end{array}$$

, where D is smooth of finite type over A.

This Proposition is the expression of the Artin Conjecture when the inverse image of the singular locus of C over A is empty in $\operatorname{Spec} B$, i.e. $\mathcal{H}_{C/A}B = B$. The Proof of the Conjecture will consist in shrinking the image in $\operatorname{Spec} B$ of the singular locus of C over A in order it becomes empty, and then to apply previous Proposition 2.2. The smothing in this degenerated case will be called *elementary smoothing.*

Now, we introduce the notion of standard elements which are much easier to deal with than general elements of the ideal of the singular locus.

Definition 2.3. *Let A be a noetherian ring. Let C be a finite A-algebra given by the presentation $C = A[u_1, \ldots, u_n]/I$. An element x of $\mathcal{H}_{C/A}$ is said to be* standard for this presentation, *if there exists $(g) = (g_1, \ldots, g_r) \subset I$ such that $x \in \sqrt{\Delta_g((g) : I)C}$. The A-algebra C is said to be* standard smooth *over A if 1 is standard for some presentation of C.*

A multiple of a standard element is still standard. The main tool to "force" an element to become standard is the Elkik Lemma [El, Lemme 3] which uses the symmetric algebra of a module.

Lemma 2.4. *Let A be a ring and M be an A-module. We denote by S_AM the symmetric algebra of M over A. Then, we have $\Omega^1_{S_AM/A} \simeq$*

$M \bigotimes_A S_A M$. Moreover, M is a projective A-module if and only if $S_A M$ is smooth over A.

Proof. For any $S_A M$-module P, we have the isomorphisms :

$$\mathrm{Hom}_{S_A M}(\Omega^1_{S_A M/A}, P) \simeq \mathrm{Der}_A(S_A M, P) \simeq \mathrm{Hom}_A(M, P) \simeq \mathrm{Hom}_{S_A M}(M \bigotimes_A S_A M, P)$$

Then we get $\Omega^1_{S_A M/A} \simeq M \bigotimes_A S_A M$.

Now, suppose M is projective and consider the commutative diagram

$$\begin{array}{ccc} S_A M & \xrightarrow{u} & C/I \\ \uparrow & & \uparrow \\ A & \rightarrow & C \end{array} .$$

We identify M with the degree one component of $S_A M$, and we denote by $u_1 : M \rightarrow C/I$ the restriction of u. Since M is projective, u_1 factorises through $M \xrightarrow{v_1} C \rightarrow C/I$. Hence, v_1 extends to a homomorphism of A-algebras $v : S_A M \rightarrow C$, which makes the diagramm commute. The infinitesimal lifting property of smoothness 1.1 is satisfied, and hence $S_A M$ is smooth over A.

Conversely, let $I = \bigoplus_{i=1}^{\infty} S^i_A M$ be the ideal of positive degree elements of $S_A M$. The second fundamental exact sequence deduced from $A \rightarrow S_A M \rightarrow S_A M/I$ is :

$$I/I^2 \rightarrow \Omega^1_{S_A M/A} \bigotimes_{S_A M} S_A M/I \rightarrow \Omega^1_{(S_A M/I)/A} \rightarrow 0$$

This exact sequence splits since $S_A M/I \simeq A$ is smooth over A. Thus, $I/I^2 \simeq M$ is a direct factor of the A-locally free module $\Omega^1_{S_A M/A} \bigotimes_{S_A M} S_A M/I$. Then M is projective.

∎

Proposition 2.5. *Consider the commutative diagram of noetherian rings* $\begin{array}{ccc} A & \xrightarrow{\phi} & B \\ \downarrow & \nearrow & \\ C & & \end{array}$ *, where C is of finite type over A. Then, there exists a factorization* $\begin{array}{ccc} A & \xrightarrow{\phi} & B \\ \downarrow & \nearrow & \uparrow \\ C & \rightarrow & D \end{array}$ *, where D is a finite type A-algebra,*

$\mathcal{H}_{C/A}D \subset \mathcal{H}_{D/C}$, and we can find a presentation of D over A such that the images in D of all the elements of $\mathcal{H}_{C/A}$ are standard.

Proof.

- We start with the presentation $C = A[u]/(f)$ where $u = (u_1, \ldots, u_n)$ and $f = (f_1, \ldots, f_k)$. We consider the symmetric algebra $D = S_C((f)/(f)^2)$. The homomorphism $C \to B$ factorizes through D by sending the elements of positive degree to 0 in B. Let $x \in \mathcal{H}_{C/A}$. The ring C_x is smooth over A, so the conormal bundle $(f)/(f)^2 \otimes_C C_x$ is locally free over $\operatorname{Spec} C_x$, and by [M1, Corollary of theorem 7.12], it is a projective C_x-module. Then the ring D_x is smooth over C_x using 2.4. Thus, the image of x in D belongs to $\mathcal{H}_{D/C}$.

- Let $z = (z_1, \ldots, z_k)$. Since the morphism $A[u]/(f)[z] \to D$ which map $(z_1, \ldots, z_k)$ to $f = (f_1, \ldots, f_k)$ is surjective, we have a presentation of D of the form $D = A[u, z]/(f, h)$. We change this presentation, by adding new variables $v = (v_1, \ldots, v_n)$ to get $D = A[u, z, v]/I$ with $I = (f, h, v)$ and we claim that the new conormal bundle $I/I^2 \otimes_D D_x$ is globally free over $\operatorname{Spec} D_x$.

- First we show that $\Omega^1_{D_x/A} \simeq D_x^n$. Since C_x is smooth over A we get a split exact sequence of C_x-modules (second fondamental exact sequence) :

$$0 \to (f)/(f)^2 \bigotimes_C C_x \to \Omega^1_{A[u]/A} \bigotimes_C C_x \to \Omega^1_{C/A} \bigotimes_C C_x \to 0$$

  Furthermore, $\Omega^1_{A[u]/A} \otimes_C C_x \simeq C_x^n$, which leads to the relation : $C_x^n \simeq ((f)/(f)^2 \bigoplus \Omega^1_{C/A}) \otimes_C C_x$. Moreover, the smoothness of D_x over C_x gives a split exact sequence and a relation $\Omega^1_{D_x/A} \simeq (\Omega^1_{C_x/A} \otimes_{C_x} D_x) \bigoplus \Omega^1_{D_x/C_x}$ (first fundamental exact sequence [M1, Th. 25.1]). On the other hand, the canonical surjection $(f)/(f)^2 \otimes_C D_x \to \Omega^1_{D_x/C_x}$ is locally injective, since $(f)/(f)^2 \otimes_C C_x$ is locally free, and we conclude $(f)/(f)^2 \otimes_{C_x} D_x \simeq \Omega^1_{D_x/C_x}$. Thus, we get $\Omega^1_{D_x/A} \simeq (\Omega^1_{C_x/A} \bigoplus (f)/(f)^2) \otimes_{C_x} D_x \simeq D_x^n$.

- Since D_x is smooth over A by transitivity, we get another split exact sequence and a relation $D_x^{n+k} \simeq ((f, h)/(f, h)^2 \bigoplus \Omega^1_{D/A}) \otimes_D D_x$.

- Since $I/I^2 = (f,h)/(f,h)^2 \bigoplus D^n$ we get :

$$I/I^2 \bigotimes_D D_x = ((f,h)/(f,h)^2 \bigotimes_D D_x) \bigoplus D_x^n \simeq$$
$$((f,h)/(f,h)^2 \bigotimes_D D_x) \bigoplus \Omega^1_{D_x/A} \simeq D_x^{n+k}$$

Which complete the fact that the new conormal bundle $I/I^2 \otimes_D D_x$ is globally free over $\operatorname{Spec} D_x$.

- Let F be a lifting of x in $A[u]$. The module $I_F/I_F^2 \simeq I/I^2 \otimes_D D_x$ is a free $D_x \simeq (A[u,z,v]/I)_F$-module. Let $(g_1,\dots,g_r)$ be a system of polynomials of I which induces a basis of I_F/I_F^2. We have $I_F = (g_1,\dots,g_r)A[u,z,v]_F + I_F^2$. Then, by Nakayama's Lemma ([M1, th. 2.2]), there exists an element $a \in 1 + I_F$ such that $aI_F \subset (g_1,\dots,g_r)A[u,z,v]_F$. So, there exists an integer t such that $F^t aI \subset (g)$, and then the image of x in D is in $\sqrt{((g):I)D}$. Moreover, $F \in \sqrt{\Delta_g}$ since D_x is smooth over A. Thus, the image of x in D is in $\sqrt{\Delta_g((g):I)D}$ and hence is standard.

Since the construction of the presentation of D does not depend on x, it is true for any element of $\mathcal{H}_{C/A}$.

■

Proposition 2.5 admits the immediate corollary which is an improvement of the Artin Conjecture. Namely, we get a smoothing by a smooth complete intersection algebra.

Corollary 2.6. *Consider the commutative diagram of noetherian rings*

$$\begin{array}{ccc} A & \xrightarrow{\phi} & B \\ \downarrow & \nearrow & \\ C & & \end{array}$$

, where C is smooth of finite type over A. Then, there exists a factorization

$$\begin{array}{ccc} A & \xrightarrow{\phi} & B \\ \downarrow & \nearrow & \uparrow \\ C & \rightarrow & D \end{array}$$

, where D is a smooth complete intersection *over A, i.e. we can find a presentation $D = A[Y_1,\dots,Y_p]/(F_1,\dots,F_m)$ with $m \le p$ such that the rank of the Jacobian matrix $(\partial F_i/\partial Y_j)$ is m everywhere on* $\operatorname{Spec} D$.

Proof. If we take $x = 1$ in Proposition 2.5, we have a desingularisation by a standard smooth A-algebra D. This means that we have a presentation $D = A[u_1, \ldots, u_n]/I$ and $(g) = (g_1, \ldots, g_m) \subset I$ satisfying $\Delta_g((g) : I)D = D$. Then $((g) : I)D = D$ and there exists $a \in 1 + I$ such that $aI \subset (g)$. The new presentation $D \simeq A[u_1, \ldots, u_n, v]/(g_1, \ldots, g_m, av-1)$ shows that D is a complete intersection over A.

■

In sections 3, 4 and 5, all the rings we will consider are noetherian rings containing Q.

3 Main ideas of the proof of Artin Conjecture

In this section, we explain the main steps of the proof and use the results of sections 4 and 5.

3.1 Central result

Let us formulate again the Artin Conjecture.

Theorem 3.1. **(Artin Conjecture).** *Consider a commutative diagram* $\begin{array}{ccc} A & \xrightarrow{\phi} & B \\ \downarrow & \nearrow & \\ C & & \end{array}$ *, where ϕ is regular and C of finite type over A. Then, there exists a factorization* $\begin{array}{ccc} A & \xrightarrow{\phi} & B \\ \downarrow & \nearrow & \uparrow \\ C & \rightarrow & D \end{array}$ *where D is smooth of finite type over A.*

In all the following, $\phi : A \to C \to B$ will denote two ring homomorphisms between noetherian rings, and C a finite type A-algebra. Let $J = \mathcal{H}_{C/A}B$ and $I = J \cap A$.

The proof consists in successive smoothings of C so that the image of the ideal of the singular locus in B increases. Since B is noetherian, it reaches the whole ring B after a finite number of smoothings. Then we conclude with Proposition 2.2.

We first reduce to a local situation, namely to the case of an "isolated singularity".

Definition 3.2. *Let a commutative diagram* $\begin{array}{ccc} A & \xrightarrow{\phi} & B \\ \downarrow & \nearrow & \\ C & & \end{array}$, *where* C *is of finite type over* A. *Let* $\mathfrak{q} \in \operatorname{Spec} B$.

The ring D *is called a* partial smoothing at $\mathfrak{q}$ *of* $A \to C \to B$, *if there exists a factorization* $\begin{array}{ccc} A & \xrightarrow{\phi} & B \\ \downarrow & \nearrow & \uparrow \\ C & \rightarrow & D \end{array}$, *where* D *is of finite type over* A *and* $\sqrt{\mathcal{H}_{C/A}B} \subset \sqrt{\mathcal{H}_{D/A}B} \not\subset \mathfrak{q}$.

Proposition 3.3. (Partial smoothing). *Consider a commutative diagram* $\begin{array}{ccc} A & \xrightarrow{\phi} & B \\ \downarrow & \nearrow & \\ C & & \end{array}$, *where* C *is of finite type over* A. *Let* $\mathfrak{q} \in \operatorname{Spec} B$, $\mathfrak{p} = \mathfrak{q} \cap A$. *Assume that* ϕ *is regular at* $\mathfrak{q}$, *i.e.* $B_\mathfrak{q}$ *is* $\mathfrak{q}$*-smooth over* $A_\mathfrak{p}$. *Assume moreover that* $\mathfrak{q}$ *is a minimal prime divisor of* $J = \mathcal{H}_{C/A}B$ *and* $\mathfrak{p}$ *is a minimal prime divisor of* $I = J \cap A$. *Then, there exists a smoothing at* $\mathfrak{q}$ *of* $A \to C \to B$.

Let us show that Proposition 3.3 implies Theorem 3.1.

Proof. If $J \neq B$ then $I \neq A$ and there exists a minimal prime divisor $\mathfrak{p}$ of I.

Let $\bar{B} = B/J$, $\bar{A} = A/I$ and $\bar{\mathfrak{p}} = \mathfrak{p}/I$. We get a morphism $\bar{A} \xrightarrow{\bar{\phi}} \bar{B}$ and $\bar{\mathfrak{p}}$ is a minimal prime ideal of $\bar{A}$. The set $\bar{A} \setminus \bar{\mathfrak{p}}$ is multiplicative in $\bar{A}$, hence we can find a prime ideal $\bar{\mathfrak{q}}$ of $\bar{B}$ which does not intersect $\bar{\phi}(\bar{A} \setminus \mathfrak{p})$. We choose such a $\mathfrak{q}$ to be minimal. Then $\bar{\mathfrak{q}} \cap \bar{A} \subset \bar{\mathfrak{p}}$ since $\bar{\phi}$ is injective and $\bar{\mathfrak{q}} \cap \bar{A} = \bar{\mathfrak{p}}$ by minimality of $\mathfrak{p}$. Let $\mathfrak{q}$ be the inverse image of $\bar{\mathfrak{q}}$ in B. Then $\mathfrak{q}$ is a minimal prime divisor of J and $\mathfrak{q} \cap A = \mathfrak{p}$. Applying Proposition 3.3, we get a smoothing D at $\mathfrak{q}$ of $A \to C \to B$.

Then we have the strict inclusion $\sqrt{\mathcal{H}_{C/A}B} \subset \sqrt{\mathcal{H}_{D/A}B}$. We replace C with D and by noetherianity of B we get $\mathcal{H}_{C/A}B = B$, after a finite number of smoothings. Then we conclude with Proposition 2.2.

■

As the previous argument shows, we will often change C with a factorization D. Here is a result about this change.

Remark 3.4. Consider the commutative diagram $\begin{array}{ccc} A & \xrightarrow{\phi} & B \\ \downarrow & & \uparrow \\ C & \rightarrow & C' \end{array}$, where C and C' are of finite type over A. Let $J' = \mathcal{H}_{C'/A}B$ and $I' = J' \cap A$. Let $\mathfrak{q} \in \operatorname{Spec} B$ be a minimal prime divisor of J and $\mathfrak{p} \in \operatorname{Spec} A$ be a minimal prime divisor of I. Assume that $\mathcal{H}_{C/A}C' \subset \mathcal{H}_{C'/A}$. Then, either $J' \not\subset \mathfrak{q}$ and C' is a smoothing at $\mathfrak{q}$ of $A \rightarrow C \rightarrow B$, or $J' \subset \mathfrak{q}$ and $\mathfrak{q}$ (resp. $\mathfrak{p}$) is a minimal prime divisor of J' (resp. I').

The proof of 3.3 is based on successive reduction on the heights of the ideals $\mathfrak{p}$ and $\mathfrak{q}$.

- In the general case we have $\operatorname{ht}\mathfrak{p} \leq \operatorname{ht}\mathfrak{q}$. Let us notice that $\dim B_\mathfrak{q}/\mathfrak{p}B_\mathfrak{q} = \dim B_\mathfrak{q} - \dim A_\mathfrak{p} = \operatorname{ht}\mathfrak{q} - \operatorname{ht}\mathfrak{p}$ since the morphism $\phi_\mathfrak{q} : A_\mathfrak{p} \rightarrow B_\mathfrak{q}$ is flat and by [M1, Th 15.1].

  First, we reduce to the case where $\operatorname{ht}\mathfrak{p} = \operatorname{ht}\mathfrak{q}$. Assume that Proposition 3.3 is true when $\operatorname{ht}\mathfrak{p} = \operatorname{ht}\mathfrak{q}$. By regularity of the morphism ϕ, the fiber $B_\mathfrak{q}/\mathfrak{p}B_\mathfrak{q}$ is a regular local ring. Let $(\omega_1, \ldots, \omega_n)$ be a system of parameters of $\mathfrak{q}B_\mathfrak{q}/\mathfrak{p}B_\mathfrak{q}$, with $\omega_i \in \mathfrak{q}$ for any $i \in \{1, \ldots, m\}$. Let $A' = A[w_1, \ldots, w_n]$. We define the morphism $A' \rightarrow B$ by sending w_i onto ω_i. Let $\mathfrak{p}' = \mathfrak{q} \cap A'$. We have $\mathfrak{p}' = (\mathfrak{p}, \omega_1, \ldots, \omega_n)$ and the isomorphisms $A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p} \simeq A'_{\mathfrak{p}'}/\mathfrak{p}'A'_{\mathfrak{p}'}$ and $B_\mathfrak{q}/\mathfrak{q}B_\mathfrak{q} \simeq B_\mathfrak{q}/\mathfrak{p}'B_\mathfrak{q}$. According to Proposition 1.15, the morphism $A'_{\mathfrak{p}'}/\mathfrak{p}'A'_{\mathfrak{p}'} \simeq k[w_1, \ldots, w_n]_{(w_1, \ldots, w_n)} \rightarrow B_\mathfrak{q}/\mathfrak{p}'B_\mathfrak{q}$ is flat. Then, by Proposition 1.14, $B_\mathfrak{q}$ is flat over $A'_{\mathfrak{p}'}$. Hence, from Theorem 1.7, we deduce that $B_\mathfrak{q}$ is $\mathfrak{q}$-smooth over $A'_{\mathfrak{p}'}$ (the separability of the residual field extension $A'_{\mathfrak{p}'}/\mathfrak{p}'A'_{\mathfrak{p}'} \rightarrow B_\mathfrak{q}/\mathfrak{q}B_\mathfrak{q}$ given by the characteristic zero hypothesis, is essential at this point). In other words, B is regular at $\mathfrak{q}$ over A'.

  Let $J' = \mathcal{H}_{C'/A'}$ and $I' = J' \cap A'$. Since $\mathfrak{q}$ is a minimal prime divisor of J, the ring $B_\mathfrak{q}/JB_\mathfrak{q}$ is an artinian local ring. Hence, there exist $s \notin \mathfrak{q}$ and a non zero integer N such that for any i, $s\omega_i^N \in J$. If we replace ω_i with $s\omega_i$, we may assume that $\omega_i \in \sqrt{J}$ and so $\sqrt{J'} = \sqrt{J}$ by Remark 1.13. We may apply Proposition 3.3 to $A' \rightarrow C' = C \bigotimes_A A' \rightarrow B$ where $\mathfrak{p}'$ is a minimal prime divisor of $I' = (I, w_1, \ldots, w_n)$ and $\mathfrak{q}$ is a minimal prime divisor of J'.

  In this case Proposition 3.3 is true since $\dim B_\mathfrak{q} = \dim A'_{\mathfrak{p}'}$. Then, there exists a factorization $\begin{array}{ccc} A' & \xrightarrow{\bar{\phi}'} & B \\ \downarrow & \nearrow & \uparrow \\ \bar{C} & \rightarrow & D \end{array}$, where D is smooth of finite type over A' and $\sqrt{\mathcal{H}_{C'/A'}B} \subset \sqrt{\mathcal{H}_{D/A'}B} \not\subset \mathfrak{q}$. We apply Proposition 1.11 to the sequence $A \rightarrow A' \rightarrow D$ to get $\mathcal{H}_{D/A'} \subset \mathcal{H}_{D/A}$ since A' is smooth over A. Thus, $\sqrt{\mathcal{H}_{C/A}B} \subset \sqrt{\mathcal{H}_{D/A}B} \not\subset \mathfrak{q}$ and the A-algebra D is a smoothing at $\mathfrak{q}$ of $A \rightarrow C \rightarrow B$.

Then, we treat the case $\operatorname{ht}\mathfrak{p} = \operatorname{ht}\mathfrak{q}$ by induction.

- The basic case of the induction where $\operatorname{ht}\mathfrak{p} = \operatorname{ht}\mathfrak{q} = 0$ is exactly Proposition 4.5.

- Let $k = \operatorname{ht}\mathfrak{p}$. Assume that $k > 0$. By hypothesis, the prime ideal $\mathfrak{p}$ of A is a minimal prime divisor of I. Let N be the union of all the minimal prime ideals of A contained in $\mathfrak{p}$. Since $\operatorname{ht}\mathfrak{p} > 0$ and $\mathfrak{p}$ is a minimal divisor of I, I is not included in N. Hence we can find $w \in I \setminus N$. Consider rings of the form $A/w^m A$. Then, for any integer $m \neq 0$, the height of the prime ideal $\mathfrak{p}/w^m A$ in $A/w^m A$ is stricly less than the height of $\mathfrak{p}$ in A.

  Since $w \in I$, we have $\phi(w) \in \sqrt{\mathcal{H}_{C/A}B}$. By Propositions 2.2 and 2.5, there exists a factorization $\begin{array}{ccc} A & \xrightarrow{\phi} & B \\ \downarrow & \nearrow & \uparrow \\ C & \rightarrow & D \end{array}$ where D is of finite type over A, $\mathcal{H}_{C/A}D \subset \mathcal{H}_{D/A}$ and w is standard in D. By remark 3.4, it suffices to desingularize at $\mathfrak{q}$ the diagram $A \rightarrow D \rightarrow B$. In other words, we may assume that w is standard in the A-algebra C.

  We use the lifting property of smoothing of section 5 which says the following :

Proposition (Lifting of smoothing) 5.8. *Consider a commutative diagram* $\begin{array}{ccc} A & \xrightarrow{\phi} & B \\ \downarrow & \nearrow & \\ C & & \end{array}$, *where C is of finite type over A. Let $\mathfrak{q}$ be a prime ideal of B and $w \in \phi^{-1}(q)$. Suppose that the image* *of w in C is a standard element for some presentation of C over A. For an integer k, and any A-algebra X, let $\bar{X} = X/w^kX$.*

Then, there exists a non zero integer k such that :

if there exist a finite type A-algebra S which factorizes $\begin{array}{ccc} \bar{A} & \xrightarrow{\bar{\phi}} & \bar{B} \\ \downarrow & \nearrow & \uparrow \\ \bar{C} & \rightarrow & S \end{array}$

with $\sqrt{\mathcal{H}_{C/A}\bar{B}} \subset \sqrt{\mathcal{H}_{S/\bar{A}}\bar{B}} \not\subset \mathfrak{q}\bar{B}$,

then, there exists a finite type A-algebra T which factorizes $\begin{array}{ccc} A & \xrightarrow{\phi} & B \\ \downarrow & \nearrow & \uparrow \\ C & \rightarrow & T \end{array}$ *, with* $\sqrt{\mathcal{H}_{C/A}B} \subset \sqrt{\mathcal{H}_{T/A}B} \not\subset \mathfrak{q}$.

Let $m = k$. We have $J\bar{C} \subset \mathcal{H}_{\bar{C}/\bar{A}}$ by 1.12. Either $\mathcal{H}_{\bar{C}/\bar{A}} \not\subset \bar{\mathfrak{q}}$ and $\bar{C}$ is a smoothing of $\bar{A} \to \bar{C} \to \bar{B}$ at $\bar{\mathfrak{q}}$, or $\bar{\mathfrak{q}}$ and $\bar{\mathfrak{p}}$ are minimal prime divisors of respectively $\mathcal{H}_{\bar{C}/\bar{A}}$ and $\mathcal{H}_{\bar{C}/\bar{A}} \cap \bar{A}$.

Since we have $\operatorname{ht}\bar{\mathfrak{p}} < \operatorname{ht}\mathfrak{p}$ and $\dim \bar{B}_{\bar{\mathfrak{q}}}/\bar{\mathfrak{p}}\bar{B}_{\bar{\mathfrak{q}}} = \dim B_{\mathfrak{q}}/\mathfrak{p}B_{\mathfrak{q}}$, the result follows by induction on $\operatorname{ht}\mathfrak{p}$.

Which complete the proof if we assume the **Basic Case** and the **Lifting of smoothing** properties of sections 4 and 5.

4 Basic case

Consider a commutative diagram $\begin{array}{ccc} A & \xrightarrow{\phi} & B \\ \downarrow & \nearrow & \\ C & & \end{array}$, where C is of finite type over A. Let $\mathfrak{q}$ be a minimal prime divisor of $\mathcal{H}_{C/A}B$ and $\mathfrak{p} = \mathfrak{q} \cap A$. We assume in this section that ϕ is regular at $\mathfrak{q}$ and $\operatorname{ht}\mathfrak{p} = \operatorname{ht}\mathfrak{q} = 0$. The aim of this section is to desingularize $A \to C \to B$ at $\mathfrak{q}$ (Proposition 4.5).

4.1 Weak version of the smoothing

The aim of this subsection is to prove 4.4 which is a weak version of the smoothing.

Proposition 4.1. *Let $A \xrightarrow{\phi} B$ be a ring homomorphism. Let $\mathfrak{q} \in \operatorname{Spec} B$ and $\mathfrak{p} = \mathfrak{q} \cap A$. Assume that $\operatorname{ht} \mathfrak{p} = \operatorname{ht} \mathfrak{q} = 0$.*

Then, $B_{\mathfrak{q}}$ is the inductive limit of local artinian essentially of finite type $A_{\mathfrak{p}}$-algebra Y such that there is a commutative diagram $\begin{array}{ccccc} A & \to & X & \to & B \\ i\downarrow & & \downarrow & & j\downarrow \\ A_{\mathfrak{p}} & \xrightarrow{\alpha} & Y & \xrightarrow{\beta} & B_{\mathfrak{q}} \end{array}$ *where α is essentially of finite type, β is faithfully flat, $X = j^{-1}(Y)$ and $X_{\mathfrak{q}} \simeq Y$.*

The first step is to reduce to the case where the morphism i, j, ϕ and $\phi_{\mathfrak{q}}$ are injective.

We use without proof the following lemma :

Lemma 4.2. *Let $R \to S$ be a ring homomorphism, where S is a local ring with maximal ideal $\mathfrak{q}$. Let $\mathfrak{p} = \mathfrak{q} \cap R$, J be an ideal of S and $I = J \cap R$. Then $R_{\mathfrak{p}} \simeq S$ if and only if $(R/I)_{\mathfrak{p}/I} \simeq S/J$ and $I_{\mathfrak{p}} \simeq J$.*

Let $\bar{A} = A/\operatorname{Ker}(\phi_{\mathfrak{q}} \circ i)$, $\bar{B} = B/\operatorname{Ker} j$ and $\bar{A}_{\mathfrak{p}} = A_{\mathfrak{p}}/\operatorname{Ker} \phi_{\mathfrak{q}}$. The morphism ϕ induce the commutative square :

$$\begin{array}{ccc} \bar{A} & \xrightarrow{\bar{\phi}} & \bar{B} \\ \downarrow & & \downarrow \\ \bar{A}_{\mathfrak{p}} & \xrightarrow{\phi_{\mathfrak{q}}} & B_{\mathfrak{q}} \end{array}$$

By Lemma 4.2 with $J = 0$, we have $\bar{B}_{\mathfrak{q}} \simeq B_{\mathfrak{q}}$. We apply Proposition 4.1 to the last square to get the diagram $\begin{array}{ccccc} \bar{A} & \to & \bar{X} & \to & \bar{B} \\ \downarrow & & \downarrow & & \downarrow \\ \bar{A}_{\mathfrak{p}} & \xrightarrow{\bar{\alpha}} & Y & \xrightarrow{\beta} & B_{\mathfrak{q}} \end{array}$ or equivalently the diagram $\begin{array}{ccccc} A & \to & \bar{X} & \to & \bar{B} \\ \downarrow & & \downarrow & & \downarrow \\ A_{\mathfrak{p}} & \xrightarrow{\alpha} & Y & \xrightarrow{\beta} & B_{\mathfrak{q}} \end{array}$.

There is $w \in B \setminus \mathfrak{q}$ such that $w \times \operatorname{Ker} j = 0$. We may assume that $j(w) \in Y$ (hence $w \in X$). Then $(X \cap \operatorname{Ker} j)_{\mathfrak{q}} = 0$ and Lemma 4.2 give the isomorphism $X_{\mathfrak{q}} \simeq Y$. Thus, we may assume now that the morphisms i, j, ϕ, $\phi_{\mathfrak{q}}$ are injective.

Set $k = A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$ and $K = B_{\mathfrak{q}}/\mathfrak{q}B_{\mathfrak{q}}$. The key point of the proof is then to find two liftings ρ and λ respectively of $A_{\mathfrak{p}} \to k$ and $B_{\mathfrak{q}} \to K$ such that $\lambda \circ \phi_{\mathfrak{q}} = \phi \circ \rho$. In fact, we will construct ρ and λ such that $\rho(k)$ (resp. $\lambda(K)$) is the fraction field of $k' = \rho(k) \cap A$ (resp. $K' = \lambda(K) \cap B$).

Suppose that we have constructed ρ and λ as above. Then we get a commutative square

$$\begin{array}{ccc} k' & \hookrightarrow & K' \\ \downarrow & & \downarrow \\ k & \hookrightarrow & K \end{array}$$

with $k = \mathrm{Frac}\,(k')$ and $K = \mathrm{Frac}\,(K')$.

Given $c_1, \ldots, c_s \in B_{\mathfrak{q}}$, we construct an artinian local $A_{\mathfrak{p}}$-algebra of finite type Y such that $c_1, \ldots, c_s \in Y$. Which will give the wanted *inductive* limit.

Let $\mathfrak{p} = (a_1, \ldots, a_m)$ and $\mathfrak{q} = (b_1, \ldots, b_n)$ with $a_i \in A$ and $b_j \in B$. Since $B_{\mathfrak{q}}$ is artinian, the elements b_i generate $B_{\mathfrak{q}}$ as a K-algebra. We write $B_{\mathfrak{q}} = K[b_1, \ldots, b_n] \simeq K[B_1, \ldots, B_n]/(F_1, \ldots, F_r)$. There exists G_l, a polynomial with coefficients in K, such that $c_l = G_l(b_1, \ldots, b_n)$ for $l = 1, \ldots, s$. Likewise, for any $i \in \{1, \ldots, m\}$, $a_i = H_i(b_1, \ldots, b_n)$, where H_i is a polynomial with coefficients in K.

There exists $\Omega'' \hookrightarrow K'$ a finite type k'-algebra such that $\mathrm{Frac}\,(\Omega'') = \Omega$ contains all the coefficients of the polynomials $G_1, \ldots, G_s, F_1, \ldots, F_r, H_1, \ldots, H_m$. Since $\Omega' = \Omega \cap K' \supset \Omega''$, we have $\mathrm{Frac}\,(\Omega') = \Omega$.

Define $Y = \Omega[b_1, \ldots, b_n] \subset B_{\mathfrak{q}}$. Then Y has the following properties :

- $c_l \in Y$ for $l \in \{1, \ldots, s\}$,
- the ring $A_{\mathfrak{p}}$ injects into Y and Y is essentially of finite type over $A_{\mathfrak{p}}$,
- since $F_1, \ldots, F_r$ are in $\Omega[B_1, \ldots, B_n]$, they generate the kernel of $\Omega[B_1, \ldots, B_n] \to \Omega[b_1, \ldots, b_n]$ and hence $\Omega[b_1, \ldots, b_n] \simeq \Omega[B_1, \ldots, B_n]/(F_1, \ldots, F_r)$. Thus, the morphism $Y \to B_{\mathfrak{q}}$ is flat. Moreover, Y is local and artinian by injectivity of $Y \to B_{\mathfrak{q}}$.

Let $X = B \cap \Omega[b_1, \ldots, b_n]$. We have $\Omega'[b_1, \ldots, b_n] \subset X \subset \Omega[b_1, \ldots, b_n] \hookrightarrow B_{\mathfrak{q}}$. Then, we get $\mathrm{Frac}\,(\Omega')[b_1, \ldots, b_n] \hookrightarrow X_{\mathfrak{q}} \hookrightarrow Y$. Since $\mathrm{Frac}\,(\Omega') = \Omega$, we have $\Omega[b_1, \ldots, b_n] \hookrightarrow X_{\mathfrak{q}} \hookrightarrow Y$ and $X_{\mathfrak{q}} \simeq Y$.

Now, it just remains to construct the liftings ρ and λ. The proof given in the following lemma is the same as proof of Cohen's structure Theorem [ZS, VIII Th 27] in which we check at each step that we have the fraction property $\rho(K) = \mathrm{Frac}\,(K')$.

Lemma 4.3. *Let C be a ring, $\mathfrak{p} \in \operatorname{Spec} C$ and $K = C_{\mathfrak{p}}/\mathfrak{p}C_{\mathfrak{p}}$. Assume that the morphism $C \to C_{\mathfrak{p}}$ is injective. Suppose that we have an integer N such that $(\mathfrak{p}C_{\mathfrak{p}})^N \simeq 0$. Let H be a subfield K. Assume that we have a lifting $\rho : H \to C_{\mathfrak{p}}$ such that $\rho(H) = \operatorname{Frac}(H')$ where $H' = \rho(H) \cap C$. Then we can extend this lifting to $\rho : K \to C_{\mathfrak{p}}$ such that $\rho(K) = \operatorname{Frac}(K')$ where $K' = \rho(K) \cap C$.*

Proof. If $H \neq K$ then there exists an element $a \in C$ such that the image $\bar{a}$ of a in K is not contained in H. Suppose that $\rho(H) = \operatorname{Frac}(\rho(H) \cap C)$. Let $\bar{H} = H(\bar{a})$. Then ρ extends to a lifting $\bar{\rho} : \bar{H} \to C_{\mathfrak{p}}$ satisfying $\bar{\rho}(\bar{H}) = \operatorname{Frac}(\bar{\rho}(\bar{H}) \cap C)$. We have two cases to consider :

- If $\bar{a}$ is transcendant over H, then we define $\bar{\rho}$ by sending $\bar{a}$ onto a. Since $a \in C$, we get $\operatorname{Frac}(\rho(H) \cap C)[a] \hookrightarrow \operatorname{Frac}(\bar{\rho}(\bar{H}) \cap C) \hookrightarrow \bar{\rho}(\bar{H})$, hence $\bar{\rho}(\bar{H}) = \rho(H)[a] \simeq \operatorname{Frac}(\bar{\rho}(\bar{H}) \cap C)$.

- If $\bar{a}$ is algebraic over H, then $\bar{H} = H[X]/S(X)$ where S is the minimal polynomial of $\bar{a}$ over H. We have $S(a) \in \mathfrak{p}C_{\mathfrak{p}}$. First we show that we may assume that $S(a) \in (\mathfrak{p}C_{\mathfrak{p}})^N$.

  Since the characteristic of H is zero, we have by Bezout theorem, the existence of U and V in $K[X]$ such that $SU + S'V = 1$. Then $S = S^2U + S'T$. We can choose an element $\nu \neq 0$ in $\rho(H) \cap C$ to get $\nu T(a) \in \rho(H) \cap C$ since $\rho(H) = \operatorname{Frac}(\rho(H) \cap C)$. Then, we replace a and $\bar{a}$ with respectively $\tilde{a} = \nu(a - T(a))$ and $\tilde{\bar{a}} = \nu\bar{a}$. From the fact that $S'(a) \notin \mathfrak{p}C_{\mathfrak{p}}$, we have $T(a) \in \mathfrak{p}C_{\mathfrak{p}}$ and $\tilde{a}$ represents $\tilde{\bar{a}}$. Moreover we have $H[\bar{a}] = H[\tilde{\bar{a}}]$ and $\tilde{a} \in C$.

  The Taylor formula gives $S(a - T(a)) - (S(a) - T(a)S'(a)) \in \mathfrak{p}^2C_{\mathfrak{p}}$. But $S(a) - T(a)S'(a) = S^2(a)U(a) \in \mathfrak{p}^2C_{\mathfrak{p}}$, hence $\tilde{S}(\tilde{a}) \in \mathfrak{p}^2C_{\mathfrak{p}}$ with $\tilde{S}(X) = S(X/\nu) \in H[X]$.

  We repeat this process with the new data $\tilde{a}$, $\tilde{\bar{a}}$, $\tilde{S}$ so that we may assume that $S(a) \in \mathfrak{p}^N C_{\mathfrak{p}}$, $\bar{H} = H[\bar{a}]$ and $a \in C$. Since $a \in C$, we have $\operatorname{Frac}(\bar{\rho}(\bar{H}) \cap C) = \bar{H}$.

  Then, we define $\bar{\rho}$ by sending $\bar{a}$ onto a since $S(a) = 0$ in $C_{\mathfrak{p}}$.

Then, we conclude with Zorn's Lemma.

∎

To get the lifting ρ, we apply Lemma 4.3 with $H = \mathbb{Q}$, $K = h$ and the ring $A_{\mathfrak{p}}$.

To get the lifting λ, we apply Lemma 4.3 with $H = h$, $K = k$ and the ring $B_{\mathfrak{q}}$.

Which complete the proof of Proposition 4.1.

Corollary 4.4. *Consider the commutative diagram* $\begin{array}{ccc} A & \xrightarrow{\phi} & B \\ \downarrow & \nearrow & \\ C & & \end{array}$ *, where C is of finite type over A. Let $\mathfrak{q}$ be a minimal prime divisor of $\mathcal{H}_{C/A}B$ and $\mathfrak{p} = \mathfrak{q} \cap A$. Assume that ϕ is regular at $\mathfrak{q}$ and* $\operatorname{ht}\mathfrak{p} = \operatorname{ht}\mathfrak{q} = 0$. *Then, there exists a factorization* $\begin{array}{ccc} A & \xrightarrow{\phi} & B \\ \downarrow & \nearrow & \uparrow \\ C & \rightarrow & D \end{array}$ *, where D is of finite type over A and $\mathcal{H}_{D/A}B \not\subset \mathfrak{q}$.*

Proof. By Proposition 4.1, we can find $A_{\mathfrak{p}} \rightarrow C_{\mathfrak{q}} \xrightarrow{\alpha} Y \xrightarrow{\beta} B_{\mathfrak{q}}$ where β is faithfully flat, α is essentially of finite type and $X_{\mathfrak{q}} = Y$ if we let $X = j^{-1}(Y)$. Hence Y is essentially of finite type over A and we may find a subalgebra D of X, of finite type over A such that D contains the image of C in B and $D_{\mathfrak{q}} \simeq Y$.

The morphism $A_{\mathfrak{p}} \rightarrow B_{\mathfrak{q}}$ is regular since $B_{\mathfrak{q}}$ is artinian and since the morphism $A \rightarrow B$ is regular at $\mathfrak{q}$ by hypothesis. Then, the morphism $A_{\mathfrak{p}} \rightarrow D_{\mathfrak{q}}$ is regular [M1, Th 32.1].

■

4.2 Smoothing in the basic case

Proposition 4.5. *Consider the commutative diagram* $\begin{array}{ccc} A & \xrightarrow{\phi} & B \\ \downarrow & \nearrow & \\ C & & \end{array}$ *, where C is of finite type over A. Let $\mathfrak{q}$ be a minimal prime divisor of $\mathcal{H}_{C/A}B$ and $\mathfrak{p} = \mathfrak{q} \cap A$. Suppose that ϕ is regular at $\mathfrak{q}$ and* $\operatorname{ht}\mathfrak{p} = \operatorname{ht}\mathfrak{q} = 0$. *Then, there exists a smoothing E at $\mathfrak{q}$, i.e. a factorization* $\begin{array}{ccc} A & \xrightarrow{\phi} & B \\ \downarrow & \nearrow & \uparrow \\ C & \rightarrow & E \end{array}$ *with E of finite type over A and* $\sqrt{\mathcal{H}_{C/A}B} \subset \sqrt{\mathcal{H}_{E/A}B} \not\subset \mathfrak{q}$.

Proof. Let $j : B \to B_{\mathfrak{q}}$ and $\operatorname{Ker} j = \{b \in B \mid \exists s \notin \mathfrak{q} \quad sb = 0\}$. The ideal $\mathfrak{q}$ is minimal, hence $\mathfrak{q}B_{\mathfrak{q}}$ is nilpotent and $\mathfrak{q} = \sqrt{\operatorname{Ker} j}$ by artinianity. Let $(b_1, \ldots, b_n)$ be a system of generators of $\operatorname{Ker} j$ and $w \notin \mathfrak{q}$ such that $w \times \operatorname{Ker} j = 0$.

By Corollary 4.4, we have a commutative diagram $\begin{matrix} A & \xrightarrow{\phi} & B \\ \downarrow & \nearrow & \uparrow \\ C & \to & D \end{matrix}$,

where $D = C[x_1, \ldots, x_h]/(f_1, \ldots, f_k)$ and $\mathcal{H}_{D_A} \not\subset \mathfrak{q}$. Let ϵ_i be the image of x_i in B. Define

$$E = C[x_1, \ldots, x_h, y, z_1, \ldots, z_n]/(yf_1, \ldots, yf_k, yz_1, \ldots, yz_n)$$

and the morphism $\psi : E \to B$ by $\psi(x_i) = \epsilon_i$, $\psi(y) = w$ and $\psi(z_j) = b_j$.

- Let us check that E is smooth at $\mathfrak{q}$. Since $y \notin \mathfrak{q}$, we deduce that y is invertible in $E_{\mathfrak{q}}$. We have $E_y \simeq C[x, y, z]/(f, h) \simeq C[x, y]/(f) \simeq D[y]$, and hence $E_{\mathfrak{q}} \simeq (D[y])_{\mathfrak{q}}$. Since $D_{\mathfrak{q}}$ is smooth over A, so is $E_{\mathfrak{q}}$.

- Let us check that $\sqrt{\mathcal{H}_{C/A}B} \subset \sqrt{\mathcal{H}_{E/A}B}$. Let $\mathfrak{r} \in \operatorname{Spec} B$ such that $\mathcal{H}_{C/A}B \not\subset \mathfrak{r}$. Since $\mathfrak{q}$ is a minimal prime divisor of $\mathcal{H}_{C_A}B$, we necessarily have $\mathfrak{q} \not\subset \mathfrak{r}$, and we get $\operatorname{Ker} j \not\subset \mathfrak{r}$. Choose an index i such that $b_i \notin \mathfrak{r}$. Then b_i is invertible in $E_{\mathfrak{r}}$. We have the isomorphism $E_{\mathfrak{r}} \simeq (C[x, y, z]/(yf, (yz_j)_{j \neq i}, y))_{\mathfrak{r}} \simeq (C[x, y, z]/(y))_{\mathfrak{r}} \simeq (C[x, z])_{\mathfrak{r}}$, and E is smooth at $\mathfrak{r}$.

Which completes the proof.

∎

5 Lifting of the smoothing

This section contains the technical part of the proof. It consists of some explicit constructions of new algebras. The main computations gives Proposition 5.2 and Proposition 5.4. The result of the lifting of the smoothing is Proposition 5.8.

5.1 First computation

We need a refinement of the notion of standard elements.

Definition 5.1. *Let A be a ring, C be a finite type A-algebra and $w \in C$. We say that w is* purely standard *over A if there exists a finite presentation $C = A[x_1, \dots, x_n]/I$, a subideal $(g_1, \dots, g_m) \subset I$ and an element w in $A[x_1, \dots, x_n]$ such that $w \in \Delta_g C$ and $w \in \Gamma_g C$ where $\Gamma_g = ((g_1, \dots, g_m) : I)$.*

It is not difficult to see that an element is standard if and only if one of its powers is purely standard.

Let A be a ring and X an A-module. Let $w \in X$. We recall that $\mathrm{Ann}_A w$ is the ideal of A defined by $\{a \in A \mid a \times w = 0\}$.

Proposition 5.2. *Consider a commutative diagram* $\begin{array}{ccc} & & D \\ & \nearrow & \downarrow \\ A & \xrightarrow{\phi} & B \\ & \searrow & \uparrow \\ & & C \end{array}$*, where C and D are of finite type over A. Let $w \in A$ such that its image in C is purely standard over A, $\mathrm{Ann}_A w = \mathrm{Ann}_A w^2$ and $\mathrm{Ann}_B \phi(w) = \mathrm{Ann}_B \phi(w^2)$. For any A-algebra X, denote by $\bar{X}$ the algebra X/w^4X. We assume that there exists a morphism $\bar{C} \to \bar{D}$ such that there is a factorization $\bar{C} \to \bar{D} \to \bar{B}$.*

Then, there exists a factorization $\phi : A \to E \to B$, where E is a finite type A-algebra, and morphims of A-algebras $C \to E$ and $D \to E$ such that the diagram $\begin{array}{ccccc} & & D & & \\ & \nearrow & \downarrow & \searrow & \\ A & \to & E & \to & B \\ & \searrow & \uparrow & \nearrow & \\ & & C & & \end{array}$ *commutes, and $\mathcal{H}_{D/A} E \subset \mathcal{H}_{E/A}$.*

Proof. Consider the presentation $C = A[x_1, \dots, x_n]/I$ and the subideal $(g_1, \dots, g_m) \subset I$ such that $w \in \Gamma_g + I$ and $w \in \Delta_g + I$. Let J be the $m \times n$-matrix whose entries are $(\partial g_i / \partial x_j) \in A[x]$, I_n the $n \times n$-identity matrix and K the $(n-m) \times n$-matrix obtained by removing the $n-m$ last lines of I_n. Let N be the set of all subsets of $n-m$ elements in $\{1, \dots, n\}$. The cardinality of N is $q = \binom{n}{m}$. Hence, we identify an element of N with an element $k \in \{1, \dots, q\}$.

We define H_k the $n \times n$-matrix obtained by adding $n-m$ lines of I_n to J. Let $M_k = \det H_k$ for $1 \le k \le q = \binom{n}{m}$. Let F_k be the tranposition of the cofactor matrix of H_k. We have $H_kF_k = M_kI_n = F_kH_k$. Since $(M_k)_{1\le k\le q}$ generate Δ_g and $w \in \Delta_g + I$, there exist some elements $L_k \in A[x]$ such that $w - \sum_k L_kM_k \in I$.

Let $c = (c_1, \dots, c_n) \in C^n$ be the image of $x = (x_1, \dots, x_n)$ in C^n. Then the image of w in C, which we still denote by w, is :

$$w = \sum_k (L_kM_k)(c) \tag{1}$$

Let G_k be the $n \times n$ matrix defined by $G_k = L_kF_k$ for $1 \le k \le q$. We get the relations :

$$H_kG_k = L_kM_kI_n = G_kH_k \quad \text{and} \quad JG_k = L_kM_kK \tag{2}$$

We introduce the new variables $x = (x_1, \dots, x_n)$, $z = (z_1, \dots, z_n)$ and
$(y_1, \dots, y_m, y_{m+1}^{(k)}, \dots, y_n^{(k)})_{1\le k\le q}$. For practical reasons, we will prefer the variables $y = (y_1^{(k)}, \dots, y_n^{(k)})_{1\le k\le q}$ with the relations $y_j^{(k)} = y_j$ for $j = 1, \dots, m$.

Denote by $d = (d_j)_{1\le j\le n}$ a lifting to D^n of the image of $c = (c_j)_{1\le j\le n}$ by the morphism $\bar{C} \to \bar{D}$. Let S and T be the multiplicative subsets of D and $D[x, y, z]$ given by $S = 1 + wD$ and $T = 1 + wD[x, y, z]$. We have $w - \sum_k (L_kM_k)(d) \in w^4D$, hence there exists $s \in 1 + w^3D \subset S$ such that in D we have :

$$sw = \sum_k (L_kM_k)(d) \tag{3}$$

We define for $j = 1, \dots, n$ the elements h_j of $D[x, y, z]$ by :

$$h_j = s(x_j - d_j) - w^2 \sum_k (G_k(d)y^{(k)})_j + w^4 z_j \tag{4}$$

Note that the h_j's are affine in the variables x, y, z.

We will need the following lemma :

Lemma 5.3.

1. *For any $f \in I$, there exist $\sigma \in S$ and $\tau \in D[y, z]$ such that :*

$$\sigma w^2 \sum_{j,k} (\partial f/\partial x_j)(d)(G_k(d)y^{(k)})_j \equiv \sigma s f + w^4 \tau \mod (h)$$

.

2. For $i = 1, \dots, m$, there exists $s_i \in S$ and $\rho_i \in D[y,z]$ such that :

$$s_i g_i \equiv w^3 l_i \mod (h) \quad \text{with} \quad l_i = s_i y_i + \rho_i w \qquad (5)$$

Proof.

1. Since the element f is in I, we have $f(d) \in w^4 D$ and it suffices to show the congruence for $f - f(d)$. We apply the Taylor formula to get :

$$s(f(x) - f(d)) \equiv s \sum_j (x_j - d_j)(\partial f / \partial x_j)(d) \mod (h, w^4)$$

(since $s^2(x_{j_1} - d_{j_1}) \times \dots \times (x_{j_\nu} - d_{j_\nu}) \equiv 0 \mod (h, w^4)$ according to (4)). Again using (4), we have :

$$s^2(f(x) - f(d)) \equiv sw^2 \sum_{j,k} (\partial f / \partial x_j)(d)(G_k(d) y^{(k)})_j \mod (h, w^4)$$

Since each sx_j is congruent to an element of $D[x,y]$ modulo h by (4), we may assume that $\tau \in D[y,z]$ after multiplying both sides of the congruence by an appropriate power of s.

2. For $i = 1, \dots, m$, we have :

$$\begin{aligned}
\sum_{j,k} (\partial g_i / \partial x_j)(d)(G_k(d) y^{(k)})_j &= \sum_k (J G_k(d) y^{(k)})_i \\
&= \sum_k (L_k M_k)(d)(K y^{(k)})_i \quad \text{by (2)} \\
&= \sum_k (L_k M_k)(d) y_i \\
&= swy_i \quad \text{by (1)}
\end{aligned}$$

Applying (1) for $f = g_i$, we get :

$$\sigma w^2 (swy_i) \equiv \sigma s g_i + w^4 \tau \mod (h) \quad \text{and} \quad \sigma s g_i \equiv w^3(\sigma s y_i - \tau w) \mod (h)$$

Which conclude the proof of the lemma.

We take $E = D[x,y,z]/U$ with the ideal U generated by I,$(h_j)_{1\leq j\leq n}$, $(l_i)_{1\leq i\leq m}$. Remenber that we included the equations $(y_j^{(k)} - y_j)$ in U.

Denote by $D \xrightarrow{\delta} B$ and $C \xrightarrow{\gamma} B$ the morphisms given in the hypothesis. For $j = 1,\dots,n$, there exists $\omega_j \in wB$ such that $\gamma(c_j) - \delta(d_j) = w^3\omega_j$. Define

$$\eta^{(k)} = H_k(\delta(d))\omega \tag{6}$$

where $\omega = (\omega_1,\dots,\omega_n)$. Note that for $j = 1,\dots m$, $\eta_j^{(k)}$ does not depend on k since $K\eta^{(k)} = KH_k(\delta(d))w = J(\delta(d))w$. Hence, we may denote by η_j the common value of $\eta_j^{(k)}$ for $j = 1,\dots,m$. Moreover, for $j = 1,\dots,n$, we have $\eta_j^{(k)} \in wB$.

The natural morphisms $D \to E$ and $C = A[x]/I \to E$ are well defined. We define the morphism $\theta : D[x,y,z] \to B$ by $x_j \mapsto \gamma(c_j)$, $y_j^{(k)} \mapsto \eta_j^{(k)}$ and $z_j \mapsto 0$. Let us check it induces a homomorphism $E \xrightarrow{\theta} B$ which we still denote by θ.

We have $\theta(I) = 0$ by definition of γ. Then :

$$\begin{aligned}
\theta(h_j) &= s(\gamma(c_j) - \delta(d_j)) - w^2\sum_k (G_k(\delta(d))\eta^{(k)})_j \quad \text{by (4)}\\
&= sw^3\omega_j - w^2\sum_k ((G_kH_k)(\delta(d)))_j \quad \text{by (6)}\\
&= sw^3\omega_j - w^2\sum_k L_kM_k\gamma(d)\omega_j \quad \text{by (2)}\\
&= 0 \quad \text{by (3)}
\end{aligned}$$

Furthermore $\theta(l_i) \in wB$ since $l_i = s_iy_i + \rho_iw$ and $\theta(y_i) = \eta_i^{(1)} \in wB$. The relation (5) and $\theta(g_i) = 0$ give $\theta(l_i)w^3 = 0$. By hypothesis, we have $\mathrm{Ann}_B w^2 = \mathrm{Ann}_B w$, hence $\mathrm{Ann}_B w^3 = \mathrm{Ann}_B w$, and $\theta(l_i) \in wB \cap \mathrm{Ann}_B w = 0$. Thus, θ factorises through the equations (l), (h) and I.

Let us check that $\mathcal{H}_{D/A}E \subset \mathcal{H}_{E/A}$.

Let $\zeta \in \mathcal{H}_{D/A}$. We have D_ζ smooth over A and we show that E_ζ is smooth over A. Since $E_\zeta = D_\zeta[x,y,z]/(I,h,l)$, it suffices to see that E_ζ

is smooth over D_ζ. Since the morphism $E \to E_w \times T^{-1}E$ is faithfully flat, if $D_\zeta \to (E_w \times T^{-1}E)_\zeta$ is regular then so is the morphism $D_\zeta \to E_\zeta$.

We first show that $E_{w,\zeta}$ is smooth over D_ζ. Since w is purely standard in C over A, C_w is smooth over A and by base change $(D[x]/I)_w$ is smooth over D. Moreover $(D[x,y,z]/(I,h))_w$ is a smooth $(D[x]/I)_w$-algebra since $\det(\partial h/\partial z) = w^{4n}$ by (4). And according to (5), the images in $(D[x,y,z]/(I,h))_w$ of the elements l_i are zero. Then, $E_w = (D[x,y,z]/(I,h,l))_w = (D[x,y,z]/(I,h))_w$ is smooth over D by transitivity.

Next, we show that $(T^{-1}E)_\zeta$ is smooth over D. Let $F = D[x,y,z]/(h,l)$ and $\tilde{y} = (y_1,\ldots,y_m)$. We have $\det \begin{pmatrix} \partial h/\partial x & \partial l/\partial x \\ \partial h/\partial \tilde{y} & \partial l/\partial \tilde{y} \end{pmatrix} = \det \begin{pmatrix} sI_n & 0 \\ \partial h/\partial \tilde{y} & I_m \bmod (w) \end{pmatrix} = t \in T$. Hence $D \to T^{-1}F$ is regular. Let us show that $I \subset (w) \cap \operatorname{Ann}(w)$ in $T^{-1}F$.

Lemma 5.3 gives the existence of $t \in T$ such that $tI \equiv 0 \mod (h,w)$.

Since w is purely standard in C, there exists $e \in A[x]$ such that $e(c) = w$, $e \in \Delta_g + I$ and $e \in \Gamma_g$. Let $f = e - w \in I$. By (5), there exists $\sigma \in S$ and $\tau \in D[x,y,z]$ such that $\sigma f \equiv w^2\tau \mod (h)$. Then $t'w \equiv 0 \mod (e,h)$ with $t' = \sigma + w\tau \in T$. Since $e \in \Gamma_g$, we have $eI \subset (g)$ and we get $twI \equiv 0 \mod (g,h)$, in $D[x,y,z]$.

Thus $I \subset (w) \cap \operatorname{Ann}(w)$ in $T^{-1}D[x,y,z]/(g,h)$ and hence $I \subset (w) \cap \operatorname{Ann}(w)$ in $T^{-1}F$. From $\operatorname{Ann}_A w = \operatorname{Ann}_A w^2$, we deduce that $(w) \cap \operatorname{Ann}(w) = 0$ in A, then also in D_ζ by flatness (smoothness) of D_ζ over A. Moreover $(w) \cap \operatorname{Ann}(w) = 0$ in $(T^{-1}F)_\zeta$ since $D \to T^{-1}F$ is regular. It shows that the image of I in $(T^{-1}F)_\zeta$ is zero. Then the isomorphism $(T^{-1}F)_\zeta \simeq (T^{-1}E)_\zeta$ allows us to conclude to the regularity of $(T^{-1}E)_\zeta$ over D.

∎

5.2 Second computation

Proposition 5.4. *Let C be a finite type A-algebra presented as the quotient $C \simeq A[x_1,\ldots,x_n]/I$ with the morphism $\Pi : A[x_1,\ldots,x_n] \to C$ and $I = \operatorname{Ker}\Pi$. Let $c_i = \Pi(x_i)$. We consider a commutative diagram*

$$\begin{array}{ccc} A & \xrightarrow{\phi} & B \\ \downarrow & \nearrow_{\theta} & \\ A[x] & & \end{array}$$

. Suppose we have $w \in A$ such that $\mathrm{Ann}_A w = \mathrm{Ann}_A w^2$ *and* $\mathrm{Ann}_B \phi(w) = \mathrm{Ann}_B \phi(w^2)$*. Assume moreover that the image in C of w^2 is zero and $\theta(I) \subset w^2 B$. Let $\bar{A} = A/w^2A$. Then, there exists a finite type A-algebra D factorizing $\theta : A[x] \xrightarrow{\alpha} D \to B$, such that $ID \subset wD$ and $\Pi^{-1}(\mathcal{H}_{C/\bar{A}})D \subset \mathcal{H}_{D/A}$.*

Proof. Let $I = (p_1, \ldots, p_m)$. For any $j \in \{1, \ldots, m\}$ there exists $\sigma_j \in wB$ such that $\theta(p_j) = w\sigma_j$. Let us introduce the new variables $(y_1, \ldots, y_m)$. We extend θ to a morphism $A[x_1, \ldots, x_n, y_1, \ldots, y_m] \to B$ such that $\theta(y_j) = \sigma_j$, and we still denote it by θ. Define :

$$\forall j = 1, \ldots, m \quad f_j = p_j - wy_j \tag{7}$$

Since the image of w^2 in C is zero, the morphism Π induces a morphism $\bar{\Pi} : \bar{A}[x_1, \ldots, x_n] \to C$. Let $\bar{I} = \mathrm{Ker}\,\bar{\Pi}$ and $\bar{p}_i = p_i \mod (w^2)$. Then $\bar{I} = (\bar{p}_1, \ldots, \bar{p}_m)$.

We choose a family $(\nu^u)_{1 \le u \le d}$ of purely standard elements in $\mathcal{H}_{C/\bar{A}}$ such that $\sqrt{(\nu^1, \ldots, \nu^d)} = \mathcal{H}_{C/\bar{A}}$. By definition, for each $u = 1, \ldots, d$, there is a family $\bar{q^u} = (\bar{q_1^u}, \ldots, \bar{q_t^u})$ in $\bar{I}$ and an element $\bar{\gamma^u} \in \Gamma_{\bar{q}}\Delta_{\bar{q}} \subset \bar{A}[x]$ such that $\nu^u = \bar{\Pi}(\bar{\gamma^u})$.

We choose liftings $q^u = (q_1^u, \ldots, q_{t_u}^u)$ of $\bar{q^u}$; we have $(q^u) \subset I$. Let $\Gamma^*_{q^u} = \{a \in A[x] \mid aI \subset (q^u, w^2)\}$ ($\Gamma^*_{q^u}$ is the inverse image of Γ_{q^u} in $A[x]$ and $\Gamma_{q^u} \subset \Gamma^*_{q^u}$). We may choose liftings $\gamma^u \in \Gamma^*_{q^u}\Delta_{q^u} \in A[x]$ of $\bar{\gamma}^u$.

There exists $\tau_k^u \in wB$ such that $\theta(q_k^u) = w\tau_k^u$, for $u = 1, \ldots, d$ and $k = 1, \ldots, t_u$. Let us introduce the new variables z_k^u, for $u = 1, \ldots, d$ and $k = 1, \ldots, t_u$. The morphism θ extends to a morphism $A[x, y, z] \xrightarrow{\theta} B$, where $\theta(z_k^u) = \tau_k^u$, and we still denote it by θ. Then $\mathrm{Ker}\,\theta$ contains the elements :

$$g_k^u = q_k^u - wz_k^u \in A[x, z] \tag{8}$$

Since $\gamma^u \in \Gamma^*_{q^u}$, we have :

$$\left\{ \begin{array}{l} \forall i = 1, \ldots, m \\ \forall v = 1, \ldots, d \quad v \neq u \quad \forall i = 1, \ldots, t_v \\ \gamma^u p_i + \sum_{1 \le k \le t_u} r_{ik}^u q_k^u + w^2 s_i^u = 0 \\ \gamma^u q_i^v + \sum_{1 \le k \le t_u} r_{ik}^{v,u} q_k^u + w^2 s_i^{v,u} = 0 \end{array} \right. \tag{9}$$

where $r_{ik}^u, s_i^u, r_{ik}^{v,u}, s_i^{v,u} \in A[x]$.
Then, we define :

$$\begin{cases} \forall i = 1, \dots, m \\ \forall v = 1, \dots, d \quad v \neq u \quad \forall i = 1, \dots, t_v \\ h_i^u = \gamma^u y_i + \sum_{1 \leq k \leq t_u} r_{ik}^u z_k^u + w s_i^u \\ h_i^{v,u} = \gamma^u z_i^v + \sum_{1 \leq k \leq t_u} r_{ik}^{v,u} z_k^u + w s_i^{v,u} \end{cases} \tag{10}$$

We clearly have the relations :

$$\begin{cases} \forall i = 1, \dots, m \\ \forall v = 1, \dots, d \quad v \neq u \quad \forall i = 1, \dots, t_v \\ w h_i^u + \gamma^u f_i + \sum_{1 \leq k \leq t_u} r_{ik}^u g_k^u = 0 \\ w h_i^{v,u} + \gamma^u g_i^v + \sum_{1 \leq k \leq t_u} r_{ik}^{v,u} g_k^u = 0 \end{cases} \tag{11}$$

We denote by h^u the family of h_i^u for all $i = 1, \dots, m$ and $h_i^{v,u}$ for all $v = 1, \dots, d$, $v \neq u$ and $i = 1, \dots, t_v$. Let h be the family of all h^u for all $u = 1, \dots, n_u = m + \sum_{v \neq u} t^v$. We set $D = A[x, y, z]/(f, g, h)$ with $h = (h^u)_{1 \leq u \leq d}$.

We have $\theta(f) = \theta(g) = 0$ by (7) and (8), and hence by (11) $w\theta(h) = 0$. Since $\theta(y_i)$ and $\theta(z_k^u)$ are elements of wB, we have $\theta(h_i^u) \in (wB) \cap \operatorname{Ann}_B(w)$ and $\theta(h_i^u) = 0$ since $\operatorname{Ann}_B w = \operatorname{Ann}_B w^2$. It shows that θ factorizes through a morphism $\mu : D \to B$. Moreover, since $p_i \equiv w y_i \mod (f_i)$ by (7), we have $\alpha(p_i) \in wD$ and $\alpha(I) \subset wD$.

It remains to check that $\Pi^{-1}(\mathcal{H}_{C/\bar{A}})D \subset \mathcal{H}_{D/A}$.

Let $D^v = A[x, y, z]/(f, g, h^v)$ for $1 \leq v \leq d$.

Lemma 5.5. *For $v = 1, \dots, d$, $u = 1, \dots, d$ and $i = 1, \dots, n^u$, we have :*

$$wh \subset (f, g, h^v) \quad \text{and} \quad (\gamma^v)^2 h \subset (g, h^v, w)$$

Proof. The first assertion is obvious by (11).

Since $\gamma^v \in \Delta_{q^v}$, there exists a $n \times t_v$-matrix (c_{jk}^v) whose entries are in $A[x]$, such that $\sum_{1 \leq j \leq n} (\partial q_k^v / \partial x_j) c_{jk'} = \begin{cases} \gamma^v & \text{if } k = k', \\ 0 & \text{otherwise.} \end{cases}$

Let $\alpha_j^v = \sum_{1 \leq k \leq t_v} c_{jk}^v z_k^v$ for $j = 1, \dots, n$. We get the relations :

$$\forall k = 1, \dots, t_v \quad \sum_{1 \leq j \leq n} \alpha_j^v (\partial q_k^v / \partial x_j) = \gamma^v z_k^v$$

From this, we deduce the following property for any linear form ϕ in the variables $z^v = (z_1^v, \ldots, z_{t_v}^u)$ with coefficients in $A[x]$:

$$\sum_{1\leq j\leq n} \alpha_j^v(\partial/\partial x_j)(\phi|_{z^v=q^v}) \equiv \gamma^v\phi \mod (q^v) \tag{12}$$

We also use the following property coming from (10) : if ψ is a linear form in the variables y and z with coefficients in $A[x]$, then we can find a linear form ϕ in the variables z^v such that $\gamma^v\psi \equiv \phi \mod (h^v, w)$. Remark that the polynomials $(h^u)_{u\neq v}$ given by (10) satisfy this last condition : they are linear homogeous in the variables y and z modulo w. Denote by ϕ^u the family of linear forms such that $\gamma^v h^u \equiv \phi^u \mod (h^v, w)$.

Moreover, using (9) they also satisfy :

$$\gamma^v h_i^u|_{y=p,z=q} \equiv \phi_i^u|_{z^v=q^v} \mod (w)$$

Then, by (12) :

$$(\gamma^v)^2 h_i^u \equiv \sum_{1\leq j\leq n} \alpha_j^v(\partial/\partial x_j)(\phi_i^u|_{z^v=q^v}) \equiv \sum_{1\leq j\leq n} \alpha_j^v(\partial/\partial x_j)(\gamma^v h_i^u|_{y=p,z=q})$$

$$\mod (q, h^v, w)$$

Furthermore, the relations (8) gives $(h^v, g, w) = (h^v, q, w)$ and for $u = 1, \ldots, d$, $i = 1, \ldots, n^u$ we have $h_i^u|_{z=q,y=p} \in (w)$ by (9) and (10). This shows that $(\gamma^v)^2 h^u \subset (g, h^v, w)$.

■

Lemma 5.6. The ring $D_{\gamma^v}^v$ is smooth over A, i.e. $\gamma^v \in \mathcal{H}_{D^v/A}$.

Proof. According to (11), we have $\gamma^v f_i \in (g^v, h^v)$ and $\gamma^v g_i^u \in (g^v, h^v)$. Then $\gamma^v \in \Gamma_{(g^v,h^v)} = \{a \in A[x,y,z] \mid a(f,g,h^v) \subset (g^v,h^v)\}$. Let us consider the matrix $\begin{pmatrix} \partial h^v/\partial y & (\partial h^v/\partial z^u)_{u\neq v} & \partial h^v/\partial x \\ \partial g^v/\partial y & (\partial g^v/\partial z^u)_{u\neq v} & \partial g^v/\partial x \end{pmatrix} = \begin{pmatrix} \gamma^v I_{n_v} & 0 & 0 \\ 0 & 0 & \partial q^v/\partial x \end{pmatrix}$, where $n_v = m + \sum_{u\neq v} t^u$. We get $(\gamma^v)^{n_v}\Delta_{q^v} \subset \Delta_{(g^v,h^v)}$. Since $\gamma^v \in \Delta_{q^v}$, we have $(\gamma^v)^{n_v+2} \in \Delta_{(g^v,h^v)}\Gamma_{(g^v,h^v)}$ and we are done.

■

It follows from Lemma 5.5 that $h_i^u \in (w) \cap \operatorname{Ann}(w)$ in $D^v_{\gamma^v}$ for $u = 1, \dots, d$ and $i = 1, \dots, n^u$. By hypothesis, $(w) \cap \operatorname{Ann}(w) = 0$ in A since $\operatorname{Ann}_A w = \operatorname{Ann}_A w^2$. Since $D^v_{\gamma^v}$ is smooth over A by Lemma 5.6, it is flat over A and $(w) \cap \operatorname{Ann}(w) = 0$ in $D^v_{\gamma^v}$. Hence $h^u = 0$ in $D^v_{\gamma^v}$ and $D_{\gamma^v} \simeq D^v_{\gamma^v}$. Thus D_{γ^v} is smooth over A for $v = 1, \dots, d$.

To establish the wanted property $\Pi^{-1}(\mathcal{H}_{C/\bar{A}})D \subset \mathcal{H}_{D/A}$, we need to show that $\alpha(I) \subset \mathcal{H}_{D/A}$ (remark that $\Pi^{-1}(\mathcal{H}_{C/\bar{A}}) \subset A[x]$ is the intersection of all prime ideals containing $I + (\gamma))$. Since $\alpha(I) \subset wD$, it suffices to see that $w \in \mathcal{H}_{D/A}$.

Consider the subfamilly (f, g) of (f, g, h). We have by (11) the relation $wh \subset (f, g)$, and hence $w \in \Gamma_{f,g}$. The determinant of the matrix $\begin{pmatrix} \partial f/\partial y & \partial f/\partial z \\ \partial g/\partial y & \partial g/\partial z \end{pmatrix}$ is equal to $(-w)^{(n+\sum_u t^u)}$ and consequently $w \in \sqrt{\Gamma_{f,g}\Delta_{f,g}}$, and hence $w \in \mathcal{H}_{D/A}$.

■

5.3 Lifting the smoothing, the result

Proposition 5.7. *Let $\phi : A \to B$ be a morphism, $\mathfrak{q}$ be a prime ideal of B and R and S be two finite type A-algebras. Suppose that we have $d \in A$, whose image in R is purely standard for some presentation and $d^8 = 0$ in S. Suppose moreover that $d \in \phi^{-1}(\mathfrak{q})$, $\operatorname{Ann}_A d = \operatorname{Ann}_A d^2$ and $\operatorname{Ann}_B \phi(d) = \operatorname{Ann}_B \phi(d^2)$. For any A-algebra X, we denote by $\bar{X}$ the algebra A/d^8A. Suppose there is a commutative diagram of A-algebras*

$$\begin{array}{ccc} R & \xrightarrow{\rho} & B \\ \downarrow & & \downarrow \\ \bar{R} & \xrightarrow{\bar{\rho}} & \bar{B} \\ & \searrow & \uparrow \\ & & S \end{array}$$

with $\mathcal{H}_{R/A}\bar{B} \subset \sqrt{\mathcal{H}_{S/\bar{A}}\bar{B}} \not\subset \mathfrak{q}\bar{B}$. Then, there exists a factorization $\rho : R \to T \to B$, where T is a finite type A-algebra and $\mathcal{H}_{R/A}B \subset \sqrt{\mathcal{H}_{T/A}B} \not\subset \mathfrak{q}$.

Proof. We choose a surjective mapping $\Pi : A[x] \to S$ and we set $I = \operatorname{Ker}\Pi$. We choose $\theta : A[x] \to B$ a lifting of $A[x] \to S \to \bar{B}$. We have $\theta(I) \subset d^8B$ in B. Then, we can apply Proposition 5.4 with $w = d^4$ and $C = S$. There exists a finite type A-algebra D such that the following diagram commutes $\begin{array}{ccc} A & \xrightarrow{\phi} & B \\ \downarrow & \overset{\theta}{\nearrow} & \uparrow \\ A[x] & \xrightarrow{\alpha} & D \end{array}$ and $ID \subset wD$, $\Pi^{-1}(\mathcal{H}_{S/\bar{A}})D \subset \mathcal{H}_{D/A}$.

Then we show that $\mathcal{H}_{R/A}B \subset \sqrt{\mathcal{H}_{D/A}B} \not\subset \mathfrak{q}$. Remark that $d^8 \in \Pi^{-1}(\mathcal{H}_{S/\bar{A}})B \subset \sqrt{\mathcal{H}_{D/A}B}$ ($d^8 = 0$ in S). Since $\mathcal{H}_{S/\bar{A}}\bar{B} = \Pi^{-1}(\mathcal{H}_{S/\bar{A}})\bar{B}$ and $\mathcal{H}_{R/A}\bar{B} \subset \sqrt{\mathcal{H}_{S/\bar{A}}\bar{B}}$, we get $\mathcal{H}_{R/A}\bar{B} \subset \sqrt{\mathcal{H}_{D/A}\bar{B}}$ and $\mathcal{H}_{R/A}B \subset \sqrt{\mathcal{H}_{D/A}B}$. Besides, if $\sqrt{\mathcal{H}_{D/A}B} \subset \mathfrak{q}$ then $\sqrt{\Pi^{-1}(\mathcal{H}_{S/\bar{A}})B} \subset \mathfrak{q}$, and hence $\sqrt{\Pi^{-1}(\mathcal{H}_{S/\bar{A}})\bar{B}} \subset \mathfrak{q}\bar{B}$ (the ideal $\mathfrak{q}\bar{B}$ is prime in $\bar{B}$ since $d \in \phi^{-1}(\mathfrak{q})$) and $\sqrt{\mathcal{H}_{S/\bar{A}}\bar{B}} \subset \mathfrak{q}\bar{B}$). Which is impossible by assumption. Then $\sqrt{\mathcal{H}_{D/A}B} \not\subset \mathfrak{q}$

The morphism $A[x] \xrightarrow{\alpha} D$ induces a morphism $\beta : S \to D/d^4D$ since $ID \subset d^4D$. Since we have a morphism $\bar{R} = R/d^8R \to S \xrightarrow{\beta} D/d^4D$, we get a factorization $R/d^4R \to S/d^4S \to D/d^4D \to B/d^4B$. Hence we may apply Proposition 5.2 to the situation $\begin{array}{ccc} & & D \\ & \nearrow & \downarrow \\ A & \xrightarrow{\phi} & B \\ & \searrow & \uparrow \\ & & R \end{array}$ with $w = d$.

Thus, there exists a finite type A-algebra T which factorizes the diagram

$$\begin{array}{ccccc} & & D & & \\ & \nearrow & \downarrow & \searrow & \\ A & \to & T & \to & B \\ & \searrow & \uparrow & \nearrow & \\ & & R & & \end{array}$$

with $\mathcal{H}_{D/A}T \subset \mathcal{H}_{T/A}$.

We finally get $\mathcal{H}_{R/A}B \subset \sqrt{\mathcal{H}_{T/A}B} \not\subset \mathfrak{q}$ and we are done.

■

We end with the wanted result on the lifting of the smoothing.

Proposition 5.8. *Consider a commutative diagram* $\begin{array}{ccc} A & \xrightarrow{\phi} & B \\ \downarrow & \nearrow & \\ C & & \end{array}$, *where* C *is of finite type over* A. *Let* $\mathfrak{q}$ *be a prime ideal of* B *and* $w \in \phi^{-1}(q)$.

Suppose that the image of w in C is a standard element for some presentation of C over A. For an integer k, and any A-algebra X, let $\bar{X} = X/w^k X$.

Then, there exists a non zero integer k such that :

if there exist a finite type A-algebra S which factorizes $\begin{array}{ccc} \bar{A} & \xrightarrow{\bar{\phi}} & \bar{B} \\ \downarrow & \nearrow & \uparrow \\ \bar{C} & \rightarrow & S \end{array}$

with $\sqrt{\mathcal{H}_{C/A}\bar{B}} \subset \sqrt{\mathcal{H}_{S/\bar{A}}\bar{B}} \not\subset \mathfrak{q}\bar{B}$,

then, there exists a finite type A-algebra T which factorizes $\begin{array}{ccc} A & \xrightarrow{\phi} & B \\ \downarrow & \nearrow & \uparrow \\ C & \rightarrow & T \end{array}$,

with $\sqrt{\mathcal{H}_{C/A}B} \subset \sqrt{\mathcal{H}_{T/A}B} \not\subset \mathfrak{q}$.

Proof. Choose an integer j such that w^j and w^{j+1} have same annihilators in A and B (it is possible by noetherianity), and such that w^j is purely standard in the A-algebra C. Set $k = 8j$ and $d = w^j$. The assumptions of Proposition 5.7 are satisfied with $R = C$, and the result follows.

■

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