

Topology of real algebraic T-surfaces.

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Abstract

The paper is devoted to algebraic surfaces which can be obtained using a simple combinatorial procedure called the T-construction. The class of T-surfaces is sufficiently rich: for example, we construct T-surfaces of an arbitrary degree in $\mathbf{R}P^3$ which are M-surfaces. We also present a construction of T-surfaces in $\mathbf{R}P^3$ with $\dim H_1(\mathbf{R}X; \mathbf{Z}/2) > h^{1,1}(CX)$, where $\mathbf{R}X$ and CX are the real and the complex point sets of the surface.

1 Introduction

The subject of the paper is T-surfaces, i. e. real algebraic surfaces which can be constructed in a simple combinatorial fashion : one can patchwork them from the pieces which essentially are planes.

The construction of combinatorial patchworking (or T-construction) works in any dimension. We restrict ourself here by the case of surfaces. The general T-construction can be formulated in a completely similar way (the combinatorial patchwork construction in the case of curves is described in [I-V], [I1], [I2]). The T-construction is a particular case of the Viro theorem (see [V2], [V3], [V5], [V6], [Ri]).

The results on topology of T-surfaces presented in the paper are concentrated around the following conjecture proposed by O. Viro ([V4]): let X be a nonsingular simply connected compact complex surface with

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an antiholomorphic involution $c : X \rightarrow X$; then $\dim H_1(\mathbf{R}X; \mathbf{Z}/2) \leq h^{1,1}(X)$, where $\mathbf{R}X$ is the fixed point set of the involution c (for a detailed information on real algebraic surfaces see [Kh], [Si], [Wi]).

This conjecture is related to the Ragsdale conjecture (see [Ra]) concerning the topology of real algebraic curves. To formulate the Ragsdale conjecture, let us denote the number of even ovals of a nonsingular real algebraic plane projective curve of degree $2k$ by p (an oval of a nonsingular curve of an even degree is called *even* (resp. *odd*), if it lies inside of even (resp. odd) number of other ovals of this curve), and denote the number of odd ovals by n .

Ragsdale conjecture. For a nonsingular real algebraic plane projective curve of degree $2k$

$$p \leq \frac{3k^2 - 3k + 2}{2}, \quad n \leq \frac{3k^2 - 3k}{2}.$$

Any counter-example to the inequality $p \leq \frac{3k^2 - 3k + 2}{2}$ produces a counter-example to Viro's conjecture: one can take a double plane ramified along the complex point set of a counter-example to the Ragsdale conjecture with an appropriate choice of a lifting of the involution of complex conjugation. Thus, the counter-examples to Ragsdale conjecture obtained in [I1] (see, also, [I2], [I-V]) show that Viro's conjecture is not true. The counter-examples to Ragsdale conjecture are constructed as T-curves. So, it is natural to try to use the combinatorial patchwork construction in order to construct counter-examples to Viro's conjecture which are real algebraic surfaces in $\mathbf{R}P^3$.

We show in sections 3 and 4 that under some conditions of "maximality" of the triangulation participating in the combinatorial patchwork construction, Viro's conjecture is true for the resulting T-surfaces. However, using a "nonmaximal" triangulation (see exact definitions in section 2), we can obtain a T-surface X in $\mathbf{R}P^3$ with $\dim H_1(\mathbf{R}X; \mathbf{Z}/2) > h^{1,1}(\mathbf{C}X)$ (see section 6).

We also construct T-surfaces of any degree in $\mathbf{R}P^3$ which are M-surfaces (it means that the total $\mathbf{Z}/2$ -homology group of the real point set has the same rank as that of the complexification; see section 5).

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2 T-construction

Let m be a positive integer number (it would be the degree of the surface under construction) and T be the tetrahedron in \mathbf{R}^3 with vertices $(0, 0, 0)$, $(0, 0, m)$, $(0, m, 0)$, $(m, 0, 0)$. Let us take a triangulation τ of T with vertices having integer coordinates. Suppose that a distribution of signs at the vertices of τ is given. The sign (plus or minus) at the vertex with coordinates (i, j, l) is denoted by $\delta_{i,j,l}$.

Take the copies

$$T_x = s_x(T), \quad T_y = s_y(T), \quad T_z = s_z(T)$$

$$T_{xy} = s_x \circ s_y(T), \quad T_{xz} = s_x \circ s_z(T), \quad T_{yz} = s_y \circ s_z(T), \quad T_{xyz} = s_x \circ s_y \circ s_z(T)$$

of T , where s_x, s_y, s_z are reflections with respect to the coordinate planes. Denote by T_* the octahedron

$$T \cup T_x \cup T_y \cup T_z \cup T_{xy} \cup T_{xz} \cup T_{yz} \cup T_{xyz}.$$

Extend the triangulation τ to a symmetric triangulation of T_* , and the distribution of signs $\delta_{i,j,l}$ to a distribution at the vertices of the extended triangulation by the following rule: passing from a vertex to its mirror image with respect to a coordinate plane we preserve its sign if the distance from the vertex to the plane is even, and change the sign if the distance is odd.

If a tetrahedron of the triangulation of T_* has vertices of different signs, select a piece of the plane (triangle or quadrangle) being the convex hull of the middle points of the edges having endpoints of opposite signs. Denote by S the union of the selected pieces. It is a piecewise-linear surface contained in T_* . Glue by $s_x \circ s_y \circ s_z$ the facets of T_* . The resulting space \tilde{T} is homeomorphic to the real projective space $\mathbf{R}P^3$. Denote by \tilde{S} the image of S in \tilde{T} .

Let us introduce an additional assumption: the triangulation τ of T is *convex*. This means that there exists a convex piecewise-linear function $\nu : T \rightarrow \mathbf{R}$ whose domains of linearity coincide with the tetrahedra of τ . Sometimes, such triangulations are also called coherent (see [GKZ]) or regular (see [Zi]).

Theorem 2.1 (O. Viro). *Under the assumptions made above on the triangulation τ of T , there exist a nonsingular real algebraic surface X*

of degree m in $\mathbf{R}P^3$ and a homeomorphism $\mathbf{R}P^3 \rightarrow \tilde{T}$ mapping the set of real points $\mathbf{R}X$ of X onto \tilde{S} .

Moreover, a polynomial defining the surface X can be written down explicitly: if t is positive and sufficiently small, the polynomial

$$\sum_{(i,j,l) \in V} \delta_{i,j,l} x_0^i x_1^j x_2^l x_3^{m-i-j-l} t^{\nu(i,j,l)}$$

(where V is the set of vertices of τ) defines a surface with the properties described in Theorem 2.1.

We consider two special types of triangulations of T . A triangulation τ of T is called *primitive* if all the tetrahedra of τ are of volume $1/6$. A T -surface constructed using a primitive triangulation is called *primitive*.

A triangulation τ' of T is called *maximal* if all the integer points of T are vertices of τ' . Clearly, any primitive triangulation is maximal. The notions of primitive and maximal triangulations coincide in dimension 2. The situation is different in dimension 3: there exist maximal triangulations of T which are not primitive.

3 Euler characteristic of T -surface

Let us consider a k -dimensional simplex Q having vertices with integer coordinates and belonging to the orthant $\{x_i \geq 0\}$ of \mathbf{R}^n . We call the simplex Q *elementary* if the reductions modulo 2 of the vertices of Q are independent (generate an affine space of dimension k over $\mathbf{Z}/2$).

Suppose that a distribution of signs at the vertices of the simplex Q is given. Let us take the distributions of signs at the vertices of the symmetric copies of Q using the following generalization of the rule formulated in section 2:

the symmetric copy of a vertex a in an orthant b gets the sign $(-1)^{\vec{a} \cdot \vec{b}} \text{sign}(a)$, where \vec{a} is the reduction modulo 2 of the vertex a ; the i -th coordinate of the vector \vec{b} in $(\mathbf{Z}/2)^n$ is equal to 0 (resp. to 1) if $x_i > 0$ (resp. $x_i < 0$) for a point (x_1, \dots, x_n) in the interior of the orthant b ; and $\vec{a} \cdot \vec{b}$ denotes the standard scalar product of two vectors in $(\mathbf{Z}/2)^n$.

We call a symmetric copy of Q *nonempty* if it has vertices of different signs.

Proposition 3.1. *If the simplex Q is elementary and does not belong to a coordinate hyperplane, then Q has exactly $2^n - 2^{n-k}$ nonempty symmetric copies.*

Proof. Let us, first, remark that the map $\vec{a} \mapsto \vec{a} \cdot \vec{b}$ is linear over $\mathbf{Z}/2$. The following operations do not change the property of any symmetric copy of Q to be nonempty:

- (1) parallel translation of Q ,
- (2) changing of signs at all the vertices of Q .

Thus, we can suppose that the reduction \vec{v}_0 modulo 2 of a vertex v_0 of Q is 0 in $(\mathbf{Z}/2)^n$, and that the vertex v_0 has the sign "+". Denote the other vertices of Q and their reductions modulo 2 by v_1, \dots, v_k and $\vec{v}_1, \dots, \vec{v}_k$, respectively. The condition that the copy of Q in an orthant b is empty (i. e. is not nonempty) can be expressed by a system of linear equations

$$\vec{v}_1 \cdot \vec{b} = \varepsilon_1, \dots, \vec{v}_k \cdot \vec{b} = \varepsilon_k,$$

where $\varepsilon_i = 0$ if the sign of the vertex v_i is positive, and $\varepsilon_i = 1$ if the sign of v_i is negative. The unknowns of the system are the coordinates of \vec{b} . A solution to the system does exist because the rank of the system is equal to k (the simplex Q is elementary). Moreover, the dimension of the space of solutions is equal to $n - k$. It means that the number of solutions is equal to 2^{n-k} , in other words, the simplex Q has exactly $2^n - 2^{n-k}$ nonempty copies. ■

Proposition 3.1 is similar to Lemma 1 in [I-R].

Now we are able to calculate the Euler characteristic of a primitive T-surface.

Theorem 3.2. *If X is a primitive T-surface in $\mathbf{R}P^3$, then the Euler characteristic $\chi(\mathbf{R}X)$ of the real point set of X is equal to the signature $\sigma(\mathbf{C}X)$ of the complex point set of X . In other words, if X is a primitive T-surface of degree m in $\mathbf{R}P^3$, then*

$$\chi(\mathbf{R}X) = -\frac{m^3}{3} + \frac{4m}{3}.$$

Proof. Let us take an arbitrary primitive triangulation τ of the tetrahedron T and an arbitrary distribution of signs at the integer points of T . The piecewise-linear surface \tilde{S} has a natural cell subdivision: each cell is the intersection of \tilde{S} with a simplex of the triangulation of \tilde{T} .

All the simplices of τ are elementary. The number of simplices of τ of any dimension is fixed (the number of simplices of any dimension contained in each face of T is also fixed). Thus, we can calculate the Euler characteristic of \tilde{S} according to Proposition 3.1.

The triangulation τ contains

m^3 tetrahedra,
 $2m^3 + 2m^2$ triangles, and $4m^2$ of them are contained in the facets of T ,
 $7m^3/6 + 3m^2 + 11m/6$ edges, $6m^2$ of them are contained in the facets of T , and $6m$ of them are contained in the edges of T ,
 $(m + 1)(m + 2)(m + 3)/6$ vertices.

We obtain that the described cell subdivision of \tilde{S} contains $7m^3$ two-dimensional cells, $12m^3$ edges and $14m^3/3 + 4m/3$ vertices. Thus,

$$\chi(\mathbf{R}X) = -\frac{m^3}{3} + \frac{4m}{3} = \sigma(\mathbf{C}X).$$

■

Theorem 3.3. *If X is a T -surface constructed using a maximal triangulation of the tetrahedron T , then $\chi(\mathbf{R}X) \geq \sigma(\mathbf{C}X)$.*

Proof. Let us, first, remark that all simplices of dimension ≤ 2 of a maximal triangulation τ' of T are elementary. Denote by q the number of tetrahedra of τ' . If any tetrahedron of τ' is elementary then, repeating the calculation of the proof of Theorem 3.2, we obtain $\chi(\tilde{S}) = 2m^3/3 - q + 4m/3$.

Each nonelementary tetrahedron of τ' has at least 6 nonempty copies, because the rank of the corresponding system of linear equations (see the proof of Proposition 3.1) is equal to 2. Thus,

$$\chi(\mathbf{R}X) = \chi(\tilde{S}) \geq \frac{2m^3}{3} - q + \frac{4m}{3} - q',$$

where q' is the number of nonelementary tetrahedra of τ' . It remains to remark that $q + q' \leq m^3$, and we obtain

$$\chi(\mathbf{R}X) \geq -\frac{m^3}{3} + \frac{4m}{3} = \sigma(\mathbf{C}X).$$

■

4 Case of primitive or maximal triangulation

As we saw in section 3, the Euler characteristic of a primitive T-surface in $\mathbf{R}P^3$ is determined by the degree and is equal to the signature $\sigma(\mathbf{C}X)$ of the complex point set of the surface.

For a real algebraic surface X (or, more generally, for a real algebraic variety of any dimension), we have Smith inequality (see, for example, [Wi]) :

$$b_*(\mathbf{R}X) \leq b_*(\mathbf{C}X)$$

between the ranks of total $\mathbf{Z}/2$ -homology groups of the real and of the complex point sets of X . If $b_*(\mathbf{R}X) = b_*(\mathbf{C}X)$, the surface X is called an *M-surface*. We denote by $b_i(Y)$ the rank of i -th homology group of Y with $\mathbf{Z}/2$ -coefficients.

Let us mention two congruences (see [Wi]).

Rokhlin congruence. *If X is an M-surface, then*

$$\chi(\mathbf{R}X) \equiv \sigma(\mathbf{C}X) \pmod{16}.$$

Kharlamov-Gudkov-Krahnov congruence. *If X is an (M-1)-surface (in other words, if $b_*(\mathbf{R}X) = b_*(\mathbf{C}X) - 2$), then*

$$\chi(\mathbf{R}X) \equiv \sigma(\mathbf{C}X) \pm 2 \pmod{16}.$$

Rokhlin congruence and Theorem 3.2 show that we can expect to construct primitive T-surfaces which are M-surfaces. We will see in section 5 that such surfaces do really exist in any degree. On the other hand, there are no (M-1)-surfaces among primitive T-surfaces in $\mathbf{R}P^3$ according to Kharlamov-Gudkov-Krahnov congruence and Theorem 3.2.

Theorem 4.1. *If X is a primitive T-surface in $\mathbf{R}P^3$ then*

$$b_1(\mathbf{R}X) \leq h^{1,1}(\mathbf{C}X), \quad b_0(\mathbf{R}X) \leq h^{2,0}(\mathbf{C}X) + 1.$$

Remarks. Theorem 4.1 states that Viro's conjecture holds in the case of primitive T-surfaces.

The inequality $b_0(\mathbf{R}X) \leq h^{2,0}(\mathbf{C}X) + 1$ for primitive T-surfaces was proved by E. Shustin in [Sh].

Proof of Theorem 4.1. Using the Smith inequality

$$b_*(\mathbf{R}X) \leq b_*(\mathbf{C}X) = m^3 - 4m^2 + 6m$$

(where m is the degree of X) and the equality

$$\chi(\mathbf{R}X) = \sigma(\mathbf{C}X) = -\frac{m^3}{3} + \frac{4m}{3}$$

proved in Theorem 3.2, we immediately obtain

$$b_1(\mathbf{R}X) \leq h^{1,1}(\mathbf{C}X) = \frac{2m^3}{3} - 2m^2 + \frac{7m}{3}$$

and

$$b_0(\mathbf{R}X) \leq h^{2,0}(\mathbf{C}X) + 1 = \frac{m^3}{6} - m^2 + \frac{11m}{6}.$$

■

Viro's conjecture also holds in the case of T-surfaces constructed using maximal triangulations.

Theorem 4.2. *If X is a T-surface constructed using a maximal triangulation of the tetrahedron T , then*

$$b_1(\mathbf{R}X) \leq h^{1,1}(\mathbf{C}X).$$

Proof. The Smith inequality and the inequality $\chi(\mathbf{R}X) \geq \sigma(\mathbf{C}X)$ proved in Theorem 3.3, give again the desired inequality

$$b_1(\mathbf{R}X) \leq h^{1,1}(\mathbf{C}X).$$

■

5 M-surfaces

We describe, first, a special primitive triangulation ρ of T suggested by O. Viro. We show that the T-construction using the triangulation ρ and an appropriate distribution of signs at the integer points of T gives an M-surface of degree m in $\mathbf{R}P^3$. In fact, the surfaces given by the procedure described below are homeomorphic to ones constructed (not as T-surfaces) by O. Viro in [V1].

Let us divide the tetrahedron T by the planes $z = l$, and denote by P_l the polytope

$$\{(x, y, z) \in T : l \leq z \leq l + 1, \quad l = 0, \dots, m - 1\}.$$

Choose an arbitrary primitive convex triangulation of each triangle

$$T_l = T \cap \{z = l\}, \quad l = 0, \dots, m - 1$$

(a triangulation of the triangle T_l is called *primitive* if all its triangles are of area $1/2$, or, equivalently, if all the integer points of T_l are vertices of the triangulation).

Each polytope P_l is triangulated as follows. If l is even, take the join J_l of the side of T_l lying in the xz -coordinate plane and of the side of T_{l+1} lying in the plane $x + y + z = m$. If l is odd, take as J_l the join of the side of T_l lying in the plane $x + y + z = m$ and of the side of T_{l+1} lying in the xz -coordinate plane. The join J_l is naturally triangulated into the joins of segments

$$[(i, 0, l), (i + 1, 0, l)], [(m - (l + 1) - j, j, l + 1), (m - (l + 1) - (j + 1), j + 1, l + 1)], \\ i = 0, \dots, m - l - 1, \quad j = 0, \dots, m - l - 2$$

if l is even, and J_l is triangulated into the joins of segments

$$[(m - l - j, j, l), (m - l - (j + 1), j + 1, l)], [(i, 0, l + 1), (i + 1, 0, l + 1)], \\ i = 0, \dots, m - l - 2, \quad j = 0, \dots, m - l - 1$$

if l is odd.

The polytope P_l is the union of J_l and of two tetrahedra $P_l^{(1)}$ and $P_l^{(2)}$. These tetrahedra can be triangulated into the cones over the triangles of the chosen triangulations of T_l and of T_{l+1} .

Clearly, the described triangulation ρ of T is primitive. To explain that ρ is convex, consider a triangulation of T formed by the tetrahedra

$$J_l, P_l^{(1)}, P_l^{(2)} \quad (l = 0, \dots, m-1).$$

The later triangulation is convex. Let $\nu' : T \rightarrow \mathbf{R}$ be a convex function certifying the convexity of this triangulation, and let $\nu_l : T_l \rightarrow \mathbf{R}$ ($l = 0, \dots, m-1$) be a convex function certifying that the chosen triangulation of T_l is convex. Consider a piecewise-linear function $\nu : T \rightarrow \mathbf{R}$ which is linear on each tetrahedron of ρ and takes the value $\nu'(r_l) + \varepsilon \nu_l(r_l)$ at an integer point r_l of T_l . It is easy to see that the function ν for a positive sufficiently small ε certifies the convexity of ρ .

Choose the following distribution of signs at the integer points of T :

- a point (i, j, l) gets the sign "+" if $i \equiv j \equiv l \equiv 0 \pmod{2}$ or $l \equiv 1 \pmod{2}$ and $ij \equiv 0 \pmod{2}$;
- and it gets the sign "-" otherwise.

Proposition 5.1. *A T -surface X constructed using the triangulation ρ and the distribution of signs described is an M -surface. The real point set $\mathbf{R}X$ of X is homeomorphic to the disjoint union of $\frac{m^3}{6} - m^2 + \frac{11m}{6} - 1$ spheres and a sphere with $\frac{m^3}{3} - m^2 + \frac{7m}{6}$ handles if m is even or a projective plane with $\frac{m^3}{3} - m^2 + \frac{7m-3}{6}$ handles if m is odd.*

Proof. It is easy to verify that any integer point r lying strongly inside T has a symmetric copy $s(r)$ with the following property: all the neighbouring vertices of $s(r)$ (i. e. vertices connected with $s(r)$ by an edge of the triangulation) have the same sign, and this sign is opposite to the sign of $s(r)$. It means that the surface \tilde{S} has a connected component homeomorphic to a sphere contained in the star of $s(r)$.

We found $\frac{m^3}{6} - m^2 + \frac{11m}{6} - 1 = h^{2,0}(\mathbf{C}X)$ components of \tilde{S} . There is at least one component of \tilde{S} more, because the surface \tilde{S} intersects the coordinate planes. On the other hand, according to Theorem 4.1, the number of connected components of $\mathbf{R}X$ does not exceed $h^{2,0}(\mathbf{C}X) + 1$. Thus, the real point set $\mathbf{R}X$ has exactly $h^{2,0}(\mathbf{C}X) + 1$ connected components.

Using the equalities

$$\chi(\mathbf{R}X) = \sigma(\mathbf{C}X), \quad b_0(\mathbf{R}X) = h^{2,0}(\mathbf{C}X) + 1,$$

we get $b_*(\mathbf{R}X) = b_*(\mathbf{C}X)$, i. e. X is an M-surface. Furthermore,

$$b_1(\mathbf{R}X) = h^{1,1}(\mathbf{C}X),$$

and, thus, the topological type of $\mathbf{R}X$ coincides with one described in the statement of Proposition.

■

6 Counter-examples to Viro's conjecture

We saw in section 4 that Viro's conjecture is true for T-surfaces constructed using a maximal triangulation. Surprisingly enough, a non-maximal triangulation of T can produce a T-surface X in $\mathbf{R}P^3$ with $b_1(\mathbf{R}X) > h^{1,1}(\mathbf{C}X)$.

Let us describe, first, the construction of an extension of a triangulation of the triangle $T_0 = T \cap \{z = 0\}$.

Suppose that m is even and that a primitive triangulation τ_0 of T_0 with the vertices having integer coordinates is given. Divide the tetrahedron T into two parts $T \cap \{z \geq 2\}$ and $T \cap \{z \leq 2\}$ by the plane $z = 2$. Take in the first part the triangulation coinciding with the triangulation ρ described in the construction of M-surfaces.

Divide now the second part $T \cap \{z \leq 2\}$ by the plane $x + y + kz = m$ (where $m = 2k$) into the tetrahedron \bar{T} with vertices $(0, 0, 0)$, $(m, 0, 0)$, $(0, m, 0)$, $(0, 0, 2)$ and the cone C with the vertex $(0, 0, 2)$ over

$$\{(x, y, z) \in T : x + y + z = m, 0 \leq z \leq 2\}$$

(see Figure 1).

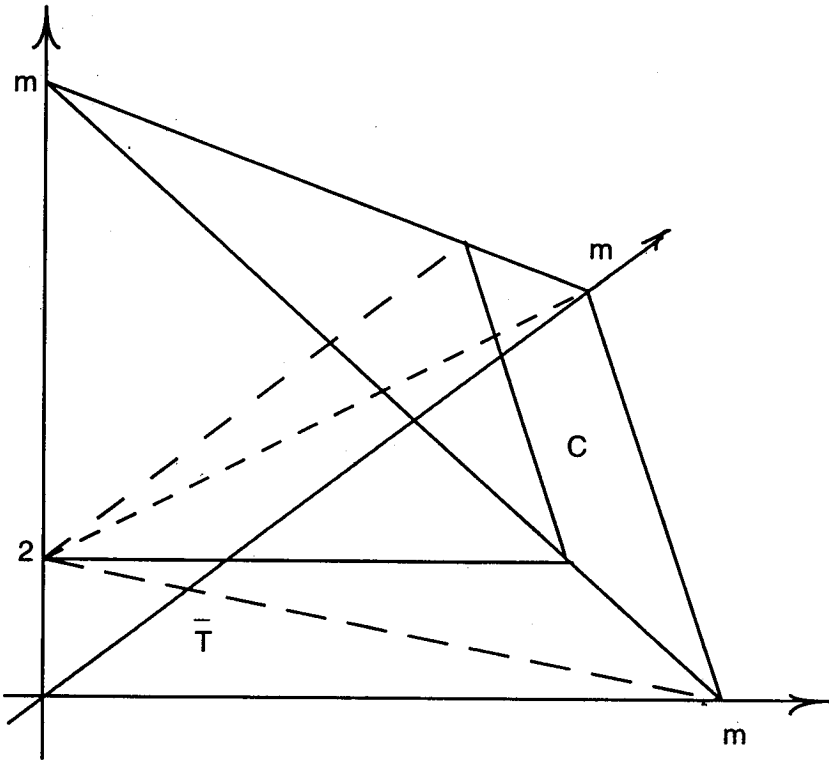


Figure 1

To triangulate the tetrahedron \bar{T} , we take the cones over all the triangles of τ_0 , and subdivide (in the unique possible way) the cones containing integer points of the plane $z = 1$ in order to obtain a maximal triangulation of \bar{T} .

To describe the triangulation of the cone C , let us consider the cone \hat{C} with the vertex $(k + 1, 0, 1)$ over the triangle $T \cap \{x + y + kz = m\}$. The rest of the cone C is divided into two parts by the plane $z = 1$ (see Figure 2). Denote the lower part (contained in $C \cap \{0 \leq z \leq 1\}$) by C_0 ,

and denote the upper part (contained in $C \cap \{1 \leq z \leq 2\}$) by C_1 .

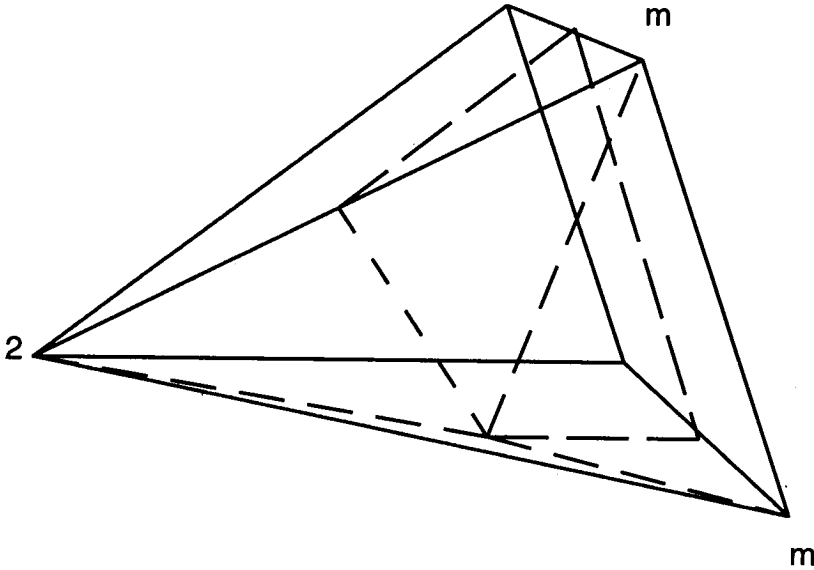


Figure 2

The triangulation of the triangle $T \cap \{x + y + kz = m\}$ is already fixed (it comes from the triangulation of \bar{T}). Thus, we can triangulate the cone \hat{C} by the cones with the vertex $(k + 1, 0, 1)$ over the triangles of the triangulation of $T \cap \{x + y + kz = m\}$.

Subdivide C_0 taking the cone C' with the vertex $(0, m, 0)$ over the facet of C_0 belonging to the plane $z = 1$, and the join J' of segments $[(m, 0, 0), (0, m, 0)]$ and $[(k + 1, 0, 1), (m - 1, 0, 1)]$. Let us choose an arbitrary primitive convex triangulation of the quadrangle $C_0 \cap \{z = 1\}$. It gives a natural primitive triangulation of C' (taking the cones over the triangles of the chosen triangulation of $C_0 \cap \{z = 1\}$). The join J' is triangulated by the joins of segments $[(m - j, j, 0), (m - j - 1, j + 1, 0)]$ and $[(i, 0, 1), (i + 1, 0, 1)]$ (where $i = k + 1, \dots, m - 2$; $j = 0, \dots, m - 1$).

It remains to triangulate the part C_1 . Subdivide C_1 into the join of segments $[(m - 1, 0, 1), (0, m - 1, 1)]$ and $[(0, 0, 2), (m - 2, 0, 2)]$ (triangulated by the joins of segments $[(m - j - 1, j, 1), (m - j - 2, j + 1, 1)]$

and $[(i, 0, 2), (i + 1, 0, 2)]$, where $i = 0, \dots, m - 3$; $j = 0, \dots, m - 2$) and the naturally triangulated cones : with the vertex $(0, 0, 2)$ (resp. $(0, m - 1, 1)$) over the quadrangle $C_1 \cap \{z = 1\}$ (resp. over the triangle $T_2 = T \cap \{z = 2\}$).

The described maximal triangulation of T is called *the extension* of the triangulation τ_0 and is denoted by $ext(\tau_0)$.

Arguments, similar to ones used in the previous section to show that the triangulation ρ is convex, prove that if τ_0 is convex then $ext(\tau_0)$ is also convex. Almost all tetrahedra of $ext(\tau_0)$ are of volume $1/6$. The only tetrahedra of a greater volume (more precisely, of volume $1/3$) are the cones with the vertex $(0, 0, 2)$ over the odd triangles of τ_0 (we call a triangle of τ_0 *odd* if it does not have a vertex with the both even coordinates).

Suppose now that a distribution δ_0 of signs at the integer points of T_0 is given. Let us describe a distribution $ext(\delta_0)$ of signs at the integer points of T which we call *an extension* of δ_0 . In the part $T \cap \{z \geq 2\}$ we take the distribution of signs described in the construction of M-surfaces. It remains, thus, to fix a distribution of signs at the integer points of $T \cap \{z = 1\}$. We do it as follows:

take an arbitrary distribution in $T \cap \{z = 1\} \cap \{x + y < k\}$,
 all the integer points of the segment $[(k, 0, 1), (0, k, 1)]$ but
 the point $(0, k, 1)$ get the sign "-",
 for the other points of T_1 we apply the rule : a point $(i, j, 1)$
 gets the sign "-" if i and j are both odd, and the sign "+"
 otherwise.

Let us take a triangulation τ_0^1 and a distribution δ_0^1 of signs at the integer points of T_0 producing a counter-example to Ragsdale conjecture with $p = \frac{3k^2 - 3k + 2}{2} + 1$ (see [I1], [I2], [I-V]). The triangulation τ_0^1 can be obtained placing the hexagon H shown in Figure 3 inside of T_0 (on suppose that $m \geq 10$) in such a way that the center of H has both the nonzero coordinates odd, and extending, then, the triangulation of H to a primitive convex triangulation of T_0 . To obtain a distribution of signs at the integer points of T_0 , we complete the distribution presented in Figure 3 by the rule :

a point $(i, j, 0)$ gets the sign "-" if i and j are even, and
 $i + j < m$,

a point $(i, j, 0)$ gets the sign "+" otherwise.

Remark that this distribution of signs at the integer points of T_0 is slightly different from the distribution described in [I1], [I2], [I-V].

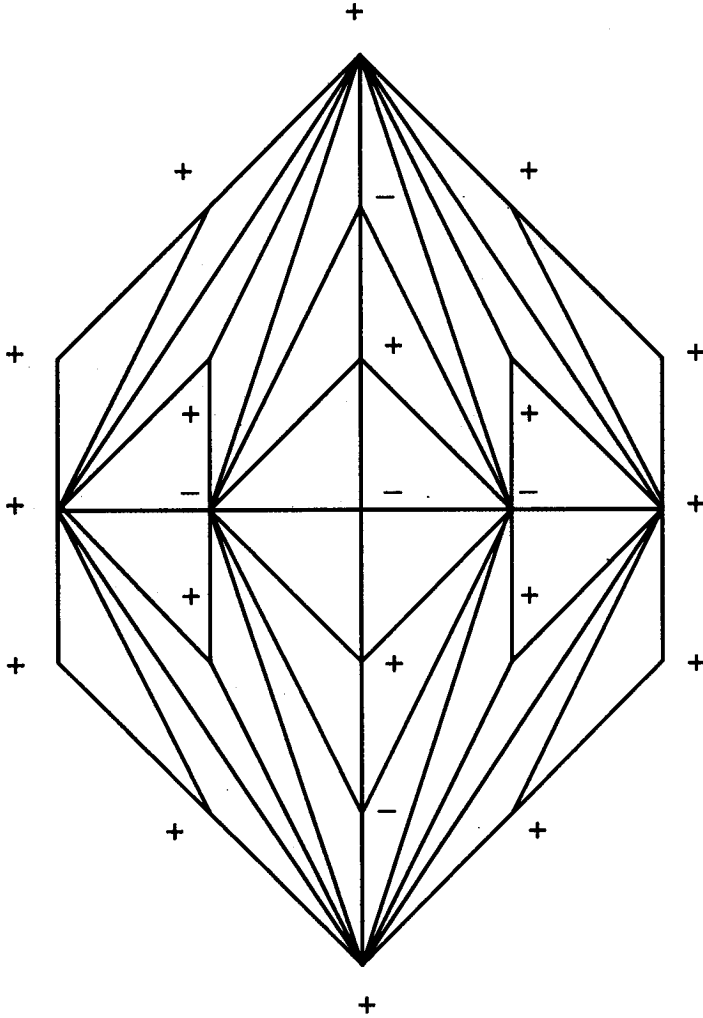


Figure 3

Proposition 6.1 *The maximal triangulation $ext(\tau_0^1)$ and a distribution*

of signs $\text{ext}(\delta_0^1)$ produce a T -surface X of degree m in $\mathbf{R}P^3$ with

$$\chi(\mathbf{R}X) = -\frac{m^3}{3} + \frac{4m}{3}, \quad b_0(\mathbf{R}X) = h^{2,0}(\mathbf{C}X) - 2.$$

The real point set $\mathbf{R}X$ of X is homeomorphic to the disjoint union

$$\left(\frac{m^3}{6} - m^2 + \frac{11m}{6} - 5 \right) S^2 \amalg S_2 \amalg S_{\frac{m^3}{3} - m^2 + \frac{7m}{6} - 5}$$

of $\frac{m^3}{6} - m^2 + \frac{11m}{6} - 5$ spheres, a sphere with 2 handles and a sphere with $\frac{m^3}{3} - m^2 + \frac{7m}{6} - 5$ handles.

Proof. Let us, first, calculate $\chi(\mathbf{R}X)$. It was already remarked that almost all tetrahedra of $\text{ext}(\tau_0^1)$ are of volume $1/6$. The only tetrahedra of greater volume (of volume $1/3$) are the cones over the odd triangles of τ_0^1 . Each of these tetrahedra of volume $1/3$ has 6 nonempty symmetric copies (a tetrahedron of volume $1/3$ of a maximal triangulation has 6 nonempty copies if the product of signs at its vertices is positive, and it has 8 nonempty copies if the product of signs is negative). Thus, the arguments of the proof of Theorems 3.2 and 3.3 show that $\chi(\mathbf{R}X) = \sigma(\mathbf{C}X)$.

Calculate now the number of connected components of \tilde{S} . Exactly as in the proof of Theorem 5.1, any integer point lying strongly inside $(T \cap \{z \geq 2\}) \cup C$ has a symmetric copy with the star containing a component of \tilde{S} homeomorphic to a sphere. It is easy to see that the stars of integer points lying strongly inside T and belonging to the segment $[(k, 0, 1), (0, k, 1)]$ also contain the components of \tilde{S} homeomorphic to a sphere. Consider the integer points lying strongly inside the tetrahedron \bar{T} . Let us call *even interior points of T_0* the integer points $(i, j, 0)$ such that $i > 0$, $j > 0$, $i + j < m$, i and j are both even. There is a correspondence between the even interior points of T_0 and the points of $\text{Int}(\bar{T}) \cap \mathbf{Z}^3$: any integer point lying strongly inside \bar{T} is a middle point of a segment joining the point $(0, 0, 2)$ and an even interior point of T_0 . We denote the middle point of a segment $[(0, 0, 2), r]$ (where r is an even interior point of T_0) by $f(r)$.

Suppose that an even interior point r does not belong to the hexagon H . Then r has the sign "-". If $f(r)$ has also the sign "-", then the union

of stars of r and of $f(r)$ (in the triangulation of T_*) contains a component of \tilde{S} homeomorphic to a sphere. If $f(r)$ has the sign "+", then the union of stars of r and of $s_z(f(r))$ contains a component of \tilde{S} homeomorphic to a sphere.

We have found $h^{2,0}(CX) - 4$ spheres of \tilde{S} (a sphere was associated to any integer point lying strongly inside of T except 4 points of the form $f(r)$, where r is an even interior point of T_0 belonging to the hexagon H). There are two connected components of \tilde{S} more. One component is homeomorphic to a sphere with two handles and lies inside of $\bar{H} \cup s_z(\bar{H})$, where \bar{H} is a cone with the vertex $(0, 0, 2)$ over H . The remaining part of \tilde{S} is connected. The number $b_1(\tilde{S})$ can be calculated via the Euler characteristic.

■

Theorem 6.2. *If m is an even integer number not less than 10, then there exists an $(M-2)$ -surface X of degree m in $\mathbf{R}P^3$ such that $b_1(\mathbf{R}X) = h^{1,1}(CX) + 2$.*

Proof. Let us take the triangulation $ext(\tau_0^1)$ of T and the distribution of signs $ext(\delta_0^1)$ at the integer points of T . According to Proposition 6.1 the resulting surface \tilde{S} is homeomorphic to

$$\left(\frac{m^3}{6} - m^2 + \frac{11m}{6} - 5 \right) S^2 \amalg S_2 \amalg S_{\frac{m^3}{3} - m^2 + \frac{7m}{6} - 5}.$$

Remove now 4 vertices of the form $f(r)$, where r is an even interior point of T_0 belonging to H (see the proof of Proposition 6.1), with all the adjacent edges. Denote the new triangulation (which is nonmaximal) by $ext'(\tau_0^1)$ and consider the surface \tilde{S}' constructed using $ext'(\tau_0^1)$ and the restriction $ext'(\delta_0^1)$ of the distribution $ext(\delta_0^1)$ to the set of vertices of $ext'(\tau_0^1)$. Clearly, the surface \tilde{S}' is homeomorphic to

$$\left(\frac{m^3}{6} - m^2 + \frac{11m}{6} - 5 \right) S^2 \amalg S_2 \amalg S_{\frac{m^3}{3} - m^2 + \frac{7m}{6} - 1}$$

because we added 4 handles to the component homeomorphic to

$S_{\frac{m^3}{3} - m^2 + \frac{7m}{6} - 5}$. Thus, the number of $b_1(\tilde{S}')$ is equal to

$$\frac{2m^3}{3} - 2m^2 + \frac{7m}{3} + 2.$$

■

Using counter-examples of degree $2k$ to Ragsdale conjecture with more than $\frac{3k^2 - 3k + 2}{2} + 1$ even ovals (see [I1], [I2], [I-V]), one can construct surfaces X of degree $2k$ in $\mathbf{R}P^3$ with $b_1(\mathbf{R}X) > h^{1,1}(\mathbf{C}X) + 2$.

Theorem 6.3. *If $m = 2k$ is an even integer not less than 10, then there exists a surface X of degree m in $\mathbf{R}P^3$ such that*

$$b_1(\mathbf{R}X) = h^{1,1}(\mathbf{C}X) + 2 \left[\frac{(k-3)^2 + 4}{8} \right]$$

(where $[u]$ denotes the greatest integer which does not exceed u).

Proof. We start from a triangulation τ_0^a and a distribution δ_0^a of signs at the integer points of T_0 giving a counter-example to Ragsdale conjecture with

$$p = \frac{3k^2 - 3k + 2}{2} + a,$$

where $a = \left[\frac{(k-3)^2 + 4}{8} \right]$ (see [I1], [I2], [I-V]). The triangulation τ_0^a can be obtained in the following way. Consider the partition of the triangle T_0 shown in Figure 4. Let us take in each shadowed hexagon the triangulation (and the signs) of the hexagon H . The triangulation of the union of the shadowed hexagons can be extended to a primitive convex triangulation τ_0^a of T_0 . To obtain the distribution δ_0^a of signs at the integer points of T_0 , we choose the signs outside of the union of the shadowed hexagons again using the rule :

a point $(i, j, 0)$ gets the sign "-" if i and j are even, and $i + j < m$,

a point $(i, j, 0)$ gets the sign "+" otherwise.

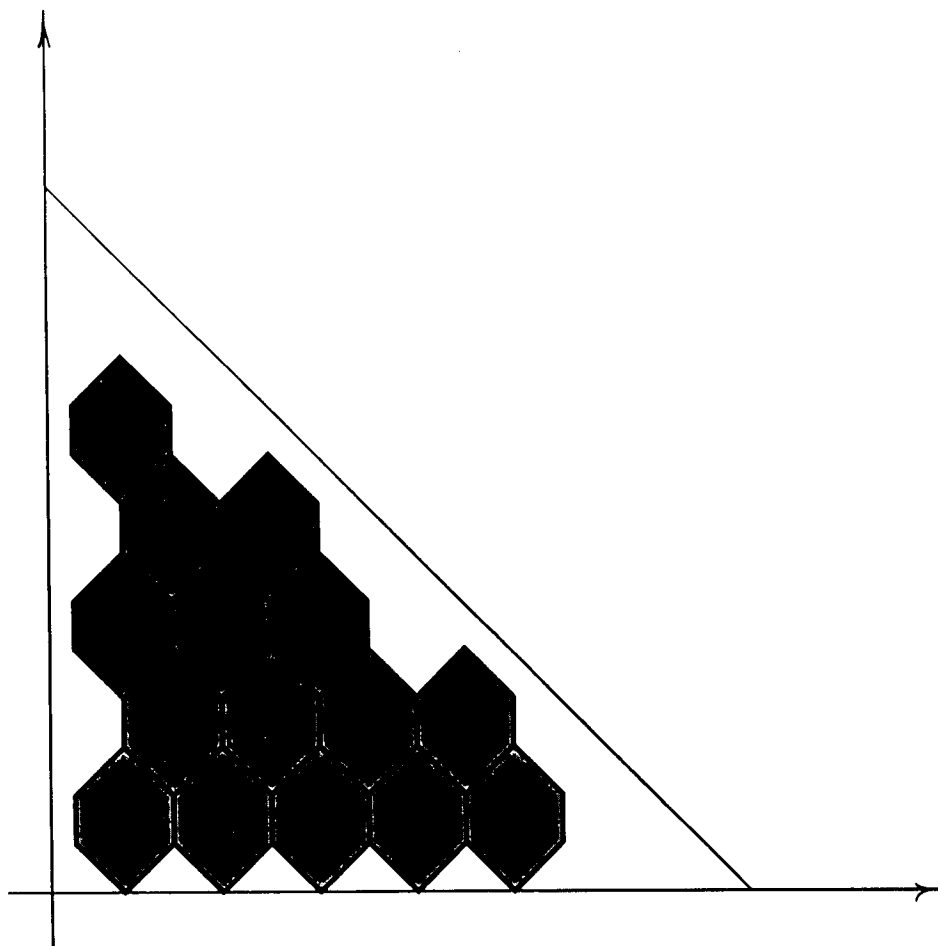


Figure 4

Consider the triangulation $ext(\tau_0^a)$ of T and the distribution $ext(\delta_0^a)$

of signs at the integer points of T . The resulting surface \tilde{S} is homeomorphic to

$$\left(\frac{m^3}{6} - m^2 + \frac{11m}{6} - 1 - 4a\right) S^2 \amalg aS_2 \amalg S_{\frac{m^3}{3} - m^2 + \frac{7m}{6} - 5a}.$$

Remove now the vertices of the triangulation $\text{ext}(\tau_0^a)$ (with adjacent edges) of the form $f(r)$, where r is an even interior point of T_0 belonging to one of the shadowed hexagons, and take the restriction $\text{ext}'(\delta_0^a)$ of the distribution $\text{ext}(\delta_0^a)$ to the vertices of the new triangulation $\text{ext}'(\tau_0^a)$. We obtain a surface S' homeomorphic to

$$\left(\frac{m^3}{6} - m^2 + \frac{11m}{6} - 1 - 4a\right) S^2 \amalg aS_2 \amalg S_{\frac{m^3}{3} - m^2 + \frac{7m}{6} - a}$$

with

$$b_1(\tilde{S}') = \frac{2m^3}{3} - 2m^2 + \frac{7m}{3} + 2a.$$

■

Remarks.

1. Removing, if necessary, some of the shadowed hexagons in the construction of Theorem 6.3, we get counter-examples to Viro's conjecture with the real point set homeomorphic to

$$\left(\frac{m^3}{6} - m^2 + \frac{11m}{6} - 1 - 4a\right) S^2 \amalg aS_2 \amalg S_{\frac{m^3}{3} - m^2 + \frac{7m}{6} - a},$$

where $a = 1, \dots, \left\lfloor \frac{(k-3)^2 + 4}{8} \right\rfloor$.

2. The counter-example of the smallest degree in $\mathbf{R}P^3$ given by Theorems 6.2 and 6.3 is a surface of degree 10. The real point set of this surface is homeomorphic to

$$80S^2 \amalg S_2 \amalg S_{244}.$$

It is unknown if there exist counter-examples of degree less than 10. The smallest degree we can expect for a counter-example to Viro's conjecture is degree 5.

3. Repeating the procedure described above for the new counter-examples to the Ragsdale conjecture constructed by B. Haas [Ha], one can construct surfaces X of degree $2k$ in $\mathbf{R}P^3$ with

$$b_1(\mathbf{R}X) = h^{1,1}(\mathbf{C}X) + 2a',$$

where $a' = \left\lceil \frac{k^2 - 7k + 16}{6} \right\rceil$.

4. We can obtain counter-examples to Viro's conjecture which are asymptotically better than the examples described above: there exist T -surfaces X of degree $2k$ in $\mathbf{R}P^3$ with $b_1(\mathbf{R}X) = h^{1,1}(\mathbf{C}X) + 2A$, where $A = k^3/24 +$ terms of smaller degrees. To construct such surfaces, we divide the tetrahedron T by the planes $z = 2l$ (where $l = 1, \dots, k-1$), and define a triangulation and a distribution of signs in each part of the subdivision using the procedure described in the proof of Theorem 6.3 for $T \cap \{0 \leq z \leq 2\}$.

References

- [GKZ] I. Gelfand, M. Kapranov and A. Zelevinski, *Discriminants, resultants and multidimensional determinants*, Birkhäuser, Boston, 1994.
- [Ha] B. Haas, *Les multilucarnes: Nouveaux contre-exemples à la conjecture de Ragsdale*, C. R. Acad. Sci. Paris (to appear).
- [I1] I. Itenberg, *Contre-exemples à la conjecture de Ragsdale*, C. R. Acad. Sci. Paris, Ser. 1. **317** (1993), 277-282.
- [I2] I. Itenberg, *Counter-examples to Ragsdale Conjecture and T-curves*, Contemporary Mathematics **182** (1995), 55-72.
- [I-R] I. Itenberg, M.-F. Roy, *Multivariate Descartes' rule*, Beiträge zur Algebra und Geometrie, (to appear).
- [I-V] I. Itenberg, O. Viro, *Patchworking algebraic curves disproves the Ragsdale conjecture*, Preprint, Uppsala University, 1995.
- [Kh] V. Kharlamov, *Real algebraic surfaces*, Proc. Internat. Congress Math., vol. 1, Helsinki, 1978, pp. 421-428. (Russian).

- [Ra] V. Ragsdale, *On the arrangement of the real branches of plane algebraic curves*, Amer. J. Math. **28** (1906), 377-404.
- [Ri] J.-J. Risler, *Construction d'hypersurfaces réelles [d'après Viro]*, Séminaire N.Bourbaki, no. 763, vol. 1992-93, Novembre 1992.
- [Sh] E. Shustin, *Critical points of real polynomials, subdivisions of Newton polyhedra and topology of real algebraic hypersurfaces*, Amer. Math., Soc. Transl. (2) **173** (1996), 203-223.
- [Si] R. Silhol, *Real Algebraic Surfaces*, Lect. Notes in Math., vol. 1392, Springer-Verlag, Berlin Heidelberg, 1989.
- [V1] O. Viro, *Construction of M-surfaces*, Functional Anal. Appl. **13** (1979).
- [V2] O. Viro, *Gluing of algebraic hypersurfaces, smoothing of singularities and construction of curves*, Proc. Leningrad Int. Topological Conf. (Leningrad, Aug. 1983), Nauka, Leningrad, 1983, pp. 149-197. (Russian).
- [V3] O. Viro, *Gluing of plane real algebraic curves and construction of curves of degrees 6 and 7*, Lect. Notes in Math., 1060, Springer-Verlag, Berlin Heidelberg, 1984, pp. 187-200.
- [V4] O. Viro, *Progress in the topology of real algebraic varieties over the last six years*, Rus. Math. Surv. **41** (1986), no. 3, 55-82.
- [V5] O. Viro, *Real algebraic plane curves: constructions with controlled topology*, Leningrad Math. J. **1** (1990), 1059-1134.
- [V6] O. Viro, *Patchworking real algebraic varieties*, Preprint, Uppsala University, 1994.
- [Wi] G. Wilson, *Hilbert's sixteenth problem*, Topology **17** (1978), no. 1, 53-73.
- [Zi] G. Ziegler, *Lectures on Polytopes*, Springer-Verlag, New York 1994.

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