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A failure of quantifier elimination.

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Abstract

We show that log is needed to eliminate quantifiers in the theory of the real numbers with restricted analytic functions and exponentiation.

We let $\mathcal{L}_{\mathrm{an}}$ be the first order language of ordered rings augmented by function symbols $\widehat{f}$ where f is an analytic function defined on an open $U \supset [0,1]^n$ for some n. We interpret $\widehat{f}$ as a function on $\mathbf{R}^n$ by

$$\widehat{f}(x) = \begin{cases} f(x) & \text{if } x \in [0,1]^n \\ 0 & \text{otherwise} \end{cases}$$

Let $\mathcal{L}_{\mathrm{an}}^{\mathbf{R}}$ be the language obtained by adding to $\mathcal{L}_{\mathrm{an}}$ unary function symbols f_r for each $r \in \mathbf{R}$. We interpret f_r as the function

$$f_r(x) = \begin{cases} x^r & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$$

and denote $f_r(x)$ by x^r. Finally we let $\mathcal{L}_{\mathrm{an,exp}}$ be the language $\mathcal{L}_{\mathrm{an}} \cup \{\exp\}$ and $\mathcal{L}_{\mathrm{an,exp}}^{\mathbf{R}} = \mathcal{L}_{\mathrm{an}}^{\mathbf{R}} \cup \{\exp\}$.

In [2] we showed that the $\mathcal{L}_{\mathrm{an,exp}}$-theory of $\mathbf{R}$ admits quantifier elimination in the language $\mathcal{L}_{\mathrm{an,exp}} \cup \{\log\}$. Indeed, we remark there that exp is unnecessary as we could actually eliminate quantifiers in the language

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$\mathcal{L}_{\text{an}} \cup \{\log\} \cup \{x^q : q \in \mathbf{Q}\}$. Here we show that although exp and log are interdefinable, log is essential for quantifier elimination.

Theorem. *Let $\phi(x, y)$ be the formula*

$$\exists z\ (\exp(\exp z) = x \wedge y = z \exp z).$$

Then $\phi(x, y)$ is not equivalent to a quantifier free $\mathcal{L}^{\mathbf{R}}_{\text{an,exp}}$-formula.

Of course $\phi(x, y)$ is equivalent to the quantifier free $\mathcal{L}_{\text{an}} \cup \{\log\}$-formula

$$x > 1 \wedge y = (\log x)(\log \log x).$$

There are several previous "failure of quantifier elimination" theorems for the reals with exponentiation. Osgood's example

$$y > 0 \wedge \exists w\ (wy = x \wedge z = ye^w)$$

is not equivalent to a quantifier free formula in the language $\{+, -, \cdot, <, 0, 1, \exp\}$ (or any expansion by total real analytic functions (see for example [1])), while, in unpublished work, van den Dries and Macintyre showed that

$$\exists z(\ z^2 = x \wedge y = e^z)$$

is not equivalent to a quantifier free formula in the language $\{+, -, \cdot, \frac{1}{x}, \exp, <, 0, 1\}$. Both of these formulas are equivalent to a quantifier free $\mathcal{L}^{\mathbf{R}}_{\text{an,exp}}$-formulas.

In [4] Gabrielov gives several "failure of quantifier elimination" results of a different spirit.

The most interesting open question of this kind is whether the theory of $(\mathbf{R}, +, \cdot, \exp)$ admits quantifier elimination in either the language $\mathcal{L} = \{+, \cdot, -, <, 0, 1\} \cup \{\exp, \log\}$ or $\mathcal{L}$ augmented by all semialgebraic functions. It seems that to eliminate quantifiers one needs to add some implicitly defined restricted analytic functions, so we expect both of these questions to have a negative answer.

Let $f(x) = (\log x)(\log \log x)$ and let Γ be the graph of f. We say that an open set $U \subseteq \mathbf{R}^2$ contains a *tail* of Γ if $(x, f(x)) \in U$ for all sufficiently large x.

Let $\phi(x,y)$ be the above formula. Suppose for purposes of contradiction that ϕ is equivalent to a quantifier free $\mathcal{L}^{\mathbf{R}}_{\text{an,exp}}$-formula, say

$$\phi(x,y) \Leftrightarrow \bigvee_{i=1}^{r} \left(F_i(x,y) = 0 \wedge \bigwedge_{j=1}^{s} G_{i,j}(x,y) > 0\right)$$

for some $\mathcal{L}^{\mathbf{R}}_{\text{an,exp}}$-terms $F_i, G_{i,j}$. Let

$$Y_i = \{(x,y) : F_i(x,y) = 0 \wedge \bigwedge_{j=1}^{s} G_{i,j}(x,y) > 0\}.$$

By o-minimality there is an i such that $(x,y) \in Y_i$ if and only if $y = f(x)$ for sufficiently large x. Fix such an i.

Let $W_0 = \{(x,y) : F_i(x,y) = 0\}$ and let $W_j = \{(x,y) : G_{i,j}(x,y) > 0\}$ for $j = 1, \ldots, s$. Each W_j contains a tail of Γ. Suppose that for each i there is an M_i such that $\{(x,y) \in \Gamma : x > M_i\}$ is in the interior of W_i. Then $\{(x,y) \in \Gamma : x > \max M_i\}$ is in the interior of Y_i, a contradiction. Thus a tail of Γ must be in the boundary of at least one of the W_j.

Thus we have shown that there is an $\mathcal{L}^{\mathbf{R}}_{\text{an,exp}}$-term F such that a tail of Γ is in either the boundary of $\{(x,y) : F(x,y) = 0\}$ or the boundary of $\{(x,y) : F(x,y) > 0\}$. Unfortunately, since our terms need not be continuous, we must consider both possibilities. The next lemma shows that we can in fact choose F such that the first possibility holds and F is analytic on a neighborhood of a tail of Γ.

Lemma 1. *Let $f(x) = (\log x)(\log\log x)$. There is an $\mathcal{L}^{\mathbf{R}}_{\text{an,exp}}$-term $F(x,y)$ which is analytic on an open $U \subseteq \mathbf{R}^2$ containing a tail of Γ such that $F(x, f(x)) = 0$ for sufficiently large x, and for all x there are at most finitely many y such that $(x,y) \in U$ and $F(x,y) = 0$. Moreover, we can choose F such that all of its subterms are analytic on U.*

Proof. We know there is an $\mathcal{L}^{\mathbf{R}}_{\text{an,exp}}$-term $F(x,y)$ with the following property:

(*) There is an open $U \subseteq \mathbf{R}^2$ containing a tail of Γ such that Γ is in the boundary of either

a) $\{(x,y) \in U : F(x,y) = 0\}$ or

b) $\{(x,y) \in U : F(x,y) > 0\}$.

We may, by induction on terms, assume that if any nonconstant subterm of F is replaced by the constant term 0 or 1, then the resulting term does not have property (*).

We next try to find an open $V \subseteq U$ containing a tail of Γ such that F and all of its subterms are analytic on V. We try to prove this by induction on subterms of F. We will see that the only obstructions to this induction will lead to a new term F_1 with property $(*)$ such that F_1 and all of its subterms are analytic on an open set containing a tail of Γ.

• If a subterm t of F is a constant or variable, it is analytic on all of U.

• Suppose t_0 and t_1 are a subterms of F and t_i is analytic on V_i where V_i is an open subset of U containing a tail of Γ. Then $V = V_0 \cap V_1$ contains a tail of Γ and $t_0 \pm t_1$, $t_0 t_1$ and $\exp(t_i)$ are analytic on V.

• Suppose $t_1, \ldots, t_n$ and $h = \widehat{g}(t_1, \ldots, t_n)$ are subterms of F, where $\widehat{g}$ is the function symbol for a restricted analytic function and $t_1, \ldots, t_n$ are analytic on an open set U_i containing a tail of Γ. Using the o-minimality of $\mathbf{R}_{\text{an,exp}}$ one of the following holds for each i.

Case 1. There is an open $V_i \subseteq U_i$ containing a tail of Γ such that $t_i(x, y) \in (-\infty, 0] \cup (1, +\infty)$ for all $(x, y) \in V_i$.

Case 2. There is an open $V_i \subseteq U_i$ containing a tail of Γ such that $t_i(x, y) = 1$ for all $(x, y) \in V_i$.

Case 3. There is an open $V_i \subseteq U_i$ containing a tail of Γ such that $0 < t_i(x, y) < 1$ for all $(x, y) \in V_i$.

If we are not in cases 1)-3) then $t_i(x, y)$ must be equal to 0 or 1 on a tail of Γ. Since $t_i(x, y)$ is analytic on an open neighborhood of a tail of Γ, we must be in one of the following two cases.

Case 4. There is an open set $V_i \subseteq U_i$ containing a tail of Γ such that $t_i(x, f(x)) = 0$ but $\{y : (x, y) \in V_i \wedge t_i(x, y) = 0\}$ is finite for sufficiently large x.

Case 5. There is an open set $V_i \subseteq U_i$ containing a tail of Γ such that $t_i(x, f(x)) = 1$ but $\{y : (x, y) \in V_i \wedge t_i(x, y) = 1\}$ is finite for sufficiently large x.

Cases 4) or 5) are the cases where our induction breaks down. In case 4) we replace F by $t_i(x, y)$. Then $t_i(x, y)$ satisfies $(*)$ and t_i and all of its subterms are analytic on V_i. In case 5) we replace F by $t_i(x, y) - 1$. In either case the new term has the desired property.

In case 1)

$$\widehat{g}(t_1, \ldots, t_n) = \widehat{g}(t_1, \ldots, t_{i-1}, 0, t_{i+1}, \ldots, t_n)$$

for all $(x, y) \in V_i$. Thus we could replace this occurence of t_i by 0 to obtain a new term F^* such that $F^* = F$ on an open set containing a tail of Γ. This contradicts our assumptions on F. Similarly in case 2) we can replace this occurence of t_i by 1 contradicting our assumptions on F.

Thus we may assume we are in case iii). Let $V = \bigcap_{i=1}^{n} V_i$. Then $(t_1(x, y), \ldots, t_n(x, y)) \in (0, 1)^n$ for all $(x, y) \in V$ and h is analytic on V.

• Suppose h and t are subterms of F, $h = t^r$ and t is analytic on an open set U containing a tail of Γ. As above, we can find an open set $V \subseteq U$ containing a tail of Γ such that one of the following holds:

Case 1. $t(x, y) \leq 0$ for all $(x, y) \in V$,

Case 2. $t(x, f(x)) = 0$ and $\{y : (x, y) \in V \wedge t(x, y) = 0\}$ for sufficiently large x, or

Case 3. $t(x, y) > 0$ for $(x, y) \in V$.

As above case 1) can not happen as we could simplify F by replacing h by 0. In case 2) we can use t instead of F and we are done. Thus we may assume that we are in case 3) and note that h is analytic on V.

This completes the induction. Either we will find a simpler term satisfying the conditions of the theorem or we will eventually thin U to an open V containing a tail of Γ such that F is analytic on V. In the later case, since F is analytic on V, $\{(x, y) \in V : F(x, y) > 0\}$ is open. Thus we must be in case a) of $(*)$ and F is the desired term.

Let $F(x, y)$ be the term guaranteed by lemma 1. Note that since F and all of its subterms are analytic on U, one can show by induction that all of the partial derivatives of F are equal to $\mathcal{L}^{\mathbf{R}}_{\text{an,exp}}$-terms on U.

Let $p \in \Gamma \cap U$. By repeated application of the Weierstrass division theorem we can find an open neighborhood V of p, $n \in \mathbf{N}$ and an analytic function g on V such that on V

$$F(x, y) = (y - f(x))^n g(x, y)$$

and there is no point $(x, y) \in V \setminus \{p\}$ such that $y = f(x)$ and $g(x, y) = 0$. Note that for each $m \leq n$ there is an analytic h_m on V such that

$$\frac{\partial^m F}{\partial y^m}(x, y) = \frac{n!}{(n-m)!}(y - f(x))^{n-m}(g(x, y) + (y - f(x))h_m(x, y)).$$

Let G be an $\mathcal{L}^{\mathbf{R}}_{\text{an,exp}}$-term such that $G = \frac{\partial^{n-1}F}{\partial y^{n-1}}$ on U. Then G vanishes identically on $\Gamma \cap V$ and $\frac{\partial G}{\partial y}$ does not vanish on $\Gamma \cap V \setminus \{p\}$. By analytic continuation and o-minimality

$$G(x, f(x)) = 0$$

and

$$\frac{\partial G}{\partial y}(x, f(x)) \neq 0$$

for sufficiently large x.

Since (e^{e^z}, ze^z) parameterizes the curve $y = f(x)$, $G(e^{e^z}, ze^z) = 0$ for sufficiently large z. Differentiating with respect to z we see that

$$0 = e^z e^{e^z} \frac{\partial G}{\partial x}(e^{e^z}, ze^z) + (z+1)e^z \frac{\partial G}{\partial y}(e^{e^z}, ze^z)$$

and

$$z = -e^{e^z} \frac{\frac{\partial G}{\partial x}(e^{e^z}, ze^z)}{\frac{\partial G}{\partial y}(e^{e^z}, ze^z)} - 1 \tag{1}$$

for sufficiently large z.

Suppose $\mathcal{M}$ is a nonstandard model of the $\mathcal{L}_{\text{an,exp}}$-theory of $\mathbf{R}$, $x \in \mathcal{M}$, and $x > \mathbf{R}$. Let N be the smallest $\mathcal{L}^{\mathbf{R}}_{\text{an,exp}}$-substructure of $\mathcal{M}$ containing $\mathbf{R}(e^{e^x}, xe^x)$, i.e. N is the smallest subset of $\mathcal{M}$ containing $\mathbf{R}(e^{e^x}, xe^x)$ and closed under $\mathcal{L}^{\mathbf{R}}_{\text{an}}$-terms and exponentiation. In fact N is the smallest $\mathcal{L}^{\mathbf{R}}_{\text{an}}$-elementary submodel of $\mathcal{M}$ containing $\mathbf{R}(e^{e^x}, xe^x)$ and closed under exp. Since G and $\frac{\partial G}{\partial y}$ are $\mathcal{L}^{\mathbf{R}}_{\text{an,exp}}$-terms, $x \in N$. We will obtain a contradiction by showing this fails when $\mathcal{M}$ is the logarithmic-exponential series field $\mathbf{R}((t))^{\text{LE}}$ constructed in [3].

For the remainder of the proof we assume familiarity with the notation and results from [3].

Lemma 2. *Let $x = t^{-1} \in \mathbf{R}((t))^{\text{LE}}$. Let $N \subset \mathbf{R}((t))^{\text{LE}}$ be the smallest $\mathcal{L}^{\mathbf{R}}_{\text{an,exp}}$-substructure of $\mathbf{R}((t))^{\text{LE}}$ containing $\mathbf{R}(e^{e^x}, xe^x)$. Then $x \notin N$.*

Proof. We first note that in fact $N \subset \mathbf{R}((t))^{\text{E}}$. We build a chain $(F_\alpha : \alpha < \lambda)$ of truncation closed $\mathcal{L}^{\mathbf{R}}_{\text{an}}$-elementary substructures of N such that:

i) $F_\alpha = \bigcup_{\beta<\alpha} F_\beta$ if α is a limit ordinal,

ii) there is $y_\alpha \in F_\alpha$ such that $F_{\alpha+1}$ is the smallest $\mathcal{L}_{\text{an}}^{\mathbf{R}}$-elementary submodel of $\mathbf{R}((t))^{\mathrm{E}}$ containing $F(e^{y_\alpha})$ for all $\alpha < \lambda$, and

iii) $N = \bigcup_{\alpha<\lambda} F_\alpha$.

Claim. Suppose F is a truncation closed $\mathcal{L}_{\text{an}}$-elementary substructure of $\mathbf{R}((t))^{\mathrm{E}}$ and the value group of F is an $\mathbf{R}$-vector space. Then F is an $\mathcal{L}_{\text{an}}^{\mathbf{R}}$-elementary substructure.

If $y \in F$ and $y > 0$, then $y = at^g(1+\epsilon)$ where $a \in \mathbf{R}$, $a > 0$, $t^g, \epsilon \in F$ and $v(\epsilon) > 0$. Then $y^r = a^r t^{rg}(1+\epsilon)^r$. Since $z \mapsto (1+z)^r$ is analytic near zero, $(1+\epsilon)^r \in F$. Since the value group of F is an $\mathbf{R}$-vector space, $t^{rg} \in F$. Thus $y^r \in F$. By the quantifier elimination from [5], F is an $\mathcal{L}_{\text{an}}^{\mathbf{R}}$-elementary submodel of $\mathbf{R}((t))^{\mathrm{E}}$.

The above claim, the truncation results of §3 of [3] and the valuation theoretic results from §3 of [2] guarantee that if F is a truncation closed $\mathcal{L}_{\text{an}}^{\mathbf{R}}$-elementary submodel of $\mathbf{R}((t))^{\mathrm{E}}$, $y \in \mathbf{R}((t))^{\mathrm{E}}$, $v(y) \notin v(F)$ and F^* is the smallest $\mathcal{L}_{\text{an}}^{\mathbf{R}}$-elementary submodel of $\mathbf{R}((t))^{\mathrm{E}}$ containing $F(y)$, then F^* is truncation closed and the value group of F^* is $v(F) \oplus \mathbf{R}v(y)$.

Let F_0 be the the smallest $\mathcal{L}_{\text{an}}^{\mathbf{R}}$-elementary submodel of $\mathbf{R}((t))^{\mathrm{E}}$ containing $\mathbf{R}(e^{e^x}, xe^x)$. By the above remarks F_0 is truncation closed. We can then build $(F_\alpha : \alpha < \lambda)$ satisfying i)-iii) above. Since

$$e^x = t^{-x} \text{ and } e^{e^x} = t^{-e^x},$$

the value group of F_0 is $\mathbf{R}(1+x) \oplus \mathbf{R}e^x$. Clearly $\mathbf{R}(1+x)$ is a convex subgroup of the value group of F_0. We argue that $\mathbf{R}(1+x)$ is a convex subgroup of the value group of F_α for all $\alpha < \lambda$. Thus $\mathbf{R}(1+x)$ is a convex subgroup of the value group of N. In particular $x \notin N$.

In fact the value group of F_0 is of the form $\mathbf{R}(1+x) \oplus H$ where $\operatorname{supp} h < \mathbf{R}$ for all $h \in H$. The next claim allows us to inductively show that this is true for the value group of F_α for all α.

Claim. Let $F \subset \mathbf{R}((t))^{\mathrm{E}}$ be a truncation closed $\mathcal{L}_{\text{an}}^{\mathbf{R}}$-elementary submodel with value group $G = \mathbf{R}(1+x) \oplus H$ where $\operatorname{supp} h < \mathbf{R}$ for all $h \in H$. Suppose $y \in F$, $e^y \notin F$ and F_1 is the smallest $\mathcal{L}_{\text{an}}^{\mathbf{R}}$-elementary submodel of $\mathbf{R}((t))^{\mathrm{E}}$ containing $F(e^y)$. Then F_1 is truncation closed and G_1, the value group of F_1, is $\mathbf{R}(1+x) \oplus H_1$ where $\operatorname{supp} h_1 < \mathbf{R}$ for all $h_1 \in H_1$.

Let $y = \alpha + \beta$ where supp $\alpha < 0$ and $v(\beta) \geq 0$. By our assumptions on G, supp $\alpha < \mathbf{R}$. Since $e^\beta \in F$, $F(e^y) = F(e^\alpha)$ and $e^\alpha = t^{-\alpha}$. Thus the value group of F_1 is $G \oplus \mathbf{R}\alpha$. Thus supp $(r\alpha + h) < 0$ for all $h \in H$. Since $H_1 = \mathbf{R}\alpha \oplus H$, this proves the claim.

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