REVISTA MATEMÁTICA de la
Universidad Complutense de Madrid
Volumen 10, número Suplementario: 1997

Levelled O-minimal structures.

David MARKER* and Chris MILLER**

Abstract

We introduce the notion of "levelled structure" and show that every structure elementarily equivalent to the real exponential field expanded by all restricted analytic functions is levelled.

An expansion $\mathfrak{R}$ of an ordered field $(R,<,+,\cdot,0,1)$ is *o-minimal* if every subset of R (parametrically) definable in $\mathfrak{R}$ is a finite union of points and open intervals; it is *exponential* if it defines an isomorphism of the ordered groups $(R,<,+)$ and $((0,\infty),<,\cdot)$, where $(0,\infty)$ denotes the positive elements of R.

Example The *ordered field of real numbers with restricted analytic functions* is the structure

$$\mathbb{R}_{\mathrm{an}} := (\mathbb{R},<,+,-,\cdot,0,1,(\tilde{f})_{f\in\mathbb{R}\{X,m\},m\in\mathbb{N}}),$$

where $\mathbb{R}\{X,m\}$ denotes the ring of all power series in $X_1,\dots,X_m$ over $\mathbb{R}$ that converge in a neighborhood of $[-1,1]^m$, and where $\tilde{f}:\mathbb{R}^m\to\mathbb{R}$ is defined for each $f\in\mathbb{R}\{X,m\}$ by

$$\tilde{f}(x) := \begin{cases} f(x), & x\in[-1,1]^m \\ 0, & \text{otherwise}\,. \end{cases}$$

We let $\mathbb{R}_{\mathrm{an,exp}}$ denote the o-minimal (see e. g. [2]) expansion of $\mathbb{R}_{\mathrm{an}}$ by the function $x\mapsto e^x:\mathbb{R}\to\mathbb{R}$.

---

*Partially supported by National Science Foundation grants DMS-9306159 and INT-922456, and an AMS Centennial Fellowship.

**Supported by National Science Foundation Postdoctoral Fellowship DMS-9407549.

Mathematics Subject Classification: 03C80-11U09

Servicio Publicaciones Univ. Complutense. Madrid, 1997.

Given an exponential o-minimal expansion $\mathfrak{R}$ of an ordered field $(R, <, +, \cdot, 0, 1)$ there is a unique definable differentiable ordered group isomorphism

$$E : (R, <, +, 0) \to ((0, \infty), <, \cdot, 1)$$

satisfying $E' = E$ on R. We denote this unique (hence 0-definable) function by exp. The function exp behaves (in $\mathfrak{R}$) to a large extent as the real exponential function e^x behaves when working over the real numbers. (See [5] for details on the above.) The compositional inverse of exp from $(0, \infty)$ onto R is denoted by log, and is called the *logarithm* function (of $\mathfrak{R}$); we extend log to be defined on R by setting $\log(x) := 0$ for $x \leq 0$. For $r \in R$ and $a > 0$, we put $a^r := \exp(r \log a)$.

Below, let $\mathfrak{R}$ denote an o-minimal expansion of an ordered exponential field $(R, <, +, -, \cdot, 0, 1, \exp)$; "definable" means "$\mathfrak{R}$-definable"—that is, "definable in $\mathfrak{R}$ with parameters from R"—unless stated otherwise. The reader is assumed to be familiar with the basic properties of o-minimal expansions of ordered exponential fields.

Whenever convenient, we regard any particular partial function as being totally defined by setting the function equal to 0 off its domain of definition.

Let e_0 denote the identity on R and put $e_{n+1}(t) := \exp(e_n(t))$ for $n \in \mathbb{N}$ and $t \in R$. Similarly, ℓ_0 denotes the identity on R and $\ell_{n+1}(t) := \log(\ell_n(t))$ for each $n \in \mathbb{N}$ and $t \in R$. We may also write ℓ_{-n} for e_n, depending on convenience; for example, ultimately we have $\ell_{j+k}(t) = \ell_j(\ell_k(t))$ for all $j, k \in \mathbb{Z}$. (*Ultimately* abbreviates "for all sufficiently large positive arguments".)

A function $f : R \to R$ is said to be *infinitely increasing* if f is ultimately strictly increasing and unbounded. Note that if f is definable, then f is infinitely increasing if and only if $\lim_{t \to +\infty} f(t) = +\infty$.

For functions $f, g : R \to R$ with g ultimately nonzero, we write $f(t) \sim g(t)$ if $\lim_{t \to +\infty} f(t)/g(t) = 1$.

Suppose that $f : R \to R$ is a definable infinitely increasing function and there exists $s \in \mathbb{Z}$ such that for some $k \in \mathbb{Z}$ we have $\ell_k(f(t)) \sim \ell_{k-s}(t)$. Then s is unique and $\ell_j(f(t)) \sim \ell_{j-s}(t)$ for all $j \geq k$. Following Rosenlicht [7], we then say that f has *level* s and we write $\text{level}(f) = s$. Equivalently, a definable infinitely increasing unary function f has level s if and only if there exists $N \in \mathbb{N}$ such that $\ell_{N+s}(f(t)) \sim \ell_N(t)$.

Definition. *The structure $\mathfrak{R}$ is levelled if every definable infinitely increasing unary function has level; its complete theory $Th(\mathfrak{R})$ is levelled if every $\mathfrak{A} \equiv \mathfrak{R}$ is levelled.*

We can now state the main result of this note.

Theorem $(\mathbb{R}_{\text{an,exp}})$ *is levelled.*

We defer the proof until later.

Levelled structures have nice properties that can show up in unexpected ways. For example, it is shown in [6] that if $\mathfrak{R}$ is levelled, and $* : R^2 \to R$ is definable, continuous and $(R, *)$ is a group, then $(R, *)$ is definably homeomorphic to $(R, +)$. (It is not known whether this property holds for $\mathfrak{R}$ without the assumption that $\mathfrak{R}$ be levelled.)

We now list some basic properties of level; the proofs are easy and we omit them.

Proposition. *Let f, f_1, f_2 be definable infinitely increasing unary functions with* $\text{level}(f) = s$, $\text{level}(f_1) = s_1$ *and* $\text{level}(f_2) = s_2$.

(1) *For each $k \in \mathbb{Z}$, ℓ_k has level k.*

(2) *If ultimately $f_1(t) \leq f_2(t)$, then $s_1 \leq s_2$.*

(3) *If $\alpha, \beta \in [1, \infty)$ are such that ultimately $f_1(t) \leq f_2(t)^\alpha$ and $f_2(t) \leq f_1(t)^\beta$, then $s_1 = s_2$.*

(4) *Both $f_1 + f_2$ and $f_1 \cdot f_2$ have level equal to* $\max(s_1, s_2)$.

(5) *The (ultimately defined) composition $f_1 \circ f_2$ has level $s_1 + s_2$.*

For $A \subseteq R^{m+1}$ and $x \in R^m$ put $A_x := \{t \in R : (x, t) \in A\}$, and for $f : A \to R$ and $x \in R^m$ define $f_x : A_x \to R$ by $f_x(t) := f(x, t)$.

Definition. *The structure $\mathfrak{R}$ is exponentially bounded, or e-bounded for short, if for each definable $f : R \to R$ there exists $n \in \mathbb{N}$ such that ultimately $|f(t)| \leq e_n(t)$.*

Note. Clearly, if $\mathfrak{R}$ is levelled then $\mathfrak{R}$ is e-bounded. On the other hand, if $\mathfrak{R}$ is e-bounded, then for every $m \in \mathbb{N}$ and definable function $f : R^{m+1} \to R$ the set

$$\{\text{level}\,(\mathrm{f_x}) : \mathrm{f_x} \text{ has level }\}$$

is finite. This follows from (1) and (2) of the previous proposition, and the fact that for f as above there is some $N \in \mathbb{N}$ such that for each $x \in R^m$ ultimately we have $|f_x(t)| \le e_N(t)$. (This fact is established over the reals using 4.18 of [4], but the proof given there goes through for o-minimal expansions of arbitrary ordered fields.)

Proposition. *The following are equivalent:*

(1) $\mathrm{Th}(\mathfrak{R})$ *is levelled.*

(2) *For every $m \in \mathbb{N}$ and definable function $f : R^{m+1} \to R$ there exist integers $N, s(1), \ldots, s(k)$ with $N \ge 0, s(1), \ldots, s(k)$ such that for every $x \in R^m$, if f_x is infinitely increasing, then $\ell_N(f_x(t)) \sim \ell_{N-s(i)}(t)$ for some $i \in \{1, \ldots, k\}$.*

(3) $\mathfrak{R}$ *is e-bounded, and for every $m \in \mathbb{N}$ and definable function $f : R^{m+1} \to R$ there exists $N \in \mathbb{N}$ such that for every $x \in R^m$, if f_x is infinitely increasing, then $\ell_{N+s}(f_x(t)) \sim \ell_N(t)$ for some integer s $(= s(x))$.*

Proof. (1) $\Rightarrow$ (2). We may assume that f is 0-definable, say by an $(m+2)$-ary formula φ in the language of $\mathfrak{R}$. Let $v = (v_1, \ldots, v_m)$, and for each pair of integers (j, s) let $\psi_{j,s}(v)$ be the m-ary formula expressing: "If $\varphi(v, t, y)$ defines an infinitely increasing function $y = F_v(t)$, then $\ell_j(F_v(t)) \sim \ell_{j-s}(t)$." Since $\mathrm{Th}(\mathfrak{R})$ is levelled, for every $\mathfrak{A} \equiv \mathfrak{R}$ and every $a \in A^m$ (where A is the underlying set of $\mathfrak{A}$) there exist $j, s \in \mathbb{Z}$ such that $\mathfrak{A} \models \psi_{j,s}(a)$. By compactness, there exist integers $j(1), \ldots, j(k), s(1), \ldots, s(k)$ such that

$$\mathfrak{R} \models \forall v \left[\psi_{j(1),s(1)}(v) \vee \ldots \vee \psi_{j(k),s(k)}(v)\right].$$

Put $N := \max\{0, j(1), \ldots, j(k), s(1), \ldots, s(k)\}$.

(2) $\Rightarrow$ (1). Let $\mathfrak{A} \equiv \mathfrak{R}$ and g be an $\mathfrak{A}$-definable infinitely increasing unary function; say that g is defined by $\varphi(a, t, y)$ with $a \in A^m$ for some $m \in \mathbb{N}$ and φ an $(m+2)$-ary formula in the language of $\mathfrak{R}$. Let X be the 0-definable set consisting of all $x \in A^m$ such that $\varphi(x, t, y)$ defines an infinitely increasing unary function $y = f_x(t)$. Now define $f : A^{m+1} \to A$ by

$$f(x,t) := \begin{cases} f_x(t), & x \in X \\ 0, & \text{otherwise}. \end{cases}$$

Then f is 0-definable, and by elementary equivalence there exist $N \in \mathbb{N}$ and $s \in \mathbb{Z}$ with $\ell_N(f_a(t)) \sim \ell_{N-s}(t)$; that is, g has level s.

That (2) $\Rightarrow$ (3) is clear, and (3) $\Rightarrow$ (2) follows from the note preceding the statement of this proposition.

■

Note. If Th($\mathfrak{R}$) is levelled and $\mathfrak{R}' := (R, <, +, \cdot, 0, 1, \exp, \ldots)$ is a reduct of $\mathfrak{R}$, then Th($\mathfrak{R}'$) is levelled.

(This is immediate from the preceding proposition, but this fact can also be established directly by a basic model-theoretic argument.)

We have no example at present of an o-minimal expansion of an ordered exponential field whose complete theory is known to be not levelled. However, Boshernitzan [1] has shown that there are real analytic functions $f : (a, \infty) \to \mathbb{R}$ satisfying $f(t+1) = e^{f(t)}$ for $t > a$ whose germs at $+\infty$ belong to Hardy fields; such a function clearly cannot be ultimately bounded by any fixed compositional iterate of e^x, hence does not have level. Also established in [1] is the existence of ultimately real analytic solutions to the functional equation $g(g(x)) = e^x$ (a so-called "half-iterate" of e^x) whose germs belong to Hardy fields. No such function could have level (otherwise, $1 = \text{level}(g \circ g) = 2\text{level}(g)$). It seems plausible that $(\mathbb{R}, <, +, \cdot, \exp)$ could be expanded by some such functions to an o-minimal structure.

Proof of the Theorem

We now fix some $\mathfrak{R} \equiv \mathbb{R}_{\text{an,exp}}$, with underlying set R. We must show that $\mathfrak{R}$ is levelled.

We let $\mathcal{L}_{\text{an}}$ and T_{an} denote respectively the language and the theory of $\mathbb{R}_{\text{an}}$, and $\mathcal{L}_{\text{an,exp}}$ and $T_{\text{an,exp}}$ denote respectively the language and the theory of $\mathbb{R}_{\text{an,exp}}$.

We assume familiarity with the main results from [2,3]; we must first modify some of the constructions from those papers.

If G is a divisible ordered abelian group, then $R((t^G))$ denotes the field of formal power series of the form $f = \sum a_g t^g$, where g ranges over G, each $a_g \in R$ and $\operatorname{supp} f := \{g : a_g \neq 0\}$ is well ordered. Since the reduct of $\mathfrak{R}$ to $\mathcal{L}_{\text{an}}$ is a model of T_{an}, we can naturally equip $R((t^G))$ with an $\mathcal{L}_{\text{an}}$-structure so that $R((t^G)) \models T_{\text{an}}$.

There is a natural valuation $v : R((t^G))^\times \to G$ given by $v(f) := \min \operatorname{supp} f$. We extend this valuation to $R((t^G))$ by putting $v(0) := \infty$, with $v(f) < \infty$ for all $f \in R((t^G))^\times$.

In the following, we say that a map F from an ordered ring D into an ordered ring D' is a *partial exponential* if F is an order-preserving homomorphism from the additive group of D into the multiplicative group of positive elements of D'.

Construction of $R((t))^E$

We construct a chain of divisible ordered abelian groups

$$\{0\} := \Gamma_{-1} \subset \Gamma_0 \subset \Gamma_1 \subset \cdots$$

such that Γ_{n-1} is a convex subgroup of Γ_n for each $n \in \mathbb{N}$. Putting $K_n := R((t^{\Gamma_n}))$ for each $n \in \mathbb{Z}$ with $n \geq -1$, we will obtain an $\mathcal{L}_{\text{an}}$-elementary chain

$$K_{-1} \prec K_0 \prec K_1 \prec \cdots$$

where Γ_{n-1} is an ordered R-subspace of K_n for each $n \in \mathbb{N}$. We identify $K_{-1} = R((t^{\{0\}}))$ with R. We will define partial exponential maps $E_{n-1} : K_{n-1} \to K_n$ such that $E_{n-1} \subset E_n$ for each $n \in \mathbb{N}$.

Let $\Gamma_0 := R$. Let $E_{-1} : R \to R((t^R))$ be given by $E_{-1}(r) := \exp(r)$. Suppose now that $n > 0$ and that Γ_m and E_{m-1} have been constructed for $m < n$. Put

$$O_n := \{x \in K_n : v(x) \geq \gamma \text{ for some } \gamma \in \Gamma_{n-1}\}$$

and

$$\mathrm{m_n} := \{\mathrm{x} \in \mathrm{K_n} : \mathrm{v(x)} > \Gamma_{\mathrm{n-1}}\}.$$

Note that $O_n = K_{n-1} \oplus \mathrm{m_n}$. We extend E_{n-1} to a partial exponential $\widehat{E}_n : O_n \to K_n$ by setting $\widehat{E}_n(x) := E_{n-1}(r) \sum_{i \in \mathbb{N}} (\alpha^i / i!)$ for $x = r + \alpha$ with $r \in K_{n-1}$ and $\alpha \in \mathrm{m_n}$. (Note that $\sum_{i \in \mathbb{N}} (\alpha^i / i!)$ is well-defined since $v(\alpha) > 0$.) Let $J_n := \{x \in K_n : \operatorname{supp} x < \Gamma_{n-1}\}$; so $K_n = J_n \oplus O_n$ as K_{n-1}-linear spaces. Then we put $\Gamma_{n+1} := J_n \oplus \Gamma_n \subseteq K_n$, ordered as an R-linear subspace of K_n, so Γ_n is convex in Γ_{n+1}.

Finally, extend $\widehat{E}_n$ to the partial exponential $E_n : K_n \to K_{n+1}$ given by
$E_n(x) := t^{-a} \widehat{E}_n(b)$ for $x = a + b$ with $a \in J_n$ and $b \in O_n$.

Put $R((t))^E := \bigcup K_n$, $\Gamma := \bigcup \Gamma_n$ and $E := \bigcup E_n$. Then $R((t))^E \models T_{\text{an}}$ and $E : R((t))^E \to R((t))^E$ is a partial exponential that agrees with the restricted exponential function on $[-1,1]$ and ultimately dominates all polynomials. Note that $R((t))^E$ is a subfield of $R((t^\Gamma))$.

Construction of $\mathrm{R}((t))^{LE}$

Similarly as in §2 of [3] we obtain an $\mathcal{L}_{\mathrm{an,exp}}$-embedding $\Phi : R((t))^E \to R((t))^E$ such that $\Phi(t^{-1}) = E(t^{-1})$. Let x denote t^{-1}. Put $L_0 := \mathrm{R}((t))^E$. We can find an $\mathcal{L}_{\mathrm{an,exp}}$-extension L_1 of L_0 and an isomorphism $\eta_1 : L_1 \to \mathrm{R}((t))^E$ such that η_1 maps $\mathrm{R}((t))^E$ onto $\Phi(\mathrm{R}((t))^E)$. Then $E(\eta_1^{-1}(x)) = x$. Indeed, every positive element g of $\mathrm{R}((t))^E$ has a logarithm in L_1 (that is, there exists $h \in L_1$ such that $E(h) = g$). We continue by constructing for each $n \in \mathbb{N}$ an $\mathcal{L}_{\mathrm{an,exp}}$-extension L_{n+1} of L_n and an isomorphism $\eta_{n+1} : L_{n+1} \to \mathrm{R}((t))^E$ such that η_{n+1} maps L_n onto $\Phi(\mathrm{R}((t))^E)$. Every element of L_n has a logarithm in L_{n+1}. Finally, put $\mathrm{R}((t))^{LE} := \bigcup L_n$.

Every positive element of $\mathrm{R}((t))^{LE}$ has a logarithm in $\mathrm{R}((t))^{LE}$. Thus, from the axiomatization of $T_{\mathrm{an,exp}}$ from [2], we see that $\mathrm{R}((t))^{LE} \models T_{\mathrm{an,exp}}$. By §5 of [2], we may identify the field $\mathcal{H}$ of germs at $+\infty$ of definable unary functions with the smallest elementary substructure of $\mathrm{R}((t))^{LE}$ containing R and the element $x = t^{-1} \in \mathrm{R}((t))^{LE}$. Therefore, in what follows we routinely identify any given definable unary function f with its germ $f \in \mathcal{H}$, which in turn is identified with the element $f \in \mathrm{R}((t))^{LE}$. In particular, note that for every definable unary function f we have $E(f) = \exp(f)$, and if f is ultimately positive then $E(\ell(f)) = f$. Thus, there is no harm in denoting the logarithm function for $\mathrm{R}((t))^{LE}$ by ℓ, and using the notation ℓ_k for $k \in \mathbb{Z}$ in the obvious fashion. Note in particular that $\eta_n^{-1}(x) = \ell_n(x)$ for all $n \geq 1$.

Given definable unary functions f and g with g ultimately nonzero, we have $f(x) \sim g(x)$ if and only if $\lim_{x\to+\infty} f(x)/g(x) = 1$ if and only if $v(f-g) > v(g)$. Thus, given nonzero $f, g \in \mathrm{R}((t))^{LE}$, we write $f \sim g$ for $v(f-g) > v(g)$, that is, $f = g(1+\epsilon)$ for some $\epsilon \in \mathrm{R}((t))^{LE}$ with $v(\epsilon) > 0$. Note also that $v(f) = v(g)$ if and only $f \sim cg$ for some nonzero $c \in R$. It is easy to see that $\sim$ is a congruence relation on the multiplicative group of nonzero elements of $\mathrm{R}((t))^{LE}$.

Lemma. *Let $f, g \in \mathrm{R}((t))^{LE}$ with $f, g > 0$, and $v(g) < 0$.*

(1) *If $f = gh$ with $g > h^r$ for all $r \in R$, then $\ell(f) \sim \ell(g)$.*

(2) *If $v(f) = v(g)$, then $\ell_k(f) \sim \ell_k(g)$ for all $k > 0$.*

Proof. For (1), note that for all positive $r \in R$ we have

$$r(\ell(f) - \ell(g)) = r\ell(h) = \ell(h^r) < \ell(g),$$

that is, $v(\ell(f) - \ell(g)) > v(\ell(g))$.

An easy induction on k yields (2).

■

Claim. Let $g \in \mathrm{R}((t))^E$ with $v(g) < 0$ and $g > 0$, and let k be the least positive integer such that $v(g) \in \Gamma_k$ (as in the construction of $\mathrm{R}((t))^E$). Then $\ell_{2+k}(g) \sim \ell_2(x)$.

Proof. We prove this by induction on k. First, suppose $k = 0$. Then $v(g) = v(x^r)$ for some positive $r \in R$, and by (2) of the Lemma we have

$$\ell_2(g) \sim \ell_2(x^r) = \ell(r) + \ell_2(x) \sim \ell_2(x).$$

Suppose now that the result holds for a certain $k > 0$ and let $v(g) \in \Gamma_{k+1} \setminus \Gamma_k$. Then $v(g) = \delta + \gamma$ where $\delta \in J_k$, $\delta < 0$ and $\gamma \in \Gamma_k$. Hence, $v(\delta) \in \Gamma_k \setminus \Gamma_{k-1}$ and $v(\delta) < \Gamma_{k-1}$, so $\ell_{2+k}(-\delta) \sim \ell_2(x)$ by the inductive assumption. Also, we have $g = t^\delta(at^\gamma + \mu)$, with $a \in R$ and $v(\mu) > \gamma$. Now $\delta < \Gamma_k$, so $\ell(g) \sim \ell(t^\delta) = -\delta$ and $\ell_{2+k+1}(g) \sim \ell_{2+k}(-\delta)$ by (1) and (2) of the Lemma, respectively. Thus, $\ell_{2+k+1}(g) \sim \ell_2(x)$.

■

Claim. Let $g \in L_n$ (as in the construction of $\mathrm{R}((t))^{LE}$), $g > 0$ and $v(g) < 0$. Then there exists $s \in \mathbb{Z}$ such that $\ell_{2+n+s}(g) \sim \ell_{2+n}(x)$.

Proof. Let $f_n : L_n \to \mathrm{R}((t))^E$ be as in the construction of $\mathrm{R}((t))^{LE}$. By the previous claim, there is some $k \in \mathbb{N}$ such that $\ell_{2+k}(f_n(g)) \sim \ell_2(x)$. Since f_n is an $\mathcal{L}_{\mathrm{an,exp}}$-isomorphism, we have

$$\ell_{2+k}(g) \sim \ell_2(f_n^{-1}(x)) = \ell_2(\ell_n(x)) = \ell_{2+n}(x).$$

Hence $\ell_{2+n+s}(g) \sim \ell_{2+n}(x)$, where $s = k - n$.

■

Definition. *An element $f \in \mathrm{R}((t))^{LE}$ has level s for $s \in \mathbb{Z}$ if $f > 0$, $v(f) < 0$ and there is an $N \in \mathbb{N}$ such that $\ell_{N+s}(f) \sim \ell_N(x)$.*

It is immediate from the preceding claim and the construction of $\mathrm{R}((t))^{LE}$ that every $g \in \mathrm{R}((t))^{LE}$ with $g > 0$ and $v(g) < 0$ has level s for some $s \in \mathbb{Z}$. Hence, $\mathfrak{R}$ is levelled.

■

References

[1] M. Boshernitzan, *Hardy fields, existence of transexponential functions, and the hypertranscendence of solutions to* $g(g(x)) = e^x$, Aequationes Math. **30** (1986), 258-280.

[2] L. van den Dries, A. Macintyre and D. Marker, *The elementary theory of restricted analytic fields with exponentiation*, Ann. of Math. **140** (1994), 183–205.

[3] ———, *Logarithmic-exponential power series*, J. London Math. Soc. (2) (to appear).

[4] L. van den Dries and C. Miller, *Geometric categories and o-minimal structures*, Duke Math. J. **84** (1996), 497–540.

[5] C. Miller, *A growth dichotomy for o-minimal expansions of ordered fields*, Logic: From Foundations to Applications (European Logic Colloquium) (W. Hodges *et al.*, eds.), Oxford University Press, (1996), pp. 385–399.

[6] C. Miller and S. Starchenko, *A growth dichotomy for o-minimal expansions of ordered groups*, (preprint, Vanderbilt Mathematics Preprint Series no. 96–001).

[7] M. Rosenlicht, *Growth properties of functions in Hardy fields*, Trans. Amer. Math. Soc. **299** (1987), 261–272.

University of Illinois at Chicago,
Chicago IL 60607-7045
E-mail address: marker@tarski.math.uic.edu

University of Illinois at Chicago,
Chicago IL 60607-7045
E-mail address: clmiller@uic.edu