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Blow-up solutions of some nonlinear elliptic problems.

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Abstract

We consider the semilinear elliptic equation

$$-\Delta u = \lambda f(u),$$

posed in a bounded domain Ω of $\mathbf{R}^n$ with smooth boundary $\partial\Omega$ with Dirichlet data $u\big|_{\partial\Omega} = 0$, and a continuous, positive, increasing and convex function f on $[0, \infty)$ such that $f(s)/s \to \infty$ as $s \to \infty$. Under these conditions there is a maximal or extremal value of the parameter $\lambda > 0$ such that the problem has a solution. We investigate the existence and properties of the corresponding extremal solutions when they are unbounded (i.e., singular or blow-up solutions). We characterize the singular H^1 extremal solutions and the extremal value by a criterion consisting of two conditions: (i) they must be energy solutions, not in L^∞; (ii) they must satisfy a Hardy inequality which translates the fact that the first eigenvalue of the linearized operator is nonnegative.

In order to apply this characterization to the typical examples arising in the literature we need an improved version of the classical Hardy inequality with best constant. We establish such a result as a simultaneous generalization of Hardy's and Poincaré's inequalities for all dimensions $n \geq 2$.

A striking property of some examples of unbounded extremal solutions is the fact that the linearization of the problem around them happens to be formally invertible and nevertheless the application of the Inverse and Implicit Function theorems fails to produce the usual existence or continuation results. We consider this question and explain the phenomenon as a lack of appropriate functional setting.

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1 Introduction

In this paper we consider the semilinear elliptic equation

$$-\Delta u = \lambda f(u), \tag{1.1}$$

posed in a bounded domain Ω of $\mathbf{R}^n$ with smooth boundary $\partial\Omega$ on which we impose Dirichlet data

$$u|_{\partial\Omega} = 0. \tag{1.2}$$

We refer to the combination of (1.1) and (1.2) as Problem (E_λ), or simply (E) when λ is understood. We assume that the nonlinearity f is a continuous, positive, increasing and convex function defined for $u \geq 0$ with $f(0) > 0$ and

$$\lim_{s\to\infty} \frac{f(s)}{s} = \infty. \tag{1.3}$$

Typical examples are $f(u) = e^u$ and $f(u) = (1+u)^p$, with $p > 1$. Equation (1.1) appears in a number of applications, like the description of a ball of isothermal gas in gravitational equilibrium, proposed by lord Kelvin [Ch]. It has been actively investigated in connection with combustion theory, [G], see also [JL]. It is well-known that there exists a finite positive number λ^*, called here the *extremal value*, such that problem (E_λ) has at least a classical positive solution $u \in C^2(\overline{\Omega})$ if $0 < \lambda < \lambda^*$, while no solution exists, even in the weak sense, for $\lambda > \lambda^*$, cf. [B4] and its references. The aim of this work is to study the properties of the solutions of problem (E) at the extremal value $\lambda = \lambda^*$, so-called *extremal solutions*.

As in [B4] we define a *weak solution* of Problem (E) as a function $u \in L^1(\Omega)$ such that

$$f(u)\,\delta \in L^1(\Omega), \tag{1.4}$$

where $\delta(x) = \mathrm{dist}(x, \partial\Omega)$ is the distance function with respect to the boundary, and (1.1)- (1.2) are satisfied in the form

$$\int_\Omega (u\,\Delta\zeta + \lambda\, f(u)\,\zeta)\,dx = 0 \tag{1.5}$$

for all $\zeta \in C^2(\overline{\Omega})$ with $\zeta = 0$ on $\partial\Omega$. It easily follows from standard regularity theory that a bounded weak solution is smooth, i.e., a classical solution. Our main interest are the unbounded or *singular* solutions. The analysis of singular extremal solutions involves an intermediate class of solutions, cf. [MP1], where $u \in H_0^1(\Omega)$. Then it also follows that $f(u)u \in L^1(\Omega)$. We call these solutions *energy solutions.*

The existence and properties of the extremal solutions depends strongly on the dimension n, domain Ω and nonlinearity f. The properties of classical extremal solutions have been well studied. We concentrate here on the analysis of singular extremal solutions for general Ω and f, which departs in many ways from the properties of the classical extremal solutions. Examples of singular solutions are well known when Ω is a ball and the reaction term is either exponential $f(u) = e^u$ or power-like $f(u) = (1+u)^p$, $p > 1$. It happens that a singular solution is not always the extremal one. Our first result characterizes the singular H^1 extremal solutions and the extremal value λ^* by a criterion consisting of two conditions:

(i) They must be energy solutions, not in L^∞.

(ii) They must satisfy the condition

$$\lambda \int_\Omega f'(u)\phi^2\,dx \le \int_\Omega |\nabla \phi|^2\,dx \tag{1.6}$$

for all $\phi \in C_0^1(\Omega)$, cf. Theorem 3.1. This formula, which roughly speaking says that the first eigenvalue of $-\Delta - \lambda\, f'(u)$ is nonnegative, is a version of Hardy's inequality. In our analysis we need an improved version of the classical Hardy inequality [HLP] with best constant, cf. Theorem 4.1. Our result is in fact a simultaneous generalization of Hardy's and Poincaré's inequalities.

A second type of result concerns the first eigenvalue of the linearized operator, $-\Delta - \lambda^*\, f'(u)$, which for classical extremal solutions is known to be zero, precisely as a consequence of the impossibility of continuing the solution branch beyond λ^*, cf. [CR]. It is quite surprising to find out that for the typical examples of singular extremal solutions with $f(u) = e^u$ and $f(u) = (1+u)^p$, cf. Sections 5 and 6, the first eigenvalue is positive, not zero. This apparently contradicts the fact that the branch of solutions cannot be continued beyond λ^*. The failure of the continuation depends on the fact that the Implicit Function Theorem cannot be applied in this singular settin even if the linearized operator is usually invertible at $\lambda = \lambda^*$ in suitable spaces, e.g., from $H_0^1(\Omega)$ onto

$H^{-1}(\Omega)$. See further details in Section 7. The same reasons explain the nonexistence results for the equation $\Delta u + \lambda^* f(u) + c = 0$ with $c > 0$. This time the Inverse Function Theorem fails.

These results are complemented by the existence of weak solutions which are not energy solutions. We note that they are not extremal solutions even if in some cases they satisfy condition (1.6). Their existence shows that condition (1.6) has to be applied to energy solutions, even if it makes sense for all solutions. See details at the end of Section 6. These examples of singular non-energy solutions are *isolated objects*, not accessible as limits of regular solutions. The existence and the role in the general theory of such solutions is not understood at this time.

We conclude in Section 8 with a list of some striking open problems.

2 Preliminaries

As we have mentioned, Problem (E) admits classical solutions for every $0 < \lambda < \lambda^*$. We summarize here the main properties, most of them well-known, see e.g. [BN1], that will be used below. Thus, for every λ in the range $0 \le \lambda < \lambda^*$ a classical solution exists which is *minimal* among all possible solutions; let us call it $\underline{u}_\lambda(x) \in C^2(\overline{\Omega})$. The family (*branch*) of such solutions depends smoothly and monotonically on f and λ, and in particular

$$\lambda < \lambda' \Longrightarrow \underline{u}_\lambda(x) < \underline{u}_{\lambda'}(x). \tag{2.1}$$

Lemma 2.1. *Minimal solutions are stable, i.e., the linearized operator*

$$L_\lambda(v) = -\Delta v - \lambda\, f'(\underline{u}_\lambda)v \tag{2.2}$$

has a positive first eigenvalue

$$\mu_1(L_\lambda) = \inf_{u \in H_0^1(\Omega)} \frac{\int_\Omega \{|\nabla \phi|^2 - \lambda\, f'(\underline{u}_\lambda)\phi^2\}\,dx}{\int_\Omega \phi^2\,dx} \tag{2.3}$$

for every $0 < \lambda < \lambda^*$. *Moreover,* $\mu_1(L_\lambda)$ *is a decreasing function of* λ.

Proof. The first assertions, including the variational characterization (2.3) are well-known. The fact that $\mu_1(L_\lambda)$ is decreasing in λ follows easily from the variational characterization of μ_1 and the convexity of f. ∎

Lemma 2.2. *Under condition (1.3) on f we obtain as $\lambda \to \lambda^*$ a finite limit a.e.*

$$u^*(x) = \lim_{\lambda \to \lambda^*} \underline{u}_\lambda(x), \tag{2.4}$$

and $u^* \in L^1(\Omega)$ is a weak solution of problem (E_{λ^*}).

We emphasize the role of condition (1.3) in this result. Thus, if we take a linear function, $f(u) = a + bu$, with $a, b > 0$, problem (E) admits a unique branch of (classical) solutions which in the limit $\lambda \to \lambda^*$ (which equals $\mu_1(-\Delta/b)$) gives a limit u^* which is infinity everywhere in Ω, i.e. there is no extremal solution.

The limit, or extremal solution, can be either classical or singular. In many cases u^* is a classical solution of (E_{λ^*}), see examples below. The following result characterizes the classical extremal solution among all classical solutions.

Lemma 2.3. *The linearized operator*

$$L_{ex}(v) = -\Delta v - \lambda^* f'(u^*)v, \tag{2.5}$$

corresponding to a classical extremal solution has zero first eigenvalue. Moreover, λ^* is a turning point for the $(\lambda\text{-}u)$ diagram.

A *turning point* means that there exists a parametrized family of (classical) solutions

$$s \mapsto (\lambda(s), u(x; s)), \qquad s \in (-\varepsilon, \varepsilon), \tag{2.6}$$

with $\lambda(0) = \lambda^*$ and $\lambda(s) < \lambda^*$ both for $s < 0$ and $s > 0$. The former branch coincides with the minimal solutions while for $s > 0$ we obtain a branch of non-minimal solutions which emanates from (λ^*, u^*). Depending on n, Ω and f this branch is continued in different ways as the examples show. The fact that

$$\mu^* = \mu_1(L_{ex}) = 0, \tag{2.7}$$

follows from a simple argument. On the one hand, $\mu_1(L_\lambda) > 0$ on the minimal branch for $\lambda < \lambda^*$, so that in the limit $\mu^* \geq 0$. On the other hand, if $\mu^* > 0$ the Implicit Function Theorem could be applied and would allow to continue the branch $\lambda \mapsto \underline{u}_\lambda$ in a classical way beyond λ^*. [Note: we use the notation L_{ex} instead of L^* to avoid confusions].

Lemma 2.4. *At a non-minimal classical solution v of (E_λ) with $0 < \lambda < \lambda^*$ the linearized operator*

$$L_v(w) = -\Delta w - \lambda f'(v)w \tag{2.8}$$

has negative first eigenvalue.

Proof. Suppose by contradiction that $\mu_1(L_v) \geq 0$. Then for every $\phi \in H_0^1(\Omega)$ we have

$$(L_v\phi, \phi) = \int (|\nabla\phi|^2 - \lambda f'(v)\phi^2)\, dx \geq 0. \tag{2.9}$$

Let now $u = \underline{u}_\lambda$ be the minimal solution with the same λ. We have $v \geq u$ and

$$-\Delta(v-u) - \lambda f'(v)(v-u) = \lambda[f(v) - f(u) - f'(v)(v-u)] \leq 0.$$

Hence, putting $\phi = v - u$ in (2.9) we get

$$\lambda \int_\Omega [f(v) - f(u) - f'(v)(v-u)](v-u)\, dx \geq 0.$$

Since f is convex the integrand is nonpositive, so that the inequality is only possible if

$$f(v) = f(u) + f'(v)(v-u) \quad \text{a.e. in } \Omega. \tag{2.10}$$

Now, when f is strictly convex we immediately conclude that $v = u$, hence v is the minimal solution, which is impossible. When f is not strictly convex the same conclusion is obtained as follows: in case $v \neq u$ the function f must be necessarily linear in any interval of the form $[u(x), v(x)]$, hence in the union of such intervals which is an interval. Then both u and v are solutions of a linear problem with $f(u) = a + bu$, i.e.,

$$-\Delta u = \lambda(a + bu),$$

for which uniqueness is known.

∎

Remarks. 2.5. a) The above results completely characterize the classical branches in terms of linearized stability. Thus, μ_1 is positive on the minimal branch, it is zero for the extremal (turning) point and negative for the non-minimal branches. Such a classification fails for singular solutions.

b) Lemma 2.4 holds for energy solutions, and this fact will be useful in the next section.

Finally, we point out that turning-point solutions are unique.

Lemma 2.6. *If u^* is classical there is a unique solution of (E_{λ^*}) even in the weak sense.*

Proof. Consider the classical and positive solution ϕ of the problem

$$-\Delta\phi = \lambda^* f'(u^*)\phi.$$

If u^* is the minimal classical solution and $v \geq u^*$ is any weak solution we have

$$\Delta(v-u^*)\phi + \lambda^*[f(v) - f(u^*)]\phi = 0.$$

Integating in Ω we get

$$\int (v-u^*)\Delta\phi\, dx + \lambda^* \int (f(v) - f(u^*))\phi\, dx = 0.$$

Thanks to the definition of ϕ we get

$$\lambda^* \int \phi\, [f(v) - f(u^*) - f'(u^*)(v-u^*)]\, dx = 0.$$

Since the integrand is nonnegative we conclude that $f(v) = f(u^*) + f'(u^*)(v-u^*)$ a.e. in Ω. If $v \in L^\infty(\Omega)$ then v is smooth and we conclude as above that $v = u^*$. Otherwise, there is a sequence $\{x_n\}$ such that $v(x_n) \to +\infty$. Since f is linear on $[u^*(x_n), v(x_n)]$ we conclude that f is linear on $[A, +\infty)$ for some A. This contradicts (1.3).

■

Remark 2.7. A delicate result of Martel [Mr] extends Lemma 2.6. It says that (E_{λ^*}) has always a unique solution, even if u^* is merely a weak solution. We shall use this fact in the next section.

3 Extremal solutions

The present work is motivated by the existence of well-known examples of singular solutions which are in some cases extremal, in other cases not. It is therefore convenient to investigate the properties of such solutions in a general setting.

Our main result characterizes the singular extremal solutions in the energy class.

Theorem 3.1. *Assume that $v \in H_0^1(\Omega)$ is an unbounded weak solution of (E_λ) for some $\lambda > 0$ (in the sense of (1.5)). Assume that*

$$\lambda \int_\Omega f'(v)\phi^2\,dx \le \int_\Omega |\nabla \phi|^2\,dx \tag{3.1}$$

for all $\phi \in C_0^1(\Omega)$. Then $\lambda = \lambda^*$ and $v = u^*$. Conversely, (3.1) holds for $\lambda = \lambda^*$ and $v = u^*$.

Proof. We begin by recalling that the extremal solution u^* is the increasing limit of classical solutions $\underline{u}_\lambda$ with positive first eigenvalue, hence

$$\lambda \int_\Omega f'(\underline{u}_\lambda)\phi^2\,dx \le \int_\Omega |\nabla \phi|^2\,dx,$$

which in the limit gives (3.1).

Let us prove the converse. We have an unbounded energy solution satisfying (3.1) and we want to conclude that it is the extremal solution u^*. We first recall (see [B4]) that no weak solution exists for $\lambda > \lambda^*$. Next we exclude the possibility $\lambda < \lambda^*$. We observe that, by a density argument plus Fatou's theorem, inequality (3.1) holds for every $\phi \in H_0^1(\Omega)$. Taking $\phi = v - \underline{u}_\lambda$ (it is at this stage that we use the assumption that $v \in H_0^1(\Omega)$) we get

$$\lambda \int_\Omega f'(v)(v - \underline{u}_\lambda)^2\,dx \le \int_\Omega |\nabla(v - \underline{u}_\lambda)|^2\,dx,$$

and we conclude as in the proof of Lemma 2.4 that

$$f(v) = f(\underline{u}_\lambda) + f'(v)(v - \underline{u}_\lambda).$$

Since $v \in L^\infty$ this yields a contradiction as in the proof of Lemma 2.6.

Hence, we have proved that $\lambda = \lambda^*$. We conclude that $v = u^*$ by invoking the uniqueness result of Martel [Mr] which says that (E_{λ^*}) has a unique weak solution.

■

Remark 3.2. Assumption (3.1) makes sense for a general solution (not necessarily in H^1). One may think that Theorem 3.1 still holds for a general weak solution. However, this is **not true**, see the example in Theorem 6.2. In that direction we also have

Remark 3.3. *Under the extra condition*

$$\liminf_{s\to\infty} \frac{f'(s)s}{f(s)} > 1, \tag{3.2}$$

any extremal solution lies in the energy class.

Proof. For all the solutions $\underline{u}_\lambda$ in the minimal branch we have

$$\lambda \int_\Omega f'(\underline{u}_\lambda)\, \underline{u}_\lambda^2 \, dx \le \int_\Omega |\nabla \underline{u}_\lambda|^2 \, dx \le \lambda \int_\Omega f(\underline{u}_\lambda)\, \underline{u}_\lambda \, dx.$$

As a consequence of (3.2) we also have

$$(1+\varepsilon) f(s) s \le f'(s) s^2 + C$$

for some $\varepsilon > 0$. It follows from both formulas that

$$\int_\Omega f(\underline{u}_\lambda)\underline{u}_\lambda \, dx \le C, \quad \int_\Omega |\nabla \underline{u}_\lambda|^2 \, dx \le C$$

with a constant C independent of λ, hence the estimates are valid for $u^* = \lim \underline{u}_\lambda$.

■

We do not know of any example where the extremal solution is not in the energy class; see Open Problem 1.

Remark 3.4. Consider

$$\mu_1^* = \lim_{\lambda \uparrow \lambda^*} \mu_1(L_\lambda).$$

As we have already pointed out, if u^* is a classical solution, then $\mu_1^* = 0$. However, if u^* is singular it may happen that $\mu_1^* > 0$ (see e.g. Theorems 5.1, 5.3 and 6.1). In that case, it has been proved in [CM] that any number $\mu \in [0, \mu_1^*]$ is an "eigenvalue" of L_{ex} associated to a weak eigenfunction $\phi \in L^1$, $\phi \geq 0$, $\phi \not\equiv 0$, with $f'(u^*)\phi \in L^1$, such that

$$-\int_\Omega \phi \, \Delta \zeta \, dx - \lambda^* \int_\Omega f'(u^*) \, \phi \, \zeta \, dx = \mu \int_\Omega \phi \, \zeta \, dx$$

for every $\zeta \in C^2(\overline{\Omega})$, $\zeta = 0$ on $\partial \Omega$.

4 Hardy inequalities

The basic ingredient in the characterization of singular extremal solutions, inequality (3.1), is actually a version of the Hardy inequality with weight, which says that for certain functions $\omega(x) \in L^1_{loc}(\Omega)$, $\omega \geq 0$, we have for all $\phi \in C_0^1(\Omega)$

$$\int_\Omega \omega(x) \, \phi(x)^2 \, dx \leq \int_\Omega |\nabla \phi(x)|^2 \, dx. \tag{4.1}$$

The classical Hardy inequality (also called the *Uncertainty Principle*) occurs for a weight of the form $\omega(x) = C/|x|^2$ when $n \geq 3$ and then it is well-known that if $0 \in \Omega$ the best value of C is

$$H = \frac{(n-2)^2}{4}, \; n \geq 3, \tag{4.2}$$

It is also known that the best constant is not attained in $H_0^1(\Omega)$. On the other hand, if we take $\omega(x) = C$ we find the standard Poincaré inequality, which is attained at the first eigenfunction of the Laplacian operator in Ω with best constant $C = \mu_1(-\Delta)$.

The Hardy inequality with best constant will play a prominent role in the analysis of the next sections. Actually, we will need an improved version of the classical Hardy inequality which includes the estimate of the error term and generalizes at the same time the Poincaré inequality.

Theorem 4.1 (Improved Hardy Inequality). *For any bounded domain Ω in $\mathbf{R}^n$, any dimension $n \geq 2$ and for every $u \in H_0^1(\Omega)$ we have*

$$\int_\Omega |\nabla u|^2 \, dx \geq H \int_\Omega \frac{u^2}{|x|^2} \, dx + H_2 \left(\frac{\omega_n}{|\Omega|} \right)^{\frac{2}{n}} \int_\Omega u^2 \, dx. \tag{4.3}$$

The value of $H = H(n)$ is given by (4.2). The constant H_2 is the first eigenvalue of the Laplacian in the unit ball in $n = 2$, hence positive and independent of n. Both constants are optimal when Ω is a ball. ω_n denotes the measure of the unit ball.

Proof. (i) The first step is to make a symmetrization that replaces Ω by a ball B_R with the same volume,

$$\omega_n R^n = |\Omega|,$$

and the function u by its symmetric rearrangement. It is well-kown that the rearrangement does not change the L^2-norm, decreases the $H_0^1(\Omega)$ norm and increases the integral $\int (u^2/|x|^2)\, dx$, cf. [B]. Hence, it is enough to prove the result in the symmetric case. Moreover, a simple scaling allows to consider the case $R = 1$.

(ii) The result for $n = 2$ is just the Poincaré inequality with corresponding eigenfunction the Bessel function $J_0(zr)$, where z is the first zero of J_0, i.e., $z \cong 2.4048$. The corresponding eigenvalue is H_2. We have

$$H_2 = z^2 \cong 5.7832. \tag{4.4}$$

(iii) Let us tackle of the main part of the proof, proving the inequality for radial functions in the ball $B = B_1(0)$ in $\mathbf{R}^n$, $n \geq 3$. The basic idea stems from the consideration of why the best constant H is not attained in $H_0^1(B_R)$. If we solve the corresponding Euler-Lagrange equation,

$$\Delta u + H\,|x|^{-2} u = 0,$$

we find the solution

$$\overline{u}(x) = |x|^{-\frac{n-2}{2}}, \tag{4.5}$$

which just does not belong to $H^1(B)$ in $\mathbf{R}^n$ if $n > 2$. Usually, perturbations of (4.5) of the type

$$\overline{u}_\varepsilon(x) = \frac{1}{(\varepsilon + |x|^2)^{(n-2)/4}} - \frac{1}{(\varepsilon + 1)^{(n-2)/4}},$$

are employed to show that H is the best constant. The proof we present uses the function $\overline{u}$ to make a **dimension reduction** of the problem from n to 2 dimensions as follows. We define the new variable

$$v(r) = u(r) r^{(n-2)/2}, \quad r = |x|. \tag{4.6}$$

The "magical" computation comes now:

$$\int_B |\nabla u|^2\,dx - H\int_B \frac{u^2}{r^2}\,dx = n\,\omega_n \left[\int_0^1 (v')^2\,r dr - (n-2)\int_0^1 v(r)v'(r)\,dr\right].$$

Taking for instance $u \in C_0^1(B)$ the last integral is zero and we get

$$\int_B |\nabla u|^2\,dx - H\int_B \frac{u^2}{r^2}\,dx = n\,\omega_n \int_0^1 (v'(r))^2\,r dr. \tag{4.7}$$

This is where Poincaré's inequality in two dimensions comes:

$$\int_0^1 (v'(r))^2\,r dr \geq H_2 \int_0^1 v(r)^2\,r dr. \tag{4.8}$$

We finally observe that

$$\int_B u^2(x)\,dx = n\,\omega_n \int_0^1 v(r)^2\,r dr. \tag{4.9}$$

The last remark consists in removing the restriction $u \in C_0^1(B)$ and this is done by density.

∎

Remark 4.2. The existence of a correction term in (4.3) explains in a concrete way why the best constant H in the classical inequality is not attained. On the other hand, H_2 is not attained either, since, by (4.7), it would correspond to equality in (4.8), which happens precisely for $v(r) = c\,J_0(zr)$, hence

$$u(x) = c\,\frac{J_0(zr)}{r^{(n-2)/2}},$$

which is not in $H^1(B)$. It seems interesting to obtain further correction terms improving formula (4.3). See Open Problem 2 in Section 8.

Extension 4.3. Theorem 4.1 is reminiscent of the improved Sobolev inequality with best constant; see [BN1] (§5 in Section 1.3) and also [BL]. In fact, the proof of Theorem 4.1 yields a stronger inequality

$$\int_\Omega |\nabla u|^2\,dx \geq H\int_\Omega \frac{u^2}{|x|^2}dx + \alpha_p\|u\|_p^2. \tag{4.10}$$

for every $u \in H_0^1$ and every $p < 2n/(n-2)$, with $\alpha_p > 0$.

Proof. We have to estimate more precisely the integral appearing in the first member of (4.8). By the Sobolev inequality we have for every $q < \infty$

$$\int_0^1 v'^2(r)\,r dr \geq C_q\Big(\int_0^1 v^q(r)\,r dr\Big)^{2/q} = C_q\Big(\int_0^1 u^q\, r^{(n-2)q/2} r dr\Big)^{2/q}. \tag{4.11}$$

On the other hand, the p-norm of u is

$$\|u\|_p^p = n\,\omega_n \int_0^1 |u(r)|^p r^{n-1} dr. \tag{4.12}$$

We want to relate both quantities for suitable p and q, $1 < p < q < \infty$. For that we write for some $\alpha > 0$

$$\begin{aligned}\int_0^1 |u|^p r^{n-1} dr &= \int_0^1 (|u|^p r^{\alpha})(r^{2-n-\alpha}) r dr \\ &\leq \left(\int_0^1 |u|^q r^{\alpha q/p} r dr\right)^{p/q} \left(r^{(n-2-\alpha)\,\gamma}\right)^{1/\gamma},\end{aligned} \tag{4.13}$$

where we have used Hölder with γ given by $(1/\gamma) + (p/q) = 1$; $\gamma > 1$ since $p < q$. In view of (4.11), (4.12) we need to choose α so that $\alpha q/p = (n-2)q/2$, i.e.

$$\alpha = \frac{p(n-2)}{2}. \tag{4.14}$$

The last factor of (4.13) is finite if

$$(n-2-\alpha)\,\gamma > -2. \tag{4.15}$$

This choice is possible if $\alpha < n$, i.e., if $p < 2n/(n-2)$. Summing up, for any given $p < 2n/(n-2)$ we define α by (4.14) and then $\gamma > 1$ satisfying (4.15). This defines $q > p$ and then (4.13) implies that

$$\int_0^1 v'^2(r)\,r dr \geq \alpha_p\Big(\int_B |u|^p dx\Big)^{2/p}$$

for some $\alpha_p > 0$.

■

Let us finally point out that the dimension reduction is a technique with precedents in the study of Schrödinger equations, cf. e.g. [RS] or [VY], but the present use seems completely different.

5 The exponential case

The results of Sections 3 and 4 allow for a quick analysis of the singular extremal solutions in the exponential case $f(u) = e^u$ posed in a ball, $\Omega = B_1(0)$. This is the most popular example, motivated by the problems of combustion theory and the λ-u diagram was studied in the classical papers by Gelfand [G] and Joseph-Lundgren [JL]. We recall that our functional approach is not confined to radially symmetric solutions, which have been more investigated. We start our review from the fact that there exists an explicit weak solution

$$U(x) = -2\log r, \quad r = |x|, \tag{5.1}$$

which is obviously in $H_0^1(B)$ for $n \geq 3$. It corresponds to the value of the parameter

$$\lambda^\sharp = 2(n-2). \tag{5.2}$$

The linearized operator is

$$L_\sharp \phi = -\Delta\phi - \frac{2(n-2)}{r^2}\phi. \tag{5.3}$$

Theorem 3.1 asserts that U is the extremal solution if and only if (3.1) holds, i.e., if

$$2(n-2)\int_B \frac{\phi^2}{r^2}\,dx \leq \int_B |\nabla\phi|^2\,dx, \tag{5.4}$$

According to Section 4 this inequality holds precisely if

$$2(n-2) \leq H = \frac{(n-2)^2}{4}, \tag{5.5}$$

i.e., if $n \geq 10$. Then $\lambda^* = 2(n-2)$ and $u^* = U$. We recall that on the other hand, if $n \leq 9$ the extremal solution u^* is smooth for any domain Ω; see [MP2]. Therefore, U cannot be the extremal solution. When $3 \leq n \leq 9$, $\lambda^* > \lambda^\sharp$ and the weak solution U lies at the "end" of the curve of unstable solutions, cf. [JL].

Let us discuss now the existence of a first eigenvalue of the linearized operator at the singular extremal solution. The analysis is different in dimensions $n \geq 11$ and $n = 10$.

Theorem 5.1. *Let $n > 10$. Then the linearized operator L_{ex} has a positive first eigenvalue μ_1^* corresponding to an eigenfunction $\phi \in H_0^1(B) \cap H^2(B)$. This eigenvalue is the limit of the eigenvalues of the operators L_λ as $\lambda \to \lambda^*$.*

Proof. As was remarked in [PV], where the dynamic stability analysis is performed, inequality (5.5) is strict, hence the bilinear form associated to L_{ex} is coercive in $H_0^1(B)$ and there is a bounded inverse operator from $H^{-1}(B)$ to $H_0^1(B)$. Moreover, the second-order Hardy estimates

$$\int_B \frac{u^2}{r^4}\, dx \le C \int_B (\Delta u)^2\, dx, \tag{5.6}$$

proved in [D4, Appendix] for $n > 4$, show that this inverse is well-defined from $L^2(B)$ into $H^2(B) \cap H_0^1(B)$. In order to characterize the first eigenvalue we consider the variational inequality

$$\int_B |\nabla \phi|^2\, dx - \lambda^* \int_B f'(u^*)\, \phi^2\, dx = \int_B (|\nabla \phi|^2 - 2(n-2)\frac{\phi^2}{r^2})\, dx \ge$$
$$\ge (1 - \frac{2(n-2)}{H}) \int_B |\nabla \phi|^2\, dx,$$

which for $\int \phi^2\, dx = 1$ gives

$$\mu_1^* \ge \frac{n-10}{n-2} \int_B |\nabla \phi|^2\, dx = \frac{n-10}{n-2}\, \mu_1(-\Delta).$$

The fact that the decreasing sequence $\mu_1(L_\lambda)$ converges towards μ_1^* comes from the monotone convergence and is left as an exercise to the reader.

■

Remark 5.2. Despite the fact that L_{ex} is formally invertible (e.g., from H_0^1 onto H^{-1}) one **cannot** apply the Implicit Function Theorem or the Inverse Function Theorem, see Section 7.

The case $n = 10$ is somewhat different.

Theorem 5.3. *For $n = 10$ the linearized operator L_{ex} does not have a first eigenfunction in $H_0^1(\Omega)$. However, the previous calculation gives a positive value for μ_1^* defined now as*

$$\mu_1^* = \lim_{\lambda \to \lambda^*} \mu_1(L_\lambda).$$

Actually, $\mu_1^* = H_2 > 0$.

Proof. We have for every $\phi \in H_0^1$ and every $\lambda < \lambda^*$

$$\int_B |\nabla \phi|^2 \, dx - \lambda \int_B f'(\underline{u}_\lambda)\, \phi^2 \, dx \geq \mu_1(L_\lambda) \int_B |\phi|^2 \, dx.$$

Passing to the limit as $\lambda \to \lambda^*$ we find

$$\int_B |\nabla \phi|^2 \, dx - \lambda^* \int_B \frac{\phi^2}{r^2} \, dx \geq \mu_1^* \int_B |\phi|^2 \, dx,$$

with $\lambda^* = 2(n-2) = 16 = (n-2)^2/4$. Since H_2 is the optimal constant in the Improved Hardy Inequality we see that $\mu_1^* \leq H_2$. On the other hand, we have

$$\underline{u}_\lambda \leq u^* = \log(1/r^2),$$

and thus $\lambda f'(\underline{u}_\lambda) \leq \lambda^* / r^2$. Consequently,

$$\int_B |\nabla \phi|^2 \, dx - \lambda \int_B f'(\underline{u}_\lambda)\, \phi^2 \, dx \geq \int_B |\nabla \phi|^2 \, dx - \lambda^* \int_B \frac{\phi^2}{r^2} \, dx \geq H_2 \int_B \phi^2 \, dx.$$

Hence, $\mu_1(L_\lambda) \geq H_2$ and passing to the limit as $\lambda \to \lambda^*$ we find that $\mu_1^* = H_2$.

∎

The behaviour of the limit of the first eigenvalues as we approach the singular extremal solution is in contrast with the behaviour near the classical extremal solutions (turning points), where the limit value is zero (cf. Lemma 2.3), and even more in contrast with the behaviour as we approach singular non-extremal solutions, like solution (5.1) for $3 \leq n \leq 9$. In that case $\lambda = 2(n-2)$ is less than λ^* and a branch of classical, unstable solutions $(\lambda_t, u_t(x))$ meanders up in the λ-u diagram towards $(2(n-2), U)$, cf [G], [JL].

Theorem 5.4. *Let* $3 \leq n \leq 9$ *and let* L_{u_t} *be the linearized operator at* (λ_t, u_t). *If as* $t \to \infty$ *we have* $\lambda_t \to 2(n-2)$ *and* $u_t \to U$ *then*

$$\lim_{t\to\infty} \mu_1(L_{u_t}) = -\infty. \tag{5.7}$$

Proof. Since now $\lambda^\sharp = 2(n-2) > H$ there exist functions $\phi \in H_0^1(B)$ such that

$$\int_B \phi^2\,dx = 1, \quad \int_B |\nabla \phi|^2\,dx - \lambda^\sharp \int_B \frac{\phi^2}{r^2}\,dx \le -\varepsilon < 0.$$

We can scale one such function in the form

$$\phi_k(x) = \begin{cases} k^{n/2}\phi(kx) & \text{if } |x| < \frac{1}{k} \\ 0 & \text{if } \frac{1}{k} \le |x| \le 1, \end{cases}$$

for some large $k > 1$. Then $\int \phi^2\,dx$ does not change but

$$\int_B |\nabla \phi_k|^2\,dx - \lambda^\flat \int_B \frac{\phi_k^2}{r^2}\,dx \le -\varepsilon k^2,$$

which can be made as negative as we please. In order to complete the proof we now observe that $u_t(x)$ converges nicely to $U(x)$, cf. [JL], so that by approximation as before

$$\mu_{1(}L_t) \to -\infty.$$

■

The last result displays the extreme instability of the singular non-extremal solutions just considered. Figure 1 displays the variation of u with λ for $n = 3, 6, 10, 11$. Previous results, mostly confined with radially symmetric solutions, can be found in [BS] or [RS].

6 Power case

We now consider the case $f(u) = (1+u)^p$ with $p > 1$ and $\Omega = B_1(0)$, a ball. In dimensions $n = 1, 2$ we have a diagram with a classical turning point and exactly two solutions for every $\lambda \in (0, \lambda^*)$. The same happens for $n > 2$ and

$$1 < p \le p_s = \frac{n+2}{n-2}, \tag{6.1}$$

cf. [JL]. For $n > 2$ and $p > n/(n-2)$ we find the explicit weak solution

$$U(x) = |x|^{-\frac{2}{p-1}} - 1, \tag{6.2}$$

corresponding to parameter

$$\lambda^\sharp = \frac{2}{p-1}\left(n - \frac{2p}{p-1}\right). \tag{6.3}$$

We always have $U \in W_0^{1,1}(B)$, but $U \in H_0^1(B)$ only if $p > p_s$, i.e., if $n-2 > 4/(p-1)$. The linearized operator at this solution is

$$L_U = -\Delta - \lambda^\sharp f'(U) = -\Delta - \frac{c_{n,p}}{r^2}, \quad c_{n,p} = \frac{2p}{p-1}\left(n - \frac{2p}{p-1}\right), \tag{6.4}$$

the same type of Laplace operator with inverse-square potential as in the exponential case. We wish to understand where the pair $(\lambda^\sharp, U)$ lies with respect to the curve of classical solutions. For this purpose we have to distinguish 3 cases.

Case 1.

$$n - 2 \geq F(p) = \frac{4p}{p-1} + 4\sqrt{\frac{p}{p-1}}. \tag{6.5}$$

Case 2.

$$\frac{4}{p-1} < n - 2 < F(p). \tag{6.6}$$

Case 3.

$$\frac{2}{p-1} < n - 2 \leq \frac{4}{p-1}. \tag{6.7}$$

Let us now proceed with the separate analysis of these cases.

Case 1. Condition (6.5) holds if and only if

$$n > 10 \qquad \text{and } p \geq p_u = \frac{n - 2\sqrt{n-1}}{n - 4 - 2\sqrt{n-1}}. \tag{6.8}$$

In this case the main result is

Theorem 6.1. *Assume (6.5). Then $\lambda^\sharp = \lambda^*$ and $U = u^*$. Moreover, the operator L_U is coercive and if the inequality in (6.5) is strict then L_U has a positive first eigenvalue. In the critical case we still have*

$$\lim_{\lambda \to \lambda^*} \mu_1(L_\lambda) > 0. \tag{6.9}$$

Proof. We apply Theorem 3.1. Note that $U \in H_0^1$ since $n-2 > 4/(p-1)$ and that $c_{n,p} \le H$, i.e.,

$$\frac{2p}{p-1}\left(n - \frac{2p}{p-1}\right) \le \frac{(n-2)^2}{4}.$$

This case is similar to the case where $f(u) = e^u$ and $n \ge 10$.

Case 2. If $(n-2) < F(p)$ holds (i.e., if $n \le 10$ and any p, or if $n > 10$ and $p < p_u$), it is known from the results of [MP2] that the extremal function u^* is smooth. Therefore, U cannot be the extremal function, hence $\lambda^\sharp < \lambda^*$ (by [B4]). If (6.6) holds the results of [JL] assert that the branch of classical unstable solutions (λ_t, U_t) meanders up in the $(\lambda\text{-}u)$ diagram towards $(\lambda^\sharp, U)$. The situation is similar to the one we have encountered in the exponential case with $3 \le n \le 9$. Here also

$$\lim_{t\to\infty} \mu_1(L_{u_t}) = -\infty.$$

Case 3. Recall that since $p \le (n+2)/(n-2)$, problem (E_λ) has exactly two classical solutions for every $\lambda \in (0, \lambda^*)$ and it has one classical solution, namely u^*, at $\lambda = \lambda^*$. Thus, $\lambda^\sharp < \lambda^*$.

Since $n - 2 \le 4/(p-1)$ the weak solution U **does not** belong to H^1. Here, it may or may not satisfy condition (3.1):

Case 3A. If

$$2/(p-1) < n - 2 \le \frac{4p}{p-1} - 4\sqrt{\frac{p}{p-1}},$$

or, in other words, if

$$\frac{n}{n-2} < p \le \tilde{p} = \frac{n + 2\sqrt{n-1}}{n - 4 + 2\sqrt{n-1}},$$

then (3.1) holds for $v = U$.

Case 3B. If

$$\frac{4p}{p-1} - 4\sqrt{\frac{p}{p-1}} < n - 2 \le \frac{4}{p-1},$$

or, in other words, if

$$\tilde{p} < p \le \frac{n+2}{n-2},$$

then (3.1) fails for $v = U$.

We may now state

Theorem 6.2. *For every p in the range (6.7), i.e.,*

$$\frac{n}{n-2} < p < p_s = \frac{n+2}{n-2}, \quad n \geq 3, \tag{6.10}$$

there exists a weak solution of problem (E) with λ given by (6.3) which is not an energy solution and hence not an extremal solution, even if in the subrange

$$\frac{n}{n-2} < p \leq \tilde{p}, \tag{6.11}$$

condition (3.1) is satisfied.

What is most remarkable about these weak solutions is that they cannot be approached by the branch of classical solutions; they are not limiting singular solutions in the terminology of [Ch] and [JL, Section III], i.e., limits of regular solutions, which leaves them in a kind of "limbo" with respect to the classical theory. The existence of additional weak solutions is an interesting open problem, see Problem 7 in Section 8.

Figure 2 summarizes graphically the three cases discussed in this section. From left to right: Case 1 for $n = 11$ and $p = 9$, Case 2 for $n = 3$ and $p = 8$, and Case 3A for $n = 3$ and $p = 4$.

7 The "failure" of the Inverse and Implicit Theorems

We return now to some peculiar properties associated with singular extremal solutions. For simplicity we consider just the case where $\Omega = B_1(0)$ and $f(u) = e^u$ in dimension $n \geq 11$. A similar phenomenon occurs when $f(u) = (1+u)^p$ and $n - 2 > F(p)$.

As we have observed in Theorem 6.1 the extremal solution u^* coincides with $U(x) = -2\log|x|$ and corresponds to $\lambda^* = 2(n-2)$. Moreover, the linearized operator

$$L_{ex} = -\Delta - \lambda^* \, f'(u^*) = -\Delta - \frac{2(n-2)}{r^2}$$

is coercive, hence bijective for example from H_0^1 onto H^{-1}. In view of the Implicit Function Theorem one might have expected Problem (E_λ) to have a solution "close" to u^* for every λ near λ^*. Of course, we cannot apply the Implicit Function Theorem to $F(u,\lambda) = -\Delta u - \lambda\, e^u$ in the space $C_0^{2,\alpha}$ for u^* does not lie in $C^{2,\alpha}$. Since $u^* \in H_0^1$ one might then try to consider F as a map from $H_0^1 \times \mathbf{R}$ into H^{-1}. But again this does not make sense since e^u need not be in H^{-1} for u in H_0^1 near u^*. In fact, there is no appropriate functional setting since we know that (E_λ) has no solution even in the weak sense for any $\lambda > \lambda^*$, see [B4].

Similarly, there is a "failure" of the Inverse Function Theorem. Consider for example the problem

$$\begin{aligned} -\Delta u &= \lambda^* e^u + c && \text{in } \Omega = B_1(0). \\ u &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{7.1}$$

where c is a constant. If $c = 0$ problem (7.1) admits the solution u^* and the linearized problem at u^* is formally bijective. Thus, it seems reasonable to expect that for every $c \in \mathbf{R}$ with $|c|$ small there is a solution u of (7.1) near u^*. This is indeed true for $c < 0$ small as may be shown using the methods of [B4]. However, we have

Theorem 7.1. *Problem (7.1) has no solution if $c > 0$.*

Proof. Suppose that there exists a weak solution for some $c > 0$. Using Lemmas 4 and 6 of [B4] we construct a *bounded* supersolution for the problem

$$\begin{cases} -\Delta v = \lambda^* e^v + c - \varepsilon & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega \end{cases}$$

for any $\varepsilon \in (0, c]$. Since $u = 0$ is a subsolution there must be a bounded positive solution. In particular, for $\varepsilon = c$ there must be a bounded extremal solution. This is impossible since the extremal solution u^* is unbounded.

∎

8 Open problems and further developments

We summarize some of the problems that have arisen or would be interesting to consider in connection with the results of this paper.

Problem 1. We have shown in Section 3 that, under a mild extra condition on f at infinity, all singular extremal solutions are energy solutions. The question is, does there exist some f and Ω for which the extremal solution is a weak solution, not in $H_0^1(\Omega)$?

Problem 2. In Section 4 we have introduced an "Improved Hardy Inequality". For example in the ball $B_1(0)$

$$-\Delta \geq \frac{(n-2)^2}{4|x|^2} + H_2,$$

where H_2 iste firteigenaue ofΔ in 2-D. Are these just the first two terms of a series? Is there a further improvement in the direction of extension 4.3?

Problem 3. Assume that Ω is a bounded smooth convex set in $\mathbf{R}^n$ with $n \geq 10$. Let $f(u) = e^u$. Is u^* always unbounded?

Problem 4. Assume $\Omega = B_1(0)$ in $\mathbf{R}^3$. Are there examples of smooth convex functions f for which u^* is unbounded?

Problem 5. For singular radial solutions in a ball the singular set of u^* is just a point, the origin (this follows from the fact that $u = \lim \underline{u}_\lambda$ and $\underline{u}_\lambda$ is radial decreasing in r). What can we say about the blow-up set when Ω is not a ball? Is it a finite set? Is it a single point for convex domains? What is the behaviour of $f'(u^*)$ near the singularities? Does it look like C/r^2?

Problem 6. Construct simple examples in 3-D domains where the Inverse Function Theorem "fails" (in the sense of Section 7) despite the fact that the linearized operator is formally bijective. For example, is the problem

$$-\Delta u = \frac{u^2}{r^2} + c \qquad \text{in } B_1(0), \qquad u = 0 \qquad \text{on } \partial B_1(0),$$

solvable in the weak sense for $|c|$ small?. (See added in proofs).

Problem 7. When $\Omega = B_1$, $n \geq 3$ and $f(u) = e^u$ we have a weak solution U of (E_λ) for the special value $\lambda = \lambda^\sharp = 2(n-2) \leq \lambda^*$. Are there other radial (resp. nonradial) weak solutions for different values of λ? for every $\lambda \in (0, \lambda^*)$? Similar question when f is a power?

H. Matano [Ma] has informed us that he is able to construct nonradial singular solutions of the exponential problem in dimension $n = 3$ with a logarithmic singularity near 0. Nonradial singular solutions have been obtained by Y. Rebaï [R] for f a power with any exponent $p \in (3, \infty)$ when $n = 3$ and $p \in (\frac{n}{n-2}, \frac{n+1}{n-3}]$ when $n \geq 4$.

Problem 8. Is it possible to prove that every singular energy solution has generalized first eigenvalue $\mu = -\infty$ if it is not extremal?

Problem 9. Dynamical instability. We have already mentioned that the singular non-extremal solutions of Sections 5, 6 are extremely unstable. On the other hand, it is rather standard that a classical extremal solution (turning-point solution) has *lateral stability* in the dynamical sense. Namely, if we consider the evolution equation

$$u_t - \Delta u = f(u) \tag{8.1}$$

supplied with zero boundary data

$$u|_S = 0, \tag{8.2}$$

where $S = \Gamma \times (0, T)$, and initial conditions

$$u(x, 0) = u_0(x) \geq 0. \tag{8.3}$$

then, for every initial data $u_0 \geq 0$ such that $u_0 \leq u^*$ the solution $u(x, t)$ of problem (8.1)-(8.3) converges to u^* as $t \to \infty$. However, for data $u_0 \geq u^*$, $u_0 \neq u^*$ there is blow-up in finite time. The study of such Problems was started by Fujita, [Fu].

The phenomenon of stability from below continues to be true for singular extremal solutions, cf. a detailed study in [B4] and in [D4]. As for the instability from above, it has been proved in [PV] in the case of the exponential case, $f(u) = e^u$, that all possible solutions above $U(x)$ **blow up instantaneously** (so that no solution can be defined even for a short time interval; the phenomenon has an obvious physical interpretation in terms of flame ignition [G] since the model is an approximation to the actual equations). This is the strongest form of instability. It is not known whether the result is general for singular extremal solutions or not.

Added in proofs. Problem 6 has been solved by H. Brezis and X. Cabre, (*"Some simple nonlinear PDE's without solutions"*, to appear). They prove that if $n \geq 3$ the equation $-\Delta u = u^2/r^2 + c$ in $B_1(0)$ with $u = 0$ in $\partial b_1(0)$ has *no* weak solution when $c > 0$ and it has a unique solution $u \leq 0$ when $c \leq 0$.

Acknowledgments

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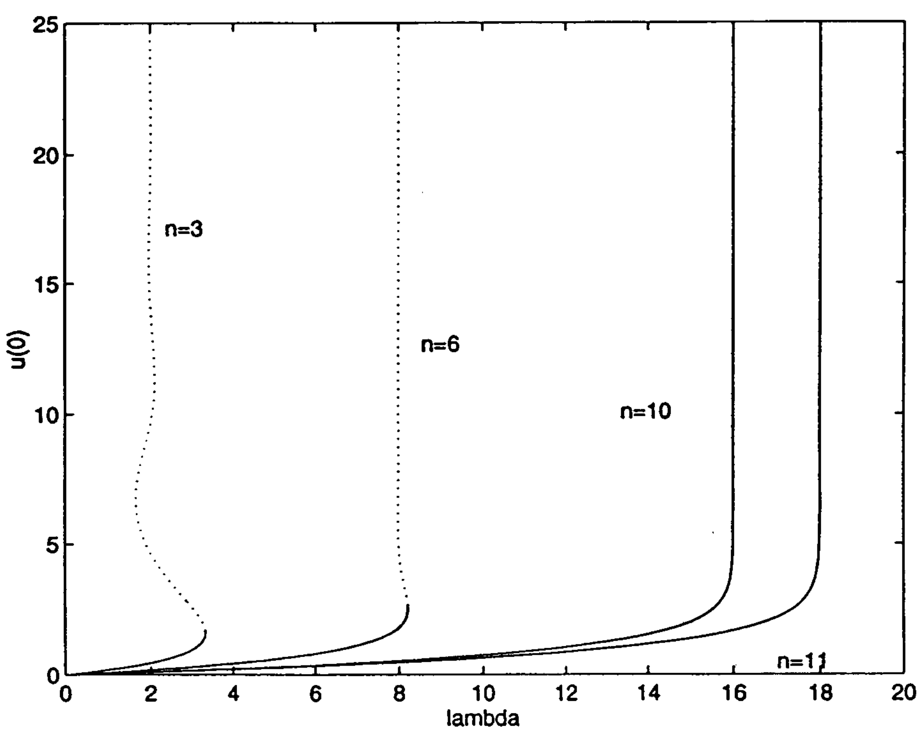

Figure 1

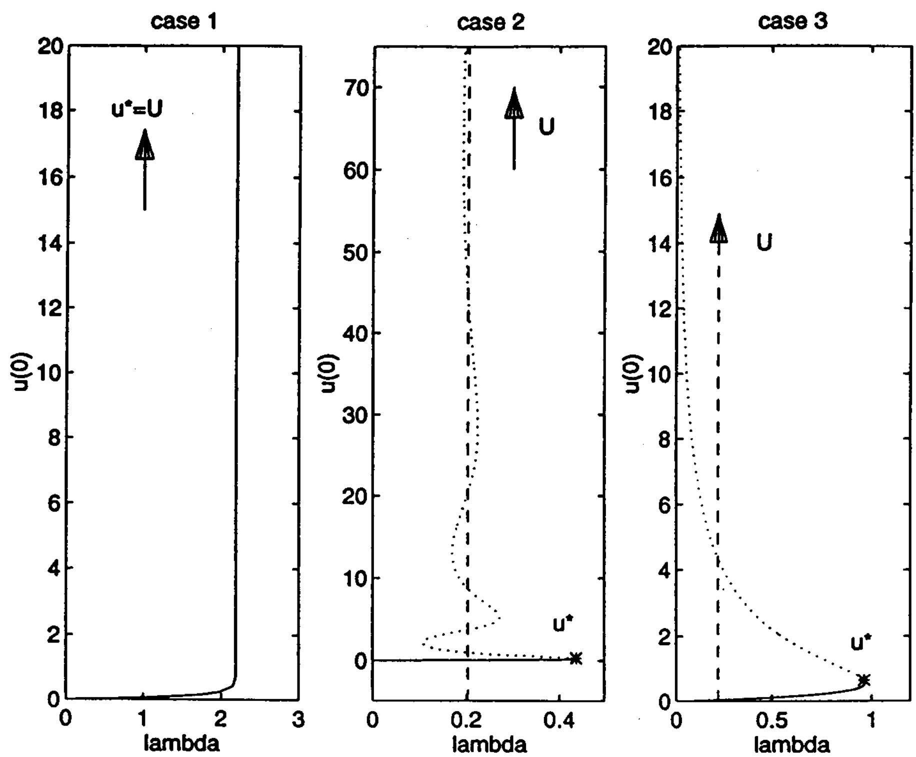

Figure 2

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