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A multiplier theorem for the Hankel transform.

Rafał KAPELKO

Abstract

Riesz function technique is used to prove a multiplier theorem for the Hankel transform, analogous to the classical Hörmander-Mihlin multiplier theorem [6].

The celebrated Hörmander-Mihlin multiplier theorem [6] says that if a function m on R^n satisfies the following condition

$$\sup_{R>0} R^{-n} \sum_{|l|\leq k_0} \int_{R<|x|<2R} |R^{|l|} D^l m(x)|^2 dx < \infty \tag{1}$$

for some integer $k_0 > \frac{n}{2}$ then the operator T_m defined by $(T_m g)\widehat{\ } = m\widehat{g}$ is bounded on every $L^p(R^n)$, $1 < p < \infty$.

Restriction of the theorem to the set of radial functions on R^n gives the multiplier theorem on spaces $L^p(R_+, x^{2\alpha+1}dx)$, $1 < p < \infty$ with $\alpha = \frac{n-2}{2}$. The ordinary Fourier transform on R^n has to be replaced by the Hankel transform

$$\widehat{f}(y) = 2^\alpha \Gamma(\alpha+1) \int_0^\infty f(x)(yx)^{-\alpha} J_\alpha(xy) x^{2\alpha+1} dx, \tag{2}$$

where J_α is the Bessel function of the first kind of order α.

The assumption (1) gets even the simpler form

$$\sup_{R>0} \left(\int_R^{2R} |x^k m^{(k)}(x)|^2 \frac{1}{x} dx \right)^{\frac{1}{2}} < \infty,$$

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where $k = 0, 1, 2, ..., k_0$ and $k_0 > \alpha + 1$.

It is quite natural to expect that the multiplier theorem should have an extension to all values $\alpha \geq \frac{1}{2}$ of the real parameter. However the exact repetition of the Hörmander proof does not lead to effect, mainly because the Hankel transform of the derivative of a function has no representation in terms of the transformation of the function. In order to omit this difficulty there were developed two technics in the literature.

The first one, [2], is indirect, uses a relation between the Jacobi polynomials and the Bessel functions but the result obtained there is weaker then expected. The proof goes under stronger assumption

$$\sup_{R>0} R^{-1} \int_R^{2R} |x^{k_0} m^{(k_0)}(x)|^2 x^{-1} dx < \infty, k_0 = [\alpha] + 2.$$

The second one, [4], developes the original Hörmander's technique but instead of the ordinary derivative of a function it makes use of the powers of a Sturm-Liouville operator. The result is like the Hörmander one, but $k_0 > \alpha + 1$ must be an even number.

The aim of the note is to prove the multiplier theorem in full generality. We assume that k_0 is the least integer greater than $\alpha + 1$. In fact k_0 may be a real number if one uses the Weyl fractional derivatives instead of ordinary derivatives. The main idea is based on the fact that the Hankel transform of Riesz function $R_u^{k_0}(x^2)$ has especially simple form. Then we follow the arguments of Gosselin and Stempak [4].

For a bounded function m on R_+ we define the multiplier operator T_m by $(T_m g)\widehat{\ } = m\widehat{g}$, where $\widehat{\ }$ denotes the Hankel transform (2).

Theorem 1. Fix $\alpha \geq \frac{1}{2}$ and let k_0 denote the least integer greater than $\alpha + 1$. Assume that a bounded function m on R_+ satisfies

$$\sup_{R>0} \left(\int_R^{2R} |x^k m^{(k)}(x)|^2 \frac{1}{x} dx \right)^{\frac{1}{2}} < \infty,$$

where $k = 0, 1, ..., k_0$. Then the operator T_m is of weak-type (1,1) and, consequently is bounded on every $L^p(R_+, x^{2\alpha+1}dx)$, $1 < p < \infty$.

In the proof we use the notion of the generalized convolution

$$f * g(x) = \int_0^\infty f(y) T_\alpha^y g(x) y^{2\alpha+1} dy,$$

where T_α^y is the generalized translation operator

$$T_\alpha^y g(x) = b(\alpha)\int_0^\pi g((x,y)_\theta)\sin^{2\alpha}(\theta)d\theta,$$

$(x,y)_\theta = (x^2+y^2-2xy\cos\theta)^{\frac{1}{2}}$, $b(\alpha) = \pi^{-\frac{1}{2}}\Gamma(\alpha+1)\left(\Gamma(\alpha+\frac{1}{2})\right)^{-1}$ and f, g are suitable functions on the half-line (cf [5]).

As usual we use C with subscripts or without subscripts for a constant which is not necessarily the same at each occurence.

Proof. The main idea of the proof is based on the fact that the Hankel transform of the function

$$R(x) = \frac{1}{\Gamma(k_0)}(u-x^2)_+^{k_0-1}$$

has a very simple form

$$\widehat{R}(x) = \Gamma(\alpha+1)2^{\alpha+k_0-1}\left(\frac{\sqrt{u}}{x}\right)^{\alpha+k_0} J_{\alpha+k_0}(\sqrt{u}x). \tag{3}$$

(cf. [7, §4 Theorem 4.15]).

As usual we cut the function m into small pieces by using a fixed bump function. Let $\Psi \in C_0^\infty(R_+)$ with support in $(1,2)$ such that $\sum_{-\infty}^{\infty}\Psi(2^{-j}x) = 1$ and $m_j(x) = m(x)\Psi(2^{-j}x)$. Define new family of functions $h(x) = m(x^2)$, $h_j(x) = m_j(x^2)$. First using (3) and applying the method of [4], we will obtain the theorem for h. More precisisely we will prove

$$||T_h g||_p \le C_{1,p}||g||_p. \tag{4}$$

Then we will show how to deduce the thesis for the function m from the thesis for the function h.

For h_j we write the reproducing formula

$$h_j(x) = \frac{1}{\Gamma(k_0)}\int_{2^j}^{2^{j+1}} m_j^{(k_0)}(u)\left(u-x^2\right)_+^{k_0-1}du.$$

By (3) we have

$$\widehat{h}_j(x) = \Gamma(\alpha+1)2^{\alpha+k_0-1}\int_{2^j}^{2^{j+1}} m_j^{(k_0)}(u)\left(\frac{\sqrt{u}}{x}\right)^{\alpha+k_0} J_{\alpha+k_0}(\sqrt{u}x)du. \tag{5}$$

Then $T_h = \sum_{-\infty}^{\infty} T_{h_j}$ where $T_{h_j} g = \widehat{h}_j * g$ and $g \in L^1(R_+, x^{2\alpha+1}dx)$. In order to prove (4) it is sufficient to establish (cf. [4, p.659] and [1, p.75]) that

$$\sum_{j=-\infty}^{\infty} \int_{|x-y_0|>2|y-y_0|} \left| T_\alpha^y \widehat{h}_j(x) - T_\alpha^{y_0} \widehat{h}_j(x) \right| x^{2\alpha+1} dx \leq C, \tag{6}$$

with $C > 0$ independent of y, y_0 0.

An application of Leibniz formula yields

$$\left(\int_{2^j}^{2^{j+1}} |m_j^{(k_0)}(x)|^2 dx \right)^{\frac{1}{2}} \leq C(2^j)^{\frac{1}{2}-k_0}, \tag{7}$$

where C does not depend on j, and $k_0 = \alpha + 1 + \epsilon$ for an $\epsilon > 0$. We prove the following estimates:

$$\int_t^\infty |\widehat{h}_j(x)| x^{2\alpha+1} dx \leq C(\sqrt{2^j}\, t)^{-\epsilon}, \tag{8}$$

$$\int_0^\infty |\widehat{h}_j(x)| x^{2\alpha+1} dx \leq C. \tag{9}$$

To prove (8) observe that by definition, $\widehat{h}_j(x)$ coincides with the Hankel transform of the function

$$H_j(y) = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+k_0+1)} \chi_{[\sqrt{2^j}, \sqrt{2^{j+1}}]}(y) m_j^{(k_0)}(y^2),$$

with respect to the measure $d_1\mu(x) = x^{4\alpha+3+2\epsilon}$.

Now Schwartz' inequality, the Plancherel formula applied to H_j and (7) give

$$\begin{aligned}
\int_t^\infty |\widehat{h}_j(x)| x^{2\alpha+1} dx &\leq \left(\int_0^\infty |\widehat{h}_j(x)|^2 (x^{2\alpha+1+\frac{1}{2}+\epsilon})^2 dx \right)^{\frac{1}{2}} \left(\int_t^\infty \frac{1}{x^{1+2\epsilon}} dx \right)^{\frac{1}{2}} \\
&= \left(\int_0^\infty |\widehat{h}_j(x)|^2 x^{4\alpha+3+2\epsilon} dx \right)^{\frac{1}{2}} t^{-\epsilon} \frac{1}{\sqrt{2\epsilon}} \\
&= C_{\alpha,k_0} \left(\int_{2^j}^{2^{j+1}} |m_j^{(k_0)}(p)|^2 p^{2\alpha+1+\epsilon} dp \right)^{\frac{1}{2}} t^{-\epsilon} \frac{1}{\sqrt{2\epsilon}} \\
&\leq C(2^j)^{\alpha+\frac{1}{2}+\frac{\epsilon}{2}} (2^j)^{\frac{1}{2}-k_0} t^{-\epsilon} = C(\sqrt{2^j} t)^{-\epsilon}.
\end{aligned}$$

To prove (9) we use (8). Now changing the variable $y = x\sqrt{u}$ in (5) we get

$$\int_0^{2^{-\frac{j}{2}}} |\widehat{h}_j(x)|x^{2\alpha+1}dx \le C_3 \int_{2^j}^{2^{j+1}} |m_j^{(k_0)}(u)|u^{k_0-1}du \int_0^{\sqrt{2}} |J_{\alpha+k_0}(y)|\frac{y^{2\alpha+1}}{y^{\alpha+k_0}}dy.$$

But Schwarz' inequality and (7) yield

$$\int_{2^j}^{2^{j+1}} |m_j^{(k_0)}(u)|u^{k_0-1}du \le C_1 \left(\int_{2^j}^{2^{j+1}} |m_j^{(k_0)}(u)|^2 du\right)^{\frac{1}{2}} (2^j)^{k_0-\frac{1}{2}} \le C_4.$$

Since $J_{\alpha+k_0}(x)$ is $x^{\alpha+k_0}$ asymptotically at 0^+ we have

$$\int_0^{2^{-\frac{j}{2}}} |\widehat{h}_j(x)|x^{2\alpha+1}dx \le C_2.$$

Also, by (8)

$$\int_0^{\infty} |\widehat{h}_j(x)|x^{2\alpha+1}dx \le C_2 + \int_{2^{-\frac{j}{2}}}^{\infty} |\widehat{h}_j(x)|x^{2\alpha+1}dx \le C.$$

Finally, to get (6) we use inequality (8) with estimates of Gosselin and Stempak (cf. [4, p.661])

$$\begin{aligned}\int_{|x-y_0|\,2|y-y_0|} &\left|T_\alpha^y\widehat{h}_j(x) - T_\alpha^{y_0}\widehat{h}_j(x)\right| x^{2\alpha+1}dx \\ &\le \textstyle\int_{|y-y_0|}^{\infty} |\widehat{h}_j(x)|x^{2\alpha+1}dx + \int_{2|y-y_0|}^{\infty} |\widehat{h}_j(x)|x^{2\alpha+1}dx \\ &\le C_1(1+2^{-\epsilon})(\sqrt{2^j}|y-y_0|)^{-\epsilon},\end{aligned}$$

which will work for $\sqrt{2^j}|y-y_0| \ge 1$.

Since h_j has support in $(0, \sqrt{2^{j+1}})$ it follows from [4, Corollary 2.2] and (9) that

$$\begin{aligned}\int_{|x-y_0|\,2|y-y_0|} &\left|T_\alpha^y\widehat{h}_j(x) - T_\alpha^{y_0}\widehat{h}_j(x)\right| x^{2\alpha+1}dx \\ &\le ||T_\alpha^y\widehat{h}_j - T_\alpha^{y_0}\widehat{h}_j||_{L^1(R_+,x^{2\alpha+1}dx)} \\ &\le C_1\sqrt{2^{j+1}}\,|y-y_0|\,||\widehat{h}_j||_{L^1(R_+,x^{2\alpha+1}dx)} \\ &\le \sqrt{2}C\,C_1\sqrt{2^j}|y-y_0|,\end{aligned}$$

which will be enough whenever $\sqrt{2^j}|y-y_0| < 1$.

This completes the proof of (6) and, consequently for the function h. The result for the function m follows than from the lemma below.

Lemma 1. For $\alpha > 0$ the transformation $x \to x^\alpha$ of $[0, \infty)$ induces the isomorphism $m(x) \to m(x^\alpha)$ of the space of all functions for which

$$||m||_{2,k_0} = \sup_{R>0} \left(\int_R^{2R} |x^k m^{(k)}(x)|^2 \frac{1}{x} dx \right)^{\frac{1}{2}} < \infty$$

for $k = 0, 1, 2, ..., k_0$.

Proof. This is a simple consequence of fact that space $||m||_{2,k_0}$ is invariant under multiplication by x^α and Lebniz formula.

Remark. The method of Riesz function works when we use the Weyl fractional derivatives instead of ordinary derivatives.

A function f on R_+ has the Weyl fractional derivative of order $v > 0$ if there exists a measurable function g on R_+ such that

$$f(x) = \frac{1}{\Gamma(v)} \int_x^\infty (t-x)^{v-1} g(t) dt$$

for almost all $x > 0$. The function g is unique up to a set of measure zero. It is denoted $f^{(v)}$ and called v-fractional derivative of order v.

The problem is that for a positive integer v there exist smooth functions in the ordinary sense but not in the Weyl sense.

Theorem 2. Let m be a bounded function on R_+ satisfies the condition

$$\sup_{R>0} \left(\int_R^{2R} |x^v m^{(v)}(x)|^2 \frac{1}{x} dx \right)^{\frac{1}{2}} < \infty,$$

where $v > \alpha + 1$, $m^{(v)}$ is the Weyl fractional derivative. Then the operator T_m is of weak-type (1,1) and, consequently is bounded on every $L^p(R_+, x^{2\alpha+1}dx)$, $1 < p < \infty$.

Proof. As in the proof of Theorem 1 we define $h(x) = m(x^2)$ and obtain the theorem for function h. To do this we don't work with bump functions and define

$$h_j(x) = \frac{1}{\Gamma(v)} \int_{2^j}^{2^{j+1}} m^{(v)}(u) \left(u - x^2\right)_+^{v-1} du.$$

Clerly $T_h = \sum_{-\infty}^{\infty} T_{h_j}$ where $T_{h_j} g = \widehat{h}_j * g$. The rest is the exact repetition of the proof of Theorem 1. Finally the result for the function m follows from lemma below.

Lemma 2. For $\alpha > 0$ the transformation $x \to x^\alpha$ of $[0, \infty)$ induces the isomorphism $m(x) \to m(x^\alpha)$ of the space of all function for which

$$||m||_{2,v} = \sup_{R>0} \left(\int_{\frac{R}{2}}^{R} |x^v m^{(v)}(x)|^2 \frac{1}{x} dx \right)^{\frac{1}{2}} < \infty.$$

Proof. The lemma is a modification of [3, Proposition 3.9]. The only difference is the norm $||.||_{(\mu),2,1}$ is changed into the norm $||.||_{2,v}$ and the proof is essentially the same.

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Instytut Matematyczny
Uniwersytet Wrocławski
pl. Grunwaldzki 2/4
50-384 Wrocław
POLAND
Rafał Kapelko
e-mail: `kapelko@math.uni.wroc.pl`

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