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Coalescence of measures and f-rearrangements of a function.

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Abstract

This paper addresses the question of characterizing optimum values in the problem

$$\sup\{\nu(E) : \mu(E) \leq C\}, \tag{1}$$

where μ and ν are measures defined on a σ-finite measurable space X. With this purpose, the f-rearrangement of a function g is introduced so as to formalize the idea of rearranging the level sets of the function g according to how these sets are arranged in a given function f. A characterization of optima of problem (1) is then obtained in terms of $d\nu/d\mu$-rearrangements, $d\nu/d\mu$ being the Radon-Nikodym derivative of the measure ν with respect to μ. When X is a topological space and μ, ν are Borel measures, we say that ν is coalescent with respect to ν when, for every $C > 0$, there exist *connected* optima solving problem (1). A general criterion for coalescence is given and some simple examples are discussed.

1 Introduction and preliminaries

As a motivation to grasp the problems this paper is concerned with, we first consider a series of simple examples. Let f be a non-negative real function such that $f \in C^0(\mathbb{R}) \cap \mathcal{L}^\infty(\mathbb{R})$ and $E = \cup_{k=1}^n (a_k, b_k)$ be a finite union of mutually disjoint open intervals. Given a real constant $C > 0$,

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we are interested in finding the maximum of the area under the graph of $f\chi_E$ when the measure of E is kept constant and equal to C; that is, we look for $\max\int_E f(x)\,dx$ subjected to the restriction $|E| = C$. This optimization problem can be posed as a standard one in Mathematical Programming. In fact, we can write it as

$$\begin{cases} \max\sum_{k=1}^{n}\int_{a_k}^{b_k} f(x)\,dx \\ \qquad \text{subjected to} \\ \sum_{k=1}^{n}(b_k - a_k) = C \\ a_1 \le b_1 \le a_2 \le b_2 \le \cdots \le a_n \le b_n \end{cases} . \tag{2}$$

Now consider a plane curve γ parametrized by $\gamma(\theta) = \rho(\theta)(\cos\theta, \sin\theta)$, $0 \le \theta < 2\pi$, being $\rho > 0$ a smooth function. An observer O placed at the origin of coordinates see the arc $\{\gamma(\theta) : \alpha < \theta < \beta\}$ of γ under an angle of measure $\beta - \alpha$. More generally, the sum $\Theta(E) = \sum_{k=1}^{n}(\beta_k - \alpha_k)$ must be taken as the total visual angle under which the finite union of arcs $E = \cup_{k=1}^{n}\{\gamma(\theta) : \alpha_k < \theta < \beta_k\}$ of γ is observed by O. The problem then consists of determining the "configurations" E on γ which realize a maximum in the visual angle $\Theta(E)$ provided that the length of these configurations is maintained equal to a constant $C > 0$. Since the length of $E = \cup_{k=1}^{n}\{\gamma(\theta) : \alpha_k < \theta < \beta_k\}$ is $\sum_{k=1}^{n}\int_{\alpha_k}^{\beta_k} ds$, where $ds = \sqrt{\rho^2(\theta) + [\rho'(\theta)]^2}d\theta$ denotes the differential of arc on γ, the following standard form can be given to our problem:

$$\begin{cases} \max\sum_{k=1}^{n}(\beta_k - \alpha_k) \\ \qquad \text{subjected to} \\ \sum_{k=1}^{n}\int_{\alpha_k}^{\beta_k}\sqrt{\rho^2(\theta) + [\rho'(\theta)]^2}\,d\theta = C \\ \alpha_1 \le \beta_1 \le \alpha_2 \le \beta_2 \le \cdots \le \alpha_n \le \beta_n \end{cases} . \tag{3}$$

Indeed, the positive constant C has to be taken lesser than $\int_0^{2\pi}\sqrt{\rho^2(\theta) + [\rho'(\theta)]^2}\,d\theta$, the length of γ, in order to obtain a nontrivial problem.

A remarkable case of the last problem occurs when γ is the real axis and the observer is placed at the point (x_0, y_0), $y_0 > 0$, on the upper half-plane. For convenience, here we choose a slightly different description of the problem. The measure of the visual angle Θ corresponding to a segment (a, b) is given (see Figure 1) by

$$\Theta = \arctan \frac{x_0 - b}{y_0} - \arctan \frac{x_0 - a}{y_0}$$

and so, a finite family of segments $E = \cup_{k=1}^{n} (a_k, b_k)$ on the x axis is seen by the observer under a total angle with measure $\Theta(E)$ expressed by

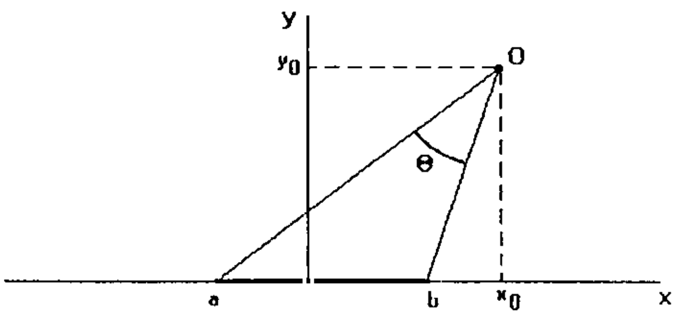

Figure 1:

$$\Theta(E) = \sum_{k=1}^{n} \left(\arctan \frac{x_0 - b_k}{y_0} - \arctan \frac{x_0 - a_k}{y_0} \right). \tag{4}$$

Taking into account that

$$\arctan \frac{x_0 - b_k}{y_0} - \arctan \frac{x_0 - a_k}{y_0} = \int_{a_k}^{b_k} \frac{y_0}{y_0^2 + (x_0 - x)^2} \, dx,$$

expression (4) for $\Theta(E)$ may be rewritten as follows

$$\Theta(E) = \sum_{k=1}^{n} \int_{a_k}^{b_k} \frac{y_0}{y_0^2 + (x_0 - x)^2} \, dx,$$

and then, our optimization problem becomes the instance of problem (2) corresponding to the function $f(x) = y_0/(y_0^2 + (x_0 - x)^2)$. Furthermore, if we take, instead of $f(x)$, the function

$$P(x_0, y_0; x) = \frac{1}{\pi} f(x) = \frac{1}{\pi} \frac{y_0}{y_0^2 + (x_0 - x)^2}, \tag{5}$$

another interpretation of the problem is feasible. In fact, the function $P(x_0, y_0; \cdot)$ given by (5) is recognized to be the Poisson kernel for the half-plane, so that (cf. [1], Example 3-1, pg. 38) the objective of the optimization problem becomes $\int_E g(x)\, dx = \omega((x_0, y_0), \mathbb{R}^2_+, E)$, the harmonic measure of E at the point (x_0, y_0) with respect to the upper half-plane $\mathbb{R}^2_+$. In this way, the initial problem of maximizing the total angle under which E is seen from the point $O = (x_0, y_0)$ provided that $|E| = C$, turns out to be equivalent to that of maximizing the harmonic measure $\omega((x_0, y_0), \mathbb{R}^2_+, E)$ of E at the point (x_0, y_0) under the same restriction $|E| = C$.

On an intuitive basis we expect the total visual angle to be a maximum for an open interval (a, b) of length C centered at x_0. An application of the Karush-Kuhn-Tucker necessary conditions of optimality to problem (2), which is made below, do confirm our intuition. In this way, the objective is maximized by a *connected* open set E of length C.

At this point, we feel ourselves inclined to relax the restriction imposed on the set E and to formulate an optimization problem like the following

$$\sup_{|E|=C} \omega((x_0, y_0), \mathbb{R}^2_+, E), \tag{6}$$

in which a Lebesgue measurable set E is now taken as the variable of the objective, being a measure the objective itself. Condition $|E| = C$ plays the role of a restriction for the optimization problem. Since no description of a general Lebesgue measurable set E exists involving only a finite number of parameters, problem (6) can not be considered as one in Mathematical Programming.

Both the structure and main characteristics of an optimization problem like (6) can be suitably generalized. To this end, we consider a topological space X and two Borel measures μ, ν on X. We assume that, for a given $0 < C < \mu(X)$, the family of measurable sets E with prescribed measure $\mu(E) = C$ is not void and therefore, it make sense to pose the problem

$$\sup_{\mu(E)=C} \nu(E). \tag{7}$$

We say that the measure ν *is coalescent with respect to* μ if, for every $0 < C < \mu(X)$, there exists a *connected* Borel set $E^* \subseteq X$ with $\mu(E^*) =$

C such that

$$\sup_{\mu(E)=C} \nu(E) = \nu(E^*).$$

In this paper, we often refer to a measurable set $E \subseteq X$ as a *configuration* and an optimal set E^* for problem (7) is said to be an *optimal configuration*. Thus, the measure ν is coalescent with respect to μ when connected optimal configurations can be found of any measure C. The term "coalescence" we use to indicate this eventual behavior of the solutions to (7) arise from the following observation: if a descent algorithm were implemented to solve problem (7), we would see to evolve a not connected initial configuration towards other configurations in which the separation between components is more and more small; in a posterior stage, these components would begin to coalesce one each other.

We have anticipated the fact that, when restricted to open sets E with a finite number of components and measure C, the harmonic measure with respect to the half-plane $\omega((x_0, y_0), \mathbb{R}^2_+, E)$ attains its maximum for a connected set of length C. Let us now prove a property of problem (2) that implies this claim. With this purpose, we first write the Lagrangian function corresponding to (2) as follows

$$\mathcal{L}(a_1, b_1, a_2, b_2, \ldots, a_n, b_n; \Lambda_0, \Lambda_1, \ldots, \Lambda_n, \Lambda'_1, \ldots, \Lambda'_n) = \sum_{k=1}^n \int_{a_k}^{b_k} f(x)\, dx$$

$$+\Lambda_0 \left[C - \sum_{k=1}^n (b_k - a_k)\right] + \sum_{k=1}^n \Lambda_k (b_k - a_k) + \sum_{k=1}^{n-1} \Lambda'_k (a_{k+1} - b_k). \tag{8}$$

For this Lagrangian function, the Karush-Kuhn-Tucker necessary conditions of optimality ([3], Theorem 9.11, pg. 50) reads

$$\left|\begin{array}{l} \frac{\partial \mathcal{L}}{\partial a_k} = -f(a_k) + \Lambda_0 - \Lambda_k + \Lambda'_{k-1} = 0,\ k = 1, 2, \ldots, n \\ \frac{\partial \mathcal{L}}{\partial b_k} = f(b_k) - \Lambda_0 + \Lambda_k - \Lambda'_k = 0,\ k = 1, 2, \ldots, n \\ \sum_{k=1}^n (b_k - a_k) = C \\ a_1 \le b_1 \le a_2 \le b_2 \le \cdots \le a_n \le b_n \\ \Lambda_k \ge 0,\ \Lambda_k (b_k - a_k) = 0,\ k = 1, 2, \ldots, n \\ \Lambda'_k \ge 0,\ \Lambda'_k (b_{k+1} - a_k) = 0,\ k = 1, 2, \ldots, n-1 \end{array}\right. , \tag{9}$$

where, for the sake of symmetry, we have put $\Lambda'_0 = \Lambda'_n = 0$. Last condition on multipliers Λ'_k shows that $\Lambda'_k = 0$ whenever (a_k, b_k) and (a_{k+1}, b_{k+1}) are consecutive components of the optimal set E. Hence, from the first and second conditions we deduce that $f(a_k) = \Lambda_0 =$

$f(b_k)$, $k = 1, 2, \ldots, n$; in other words, at the end points of the components of the optimal set E, the function f assumes the same value Λ_0. Now suppose that, for a given y, the equation $f(x) = y$ admits two solutions at most. In this case, we must conclude that an optimal set E possesses only one component and therefore, it is connected. An example of this behavior is provided by the Poisson kernel (5), which prove our claim. That the optimal interval is symmetric with respect to x_0 it is easily deduced from the corresponding symmetry of the Poisson kernel.

Indeed, the harmonic measure $\omega((x_0, y_0), \mathbb{R}^2_+, \cdot)$ is coalescent with respect to the Lebesgue measure in the general sense above defined. The proof of this assertion involves a standard property of the Lebesgue measure. Let C be a positive constant and E be a Borel subset of $\mathbb{R}$ with $|E| = C$. For every $n \in \mathbb{N}$, there exists an open set $E_n \subseteq \mathbb{R}$ with a finite number of components such that $|E \,\Delta\, E_n| < 1/n$ (cf. [7], pg. 62). Since the harmonic measure $\omega((x_0, y_0), \mathbb{R}^2_+, \cdot)$ is absolutely continuous with respect to the Lebesgue measure and $\lim_{n\uparrow+\infty} |E_n| = |E| = C$, we have

$$\omega((x_0, y_0), \mathbb{R}^2_+, E) = \lim_{n\uparrow+\infty} \omega((x_0, y_0), \mathbb{R}^2_+, E_n). \tag{10}$$

But, as we know,

$$\omega((x_0, y_0), \mathbb{R}^2_+, E_n) \leq \omega((x_0, y_0), \mathbb{R}^2_+, E^*_n), \tag{11}$$

where $E^*_n = (x_0 - |E_n|/2, x_0 + |E_n|/2)$ so that, by taking limits in (11) and using (10), we deduce

$$\omega((x_0, y_0), \mathbb{R}^2_+, E) \leq \lim_{n\uparrow+\infty} \omega((x_0, y_0), \mathbb{R}^2_+, E^*_n) = \omega((x_0, y_0), \mathbb{R}^2_+, E^*), \tag{12}$$

where $E^* = (x_0 - C/2, x_0 + C/2)$. Inequality (12) shows that $\sup_{|E|=C} \omega((x_0, y_0), \mathbb{R}^2_+, E)$ is realized by the connected set $(x_0 - C/2, x_0 + C/2)$; then the harmonic measure $\omega((x_0, y_0), \mathbb{R}^2_+, \cdot)$ is coalescent with respect to the Lebesgue measure, as we claimed. Note that the crucial step to establish the coalescence of $\omega((x_0, y_0), \mathbb{R}^2_+, \cdot)$ was the proof of inequality (11). This proof was based on a particular property of the Poisson kernel (5), which is the Radon-Nikodym derivative of $\omega((x_0, y_0), \mathbb{R}^2_+, \cdot)$ with respect to the Lebesgue measure.

This paper is concerned with a study of problem (7) in the case in which μ is a σ-finite measure and ν is an absolutely continuous measure with respect to μ. In this case, there exists a Radon-Nikodym derivative $d\nu/d\mu$ and the relevant characteristics of optimal configurations are shown to depend almost exclusively on the properties of this derivative. The plan of the paper is the following: the concept and basic properties of the f-rearrangement of sets and functions are first introduced in section 2. From an informal viewpoint, the classical rearrangements of a function g have to do with special redistributions of the 'mass under the graph of g'. Given a function f, we roughly define the f-rearrangement of g as a new function obtained by redistributing the mass under the graph of g in the same way as the mass under the graph of f is distributed. For the sake of clearness, the material of section 2 is split in five subsections: the introduction of f-rearrangements is performed in the second and third of them, while basic material on the distribution functions is presented in the first subsection. The two remaining subsections are respectively devoted to study f-rearrangements of simple functions and to extend the Hardy-Littlewood inequality to f -rearrangements. The tools developed in section 2 are employed in section 3 to derive a general characterization (Theorem 11) of optimal configurations for problem (7). From this characterization, a criterion for coalescence of measures finally emerges (Theorem 12).

2 f-rearrangement of sets and functions

Throughout this section, $(X, \mathcal{A}, \mu)$ will denote a measure space and $f : X \to \overline{\mathbb{R}}$ will indicate a function defined on X and taking its values in the set of extended real numbers $\overline{\mathbb{R}}$. The subset of X where f assumes finite values will be denoted by $\mathfrak{F}(f)$; i.e., $\mathfrak{F}(f) = f^{-1}(\mathbb{R})$. The set $\{x \in X : f(x) \neq 0\}$ (or, when defined, its closure) is usually called the support of the function f. For functions taking values in $\overline{\mathbb{R}}$, we find useful to exclude from the support that points where $f = \pm\infty$. Accordingly, we define $\operatorname{supp} f = \{x \in \mathfrak{F}(f) : f(x) \neq 0\}$. As usually, the Lebesgue measure on a subset $X \subseteq \mathbb{R}^n$ will be indicated by $|\cdot|$.

2.1 The distribution graph

For every $\lambda \in \mathbb{R}$, the *level set* F_λ^+ of the function f is defined as

$$F_\lambda^+ = \{x \in X : f(x) \geq \lambda\},$$

while the *strict level set* F_λ^- of f is

$$F_\lambda^- = \{x \in X : f(x) > \lambda\}.$$

To distinguish among level and strict level set of a function f turns out to be irrelevant in many situations. In these cases, we will write F_λ to denote F_λ^+ or F_λ^- indistinctly. Note that the inclusion

$$F_\lambda^- \subseteq F_\lambda^+ \tag{13}$$

holds for every $\lambda \in \mathbb{R}$. In what follows, the difference $F_\lambda^+ \backslash F_\lambda^- = \{x \in X : f(x) > \lambda\}$ is denoted by N_λ. The families $\{F_\lambda^+ : \lambda \in \mathbb{R}\}$ and $\{F_\lambda^- : \lambda \in \mathbb{R}\}$ are decreasing in the sense that

$$F_{\lambda_1}^+ \supseteq F_{\lambda_2}^+ \text{ and } F_{\lambda_1}^- \supseteq F_{\lambda_2}^- \tag{14}$$

when $\lambda_1 < \lambda_2$. Conversely, given a decreasing family $\{G(\lambda) : \lambda \in \mathbb{R}\}$ of subsets of X, a function $g : X \to \overline{\mathbb{R}}$ can be defined by

$$g(x) = \sup\{\lambda \in \mathbb{R} : x \in G(\lambda)\}, \tag{15}$$

with the convention of setting $g(x) = +\infty$ when $\{\lambda \in \mathbb{R} : x \in G(\lambda)\} = \emptyset$. This construction allows us to recover a function f from its level sets. For instance, if the family $\{F_\lambda^-\}$ of strict level sets of f is known, we have

$$\sup\{\lambda \in \mathbb{R} : x \in F_\lambda^-\} = \sup\{\lambda \in \mathbb{R} : f(x) > \lambda\} = f(x) \tag{16}$$

and, since

$$\sup\{\lambda \in \mathbb{R} : x \in F_\lambda^+\} = \sup\{\lambda \in \mathbb{R} : f(x) \geq \lambda\} = f(x), \tag{17}$$

the same is true for the family $\{F_\lambda^+\}$.

The following equalities related to union and intersections of level sets are immediate

$$\bigcap_{\lambda < \lambda_0} F_\lambda^\pm = F_{\lambda_0}^+, \quad \bigcup_{\lambda > \lambda_0} F_\lambda^\pm = F_{\lambda_0}^-. \tag{18}$$

In view of the inclusions (14), these equalities can be respectively written in the form

$$F_\lambda^\pm \nearrow F_{\lambda_0}^+ \text{ when } \lambda \uparrow \lambda_0, \quad F_\lambda^\pm \searrow F_{\lambda_0}^- \text{ when } \lambda \downarrow \lambda_0. \tag{19}$$

If f is a measurable function, then F_λ^+ and F_λ^- become measurable subsets of X and a pair of functions δ_f^+, δ_f^- can be correspondingly defined as the measure of each of these sets; namely, for $\lambda \in \mathbf{R}$, we define

$$\delta_f^+(\lambda) = \mu\left(F_\lambda^+\right), \ \delta_f^-(\lambda) = \mu\left(F_\lambda^-\right).$$

Both functions δ_f^+ and δ_f^- are commonly called *distribution function of* f, but δ_f^- is to be named *strict distribution function of* f when it becomes useful to distinguish among them. In other cases, this distinction is not significative and we write δ_f to denote δ_f^+ or δ_f^-, indistinctly.

It may well occurs $\delta_f^+ = \delta_f^- \equiv +\infty$, as the trivial example $X = \mathbf{R}$, $f(x) \equiv x$, shows. Excepting this case, we have $\delta_f^\pm(+\infty) = \lim_{\lambda\uparrow+\infty}\delta_f^\pm(\lambda) = 0$. Of course, $\delta_f^+ < +\infty$ for any measurable function f when $\mu(X) < +\infty$ but, at all events, δ_f^+ is finite on the eventually empty interval $(\inf_{\delta_f^+(\lambda)<+\infty}\lambda, +\infty)$; i.e.

$$\mathfrak{F}(\delta_f^+) = (\inf_{\delta_f^+(\lambda)<+\infty}\lambda, +\infty). \tag{20}$$

Analogously, we can write

$$\mathfrak{F}(\delta_f^-) = [\inf_{\delta_f^+(\lambda)<+\infty}\lambda, +\infty). \tag{21}$$

As for the supports of δ_f^+ and δ_f^-, we have

$$\begin{aligned} &(\inf_{\delta_f^+(\lambda)<+\infty}\lambda, \operatorname{essup} f) \subseteq \operatorname{supp}\delta_f^+ \subseteq (\inf_{\delta_f^+(\lambda)<+\infty}\lambda, \operatorname{essup} f], \\ &\operatorname{supp}\delta_f^- = [\inf_{\delta_f^+(\lambda)<+\infty}\lambda, \operatorname{essup} f), \end{aligned} \tag{22}$$

where essup$f$ denotes the essential supremum of f.

We remark that other notions of distribution function appear in the literature which turn out to be better adapted to specific purposes (see, for instance, [8], pg. 4, [9] and [10]).

Next lemma states some elementary properties of distribution functions.

Lemma 1. *Let $f : X \to \overline{\mathbf{R}}$ be a measurable function. The distribution functions δ_f^+ and δ_f^- satisfy the following properties.*

i) $\delta_f^- \leq \delta_f^+$.

ii) *δ_f^+ and δ_f^- are decreasing (and therefore measurable) functions. Moreover, δ_f^+ and δ_f^- are strictly decreasing on their respective supports if and only if, for every $\lambda_1, \lambda_2 \in \text{supp}\delta_f^\pm$ such that $\lambda_1 < \lambda_2$, the condition*

$$\mu(F_{\lambda_1}^\pm \backslash F_{\lambda_2}^\pm) > 0 \tag{23}$$

is satisfied by the corresponding level sets of the function f.

iii) *δ_f^+ is left-continuous while δ_f^- is right-continuous. Moreover,*

$$\delta_f^+(\lambda_0^+) = \lim_{\lambda \downarrow \lambda_0} \delta_f^+(\lambda) = \sup_{\lambda > \lambda_0} \delta_f^+(\lambda) = \delta_f^-(\lambda_0) \tag{24}$$

and

$$\delta_f^-(\lambda_0^-) = \lim_{\lambda \uparrow \lambda_0} \delta_f^-(\lambda) = \inf_{\lambda < \lambda_0} \delta_f^-(\lambda) = \delta_f^+(\lambda_0). \tag{25}$$

In a point $\lambda \in \mathfrak{F}(\delta_f^+)$, δ_f^+ and δ_f^- are continuous if and only if

$$\mu(N_\lambda) = \mu(F_\lambda^+ \backslash F_\lambda^-) = 0. \tag{26}$$

iv) *If $f \in L^1(X, \mathcal{A}, \mu)$, then $\delta_f^+(\lambda) < +\infty$ for every $\lambda > 0$. Furthermore, if $f \geq 0$; then*

$$\int_X f(x)\, d\mu(x) = \int_0^{+\infty} \delta_f^\pm(\lambda)\, d\lambda. \tag{27}$$

Proof. Properties **i)-iv)** of distribution functions are rather standard (cf. pg. 76 of [11] for example), and a proof of them is included here only for the sake of completeness. The inequality $\delta_f^- \leq \delta_f^+$ follows from the inclusion (13) just as the monotonicity of δ_f^+ and δ_f^- follows from (14). The distribution function $\delta_f^\pm$ is strictly decreasing on the interval supp$\delta_f^\pm$ given by (22) if and only if $0 > \delta_f^\pm(\lambda_1) - \delta_f^\pm(\lambda_2) = \mu\left(F_{\lambda_1}^\pm\right) - \mu\left(F_{\lambda_2}^\pm\right) = \mu\left(F_{\lambda_1}^\pm \backslash F_{\lambda_2}^\pm\right)$ for every $\lambda_1, \lambda_2 \in \text{supp}\delta_f^\pm$ such that $\lambda_1 < \lambda_2$. This completes the proof of **i)** and **ii)**. Lateral continuity of δ_f^+ and δ_f^- is a consequence of (19) and of a well-known result on the measure of limits of monotone sequences of measurable sets (cf. [11], Theorem 10.11, pg. 166). Lateral limits given by (24) and (25) can be analogously established. From these expression of lateral limits we deduce that $\delta_f^{\pm}(\lambda^+) = \delta_f^{\pm}(\lambda^-)$ if and only if $\delta_f^-(\lambda) = \delta_f^+(\lambda)$; therefore, condition (26) is necessary and sufficient in order that $\delta_f^{\pm}$ to be continuous in a point λ such that $\delta_f^+(\lambda) < +\infty$. As for property **iv)**, for an integrable function f and for $\lambda > 0$, we have

$$\lambda\delta_f^+(\lambda) = \int_{F_\lambda^+} \lambda\, d\mu(x) \le \int_{F_\lambda^+} f(x)\, d\mu(x) \le \|f\|_1 < +\infty;$$

hence $\delta_f^+(\lambda) < +\infty$. If, apart from the integrability of f we assume that $f \ge 0$, then we can write

$$\begin{aligned} \int_X f(x)\, d\mu(x) &= \int_X \left(\int_0^{+\infty} \chi_{[0,f(x)]} d\lambda\right) d\mu(x) \\ &= \int_0^{+\infty} \left(\int_X \chi_{[0,f(x)]}\, d\mu(x)\right) d\lambda, \qquad (28) \end{aligned}$$

where the last equality is justified by the Fubini-Tonelli Theorem. By observing that $\int_X \chi_{[0,f(x)]}\, d\mu(x) = \int_{F_\lambda^+} d\mu(x) = \delta_f^+(\lambda)$, we obtain the identity (27) for δ_f^+. The corresponding identity for δ_f^- can be derived, for example, by taking $\chi_{[0,f(x))}$ instead of $\chi_{[0,f(x)]}$ in (28).

■

Assume for a moment that X is a topological space and that $\mathcal{A}$ is the Borel σ-algebra on X. Moreover, suppose that there exist no μ-negligible open subsets of X; i.e., that $\mu(U) > 0$ for every open set U. Then condition (23) is satisfied by continuous functions and therefore, δ_f^+ is a strictly decreasing function on its support (cf. [5], pg. 27). Indeed, the set $U = \{x \in X : \lambda_1 < f(x) < \lambda_2\} \subseteq F_{\lambda_1}^+ \setminus F_{\lambda_2}^+$ is open for every $\lambda_1, \lambda_2 \in \mathrm{supp}\delta_f^+$ such that $\lambda_1 < \lambda_2$; therefore, $\mu(F_{\lambda_1}^+ \setminus F_{\lambda_2}^+) \ge \mu(U) > 0$.

Other properties of distribution functions are put together in the following result, whose immediate proof will be omitted.

Lemma 2.

i) *If* $f \le g$, *then* $\delta_f^{\pm} \le \delta_g^{\pm}$;

ii) *For every real constant* c, $\delta_{f+c}^{\pm}(\lambda) = \delta_f^{\pm}(\lambda - c)$, $\lambda \in \mathbb{R}$.

iii) *If* $t > 0$, *then* $\delta^{\pm}_{tf}(\lambda) = \delta^{\pm}_{f}(\lambda/t)$, $\lambda \in \mathbb{R}$.

Now we consider the set valued map Δ_f obtained by "filling the gaps" of the distributions functions; that is, we define $\Delta_f : \mathbb{R} \to 2^{\mathbb{R}}$ by means of

$$\Delta_f(\lambda) = [\delta^-_f(\lambda), \delta^+_f(\lambda)],\ \lambda \in \mathbb{R}. \tag{29}$$

Note that $\Delta_f(\lambda) = \{+\infty\}$ whenever $\delta^-_f(\lambda) = +\infty$. From now on, the map Δ_f will be called the *distribution graph of* f. After Lemma 1- **ii)** and **iii)**, we can also write $\Delta_f(\lambda) = [\sup_{\alpha>\lambda} \delta^+_f(\alpha), \delta^+_f(\lambda)] = [\delta^-_f(\lambda), \inf_{\alpha<\lambda} \delta^-_f(\alpha)] = [\sup_{\alpha>\lambda} \delta^+_f(\alpha), \inf_{\alpha<\lambda} \delta^-_f(\alpha)]$. From the last equality we see that the inverse map Δ^{-1}_f, defined for $\lambda \geq 0$ by

$$\lambda \in \Delta^{-1}_f(y) \Leftrightarrow y \in \Delta_f(\lambda),$$

can be expressed in the form

$$\Delta^{-1}_f(y) = [\sup_{\delta_f(\alpha)>y} \alpha, \inf_{\delta_f(\alpha)<y} \alpha], \tag{30}$$

where we agree in defining $\inf_{\emptyset} \alpha = +\infty$ so that $\Delta^{-1}_f(0) = [\sup_{\delta_f(\alpha)>y} \alpha, +\infty)$. In fact, the inclusion $\lambda \in \Delta^{-1}_f(y)$ holds if and only if $\sup_{\alpha>\lambda} \delta^+_f(\alpha) \leq \lambda \leq \inf_{\alpha<\lambda} \delta^-_f(\alpha)$ or, equivalently, $\sup_{\delta^+_f(\alpha)>y} \alpha \leq \lambda \leq \inf_{\delta^-_f(\alpha)<y} \alpha$. By realizing that

$$\sup_{\delta^+_f(\alpha)>y} \alpha = \sup_{\delta^-_f(\alpha)>y} \alpha \quad \text{and} \quad \inf_{\delta^+_f(\alpha)<y} \alpha = \inf_{\delta^-_f(\alpha)<y} \alpha, \tag{31}$$

expression (30) for $\Delta^{-1}_f(y)$ is obtained.

In addition, if we coincide in defining $\sup_{\emptyset} = -\infty$, then expression (30) for $\Delta^{-1}_f(y)$ provides the values of λ that solve the inclusion $y \in \Delta_f(\lambda)$ for a given $0 \leq y \leq +\infty$. By observing that $\Delta^{-1}_f \equiv \{+\infty\}$ if and only if $\delta_f \equiv +\infty$, we conclude that the inclusion $y \in \Delta_f(\lambda)$ has a real solution unless $\delta_f \equiv +\infty$.

2.2 f-rearrangement of sets

Loosely speaking, given a measurable function $f : X \to \overline{\mathbb{R}}$ and a measurable set $E \subsetneq X$, we say that a level set F_λ of f is the f-rearrangement of E when $\mu(F_\lambda) = \mu(E)$, so that the equation

$$\delta_f(\lambda) = \mu(E) \tag{32}$$

is satisfied. This ingenuous idea can not be plainly taken as a definition since, on one hand, equation (32) is not generally solvable and on the other hand, if a solution exists at all, this may be not unique. Indeed, the last problem is not a serious one: whenever equation (32) admits two different solutions λ_1 and λ_2, both level sets F_{λ_1} and F_{λ_2} have a common measure $\mu(E)$. If $\mu(E) < +\infty$ then $\mu(F_{\lambda_1} \backslash F_{\lambda_2}) = 0$, so that F_{λ_1} differs from F_{λ_2} only in a null set and a theory is constructed by taking any level set F_λ with λ solving (32) as the f-rearrangement of the set E. This theory will enjoy of the usual "resolution power" of Measure Theory: two sets A, B are considered identical when $A \subseteq B$ and $\mu(B \setminus A) = 0$. Neither can two sets A, B such that $A \subseteq B$ and $\mu(A) = +\infty = \mu(B)$ be substantially distinguished by mean of measure-theoretical arguments, so that there is no problem in defining the f-rearrangement of a set E with $\mu(E) = +\infty$ to be any level set F_λ with $\mu(F_\lambda) = +\infty$.

In order to overcome the first difficulty, we replace equation (32) by the inclusion

$$\mu(E) \in \Delta_f(\lambda). \tag{33}$$

As we have said at the end of previous subsection, unless $\delta_f \equiv +\infty$, the inclusion (33) admits real solutions. In fact, with the aid of (30) the set of their solutions is expressed by

$$\Delta_f^{-1}(\mu(E)) = [\sup_{\delta_f(\alpha) > \mu(E)} \alpha, \inf_{\delta_f(\alpha) < \mu(E)} \alpha],$$

so that we recognize in the level sets F_λ with $\sup_{\delta_f(\alpha) > \mu(E)} \alpha \leq \lambda \leq \inf_{\delta_f(\alpha) < \mu(E)} \alpha$ the natural objects to be considered as f-rearrangements of the set E. Concretely, given a measurable set $E \subseteq X$ and a measurable function $f : X \to \overline{\mathbf{R}}$ such that δ_f is not identically $+\infty$, we define the *superior f-rearrangement* E^{*f} *of* E to be the level set

$$E^{*f} = F^{+}_{\sup_{\delta_f(\alpha) > \mu(E)} \alpha} \tag{34}$$

while the level set

$$E_{*f} = F^{-}_{\inf_{\delta_f(\alpha) < \mu(E)} \alpha}. \tag{35}$$

is defined to be the *inferior f-rearrangement of E*.

Recalling that $\sup_{\delta_f(\alpha)>y}\alpha \leq \inf_{\delta_f(\alpha)<y}\alpha$, $y \geq 0$, we have

$$E_{*f} = F^-_{\inf_{\delta_f(\alpha)<\mu(E)}\alpha} \subseteq F^+_{\sup_{\delta_f(\alpha)>\mu(E)}\alpha} = E^{*f}. \tag{36}$$

Moreover, in view of δ_f^+ is a left continuous and decreasing function, we can write

$$\mu(E^{*f}) = \delta_f^+\left(\sup_{\delta_f(\alpha)>\mu(E)}\alpha\right) = \inf_{\delta_f(\alpha)>\mu(E)}\delta_f^+(\alpha) \geq \mu(E) \tag{37}$$

and, in a similar way,

$$\mu(E_{*f}) = \delta_f^-\left(\inf_{\delta_f(\alpha)<\mu(E)}\alpha\right) = \sup_{\delta_f(\alpha)<\mu(E)}\delta_f^-(\alpha) \leq \mu(E). \tag{38}$$

If $\mu(E) = +\infty$, then $\mu(E^{*f}) = +\infty = \mu(E)$ by (37). Analogously, if $\mu(E) = 0$, then $\mu(E_{*f}) = 0 = \mu(E)$ by (38). However, simple examples show that inequalities (37) and (38) are generally strict. Sufficient conditions in order that these inequalities become equalities are provided by the following:

Lemma 3. *Let E be a measurable set with $0 < \mu(E) < +\infty$; then*

i) $\mu(E^{*f}) = \mu(E)$ *if δ_f is continuous at the point* $\lambda = \sup_{\delta_f(\alpha)>\mu(E)}\alpha$*;*

ii) $\mu(E_{*f}) = \mu(E)$ *if δ_f is continuous at the point* $\lambda = \inf_{\delta_f(\alpha)<\mu(E)}\alpha$.

Proof. To prove **i)**, assume that $\lambda = \sup_{\delta_f(\alpha)>\mu(E)}\alpha$ is a point of continuity of δ_f; then

$$\mu(E^{*f}) = \delta_f^+\left(\sup_{\delta_f(\alpha)>\mu(E)}\alpha\right) = \delta_f^-\left(\sup_{\delta_f(\alpha)>\mu(E)}\alpha\right).$$

Taking into account that $\sup_{\delta_f(\alpha)>\mu(E)}\alpha = \inf_{\delta_f(\alpha)\leq\mu(E)}\alpha$, we have

$$\delta_f^-\left(\sup_{\delta_f(\alpha)>\mu(E)}\alpha\right) = \delta_f^-\left(\inf_{\delta_f(\alpha)\leq\mu(E)}\alpha\right) = \sup_{\delta_f(\alpha)\leq\mu(E)}\delta_f^-(\alpha) \leq \mu(E)$$

and therefore $\mu(E^{*f}) \leq \mu(E)$, which, together with (37), gives $\mu(E^{*f}) = \mu(E)$.

The proof of **ii)** is similar. ∎

From the example in which $X = \mathbb{R}$ and $f = \chi_{(0,1)}$ we see that sufficient conditions furnished by Lemma 3 are not necessary. In effect, the distribution function δ_f^+ is here given by

$$\delta_f^+(\lambda) = \left| \begin{array}{ll} +\infty, & \lambda \leq 0 \\ 1, & 0 < \lambda \leq 1 \\ 0, & \lambda > 1 \end{array} \right. ,$$

and for any measurable set $E \subseteq \mathbb{R}$ with $0 \leq |E| \leq 1$ we have $E^{*f} = F_1^+$ and

$$E_{*f} = \left| \begin{array}{ll} F_{+\infty}^-, & |E| = 0 \\ F_1^-, & 0 < |E| \leq 1 \end{array} \right. ,$$

whence $\left|E^{*f}\right| = 1$ and $|E_{*f}| = 0$.

We will say that a measurable function f is a *regular function* when δ_f is a not identically $+\infty$ strictly decreasing and continuous function on suppδ_f. Lemma 1- **ii),iii)** gives necessary and sufficient conditions in order that a function f to be regular. For a regular function f, the inverse δ_f^{-1} of the distribution function is defined on the interval $\left[0, \delta_f\left(\inf_{\delta_f(\alpha)<+\infty} \alpha\right)\right]$ and it is a strictly decreasing and continuous function there. Hence, for every measurable set E with $0 \leq \mu(E) \leq \delta_f\left(\inf_{\delta_f(\alpha)<+\infty} \alpha\right)$, we have

$$\sup_{\delta_f(\alpha)>\mu(E)} \alpha = \sup_{\alpha<\delta_f^{-1}(\mu(E))} \alpha = \delta_f^{-1}(\mu(E))$$

and $\inf_{\delta_f(\alpha)<\mu(E)} \alpha = \inf_{\alpha>\delta_f^{-1}(\mu(E))} \alpha = \delta_f^{-1}(\mu(E))$, so that

$$E^{*f} = F^+_{\sup_{\delta_f(\alpha)>\mu(E)} \alpha} = F^+_{\delta_f^{-1}(\mu(E))}$$

and

$$E_{*f} = F^-_{\inf_{\delta_f(\alpha)<\mu(E)} \alpha} = F^-_{\delta_f^{-1}(\mu(E))}.$$

Thus, $\mu(E^{*f} \setminus E_{*f}) = \mu(N_{\delta_f^{-1}(\mu(E))}) = 0$ and we have proved the following theorem.

Theorem 4. *Let f be a regular function; then, for every measurable set $E \subseteq X$ such that $0 \leq \mu(E) \leq \delta_f\left(\inf_{\delta_f(\alpha)<+\infty} \alpha\right)$, the equality*

$$E^{*f} = E_{*f}$$

holds up to a μ-negligible set.

Proof. See the previous discussion.

■

In our next result, the main properties of f-rearrangement of sets are collected.

Theorem 5. *Let $f : X \to \overline{\mathbb{R}}$ be a measurable function such that δ_f is not identically $+\infty$, and E, E_1, E_2 be measurable subset of X. Then, the following statements hold for superior f-rearrangements:*

i) *If $E_1 \subseteq E_2$, then $(E_1)^{*f} \subseteq (E_2)^{*f}$.*

ii) $(E_1 \cap E_2)^{*f} \subseteq (E_1)^{*f} \cap (E_2)^{*f}$ *and* $(E_1 \cup E_2)^{*f} \supseteq (E_1)^{*f} \cup (E_2)^{*f}$.

iii) $\mu\left((E_1)^{*f} \cup (E_2)^{*f}\right) = \max\left\{\mu\left((E_1)^{*f}\right), \mu\left((E_2)^{*f}\right)\right\}$ *and* $\mu\left((E_1)^{*f} \cap (E_2)^{*f}\right) = \min\left\{\mu\left((E_1)^{*f}\right), \mu\left((E_2)^{*f}\right)\right\}$.

iv) $E^{*(f+c)} = E^{*f}$, $c \in \mathbb{R}$.

v) $E^{*(tf)} = E^{*f}$, $t > 0$.

Statements **i)-v)** *hold for inferior f-rearrangements as well. Moreover, the following properties relate superior rearrangements to inferior rearrangements:*

vi) *If $E = F_\lambda^{\pm}$ is a level set corresponding to f; then $E^{*f} = F_\lambda^{+}$ and $E_{*f} = F_\lambda^{-}$.*

vii) $(E^{*f})^{*f} = E^{*f}$, $(E_{*f})_{*f} = E_{*f}$, $(E^{*f})_{*f} = E_{*f}$ *and* $(E_{*f})^{*f} = E^{*f}$.

Proof. We will prove asserts **i)-v)** only for superior f-rearrangements. If $E_1 \subseteq E_2$, then $\mu(E_1) \leq \mu(E_2)$ and so the inclusion $(E_1)^{*f} \subseteq (E_2)^{*f}$ is immediate from the inequalities $\sup_{\delta_f(\alpha)>\mu(E_1)} \alpha \geq \sup_{\delta_f(\alpha)>\mu(E_2)} \alpha$

and $\inf_{\delta_f(\alpha)<\mu(E_1)} \alpha \geq \inf_{\delta_f(\alpha)<\mu(E_2)} \alpha$. This proves **i)**. Assert **ii)** follows from **i)** by simply observing that $E_1 \cap E_2 \subseteq E_i$ and $E_1 \cup E_2 \supseteq E_i$, $i = 1, 2$. **iii)** is clear from the fact that $(E_1)^{*f}$ and $(E_2)^{*f}$ are level set of the function f. Statements **iv)** and **v)** are simple consequence of the properties of the distribution function which were established in Lemma 2-**ii)**,**iii)**. To prove **vi)**, we simply note that $(F_\lambda^\pm)^{*f} = F^+_{\sup_{\delta_f(\alpha)>\delta_f^\pm(\lambda)} \alpha} = F_\lambda^+$ and that $(F_\lambda^\pm)^{*f} = F^-_{\inf_{\delta_f(\alpha)<\delta_f^\pm(\lambda)} \alpha} = F_\lambda^-$. Finally, property **vii)** follows from **vi)** by taking into account that E^{*f} and E_{*f} are levelsets of f.

■

2.3 f-rearrangement of functions

We are now ready to introduce f-rearrangements of a function g by rearranging its level sets. Let us begin by considering two measurable functions $f, g : X \to \overline{\mathbb{R}}$ such that δ_f is not identically $+\infty$. The families $\{(G_\lambda)^{*f} : \lambda \in \mathbb{R}\}$ and $\{(G_\lambda)_{*f} : \lambda \in \mathbb{R}\}$ are decreasing by Theorem 5 -**i)** and so, using expression (15), certain functions can be recovered from them; namely, the *superior f-rearrangement* g^{*f} and the *inferior f-rearrangement* g_{*f} of a function g are respectively defined, for every $x \in X$, by

$$g^{*f}(x) = \sup\{\lambda \in \mathbb{R} : x \in (G_\lambda)^{*f}\} \tag{39}$$

and

$$g_{*f}(x) = \sup\{\lambda \in \mathbb{R} : x \in (G_\lambda)_{*f}\}. \tag{40}$$

The fact that rearranging strict level sets G_λ^- amounts the same of rearranging level sets G_λ^+ is emphasized by definitions (39) and (40). Take for instance superior f-rearrangements. In this case, the inclusions

$$\Gamma^-(x) = \{\lambda \in \mathbb{R} : x \in (G_\lambda^-)^{*f}\} \subseteq \{\lambda \in \mathbb{R} : x \in (G_\lambda^+)^{*f}\} = \Gamma^+(x),\ x \in X, \tag{41}$$

show that

$$\sup\{\lambda \in \mathbb{R} : x \in (G_\lambda^-)^{*f}\} \leq \sup\{\lambda \in \mathbb{R} : x \in (G_\lambda^+)^{*f}\},\ x \in X. \tag{42}$$

From applying the definition of superior f-rearrangement, we see that $x \in (G_\lambda^-)^{*f}$ if and only if $f(x) \geq \sup_{\delta_f(\alpha)>\delta_g^-(\lambda)} \alpha$ and that $x \in (G_\lambda^+)^{*f}$

if and only if $f(x) \geq \sup_{\delta_f(\alpha)>\delta_g^+(\lambda)} \alpha$. If we recall that $\delta_g^+(\lambda)$ may differ from $\delta_f^-(\lambda)$ in a numerable set of values of λ at most, we conclude that, for every $x \in X$, the set $\Gamma^+(x)\backslash\Gamma^-(x)$ is at most numerable, so that inequality (42) is really an equality. A similar argument shows that definition (40) of inferior f-rearrangement does not depend on what type of level sets are taken for function g.

In the light of previous discussion, we can write

$$g^{*f}(x) = \sup\{\lambda \in \mathbb{R} : f(x) \geq \sup_{\delta_f(\alpha)>\delta_g(\lambda)} \alpha\}, \ x \in X, \tag{43}$$

and

$$g_{*f}(x) = \sup\{\lambda \in \mathbb{R} : f(x) > \inf_{\delta_f(\alpha)<\delta_g(\lambda)} \alpha\}, \ x \in X. \tag{44}$$

It follows from inclusion (36) that $(G_\lambda)^{*f} \subseteq (G_\lambda)_{*f}$, $\lambda \in \mathbb{R}$; whence the following inequality

$$g_{*f}(x) \leq g^{*f}(x) \tag{45}$$

holds for every $x \in X$ and somewhat justifies the terminology we employ for f-rearrangements.

Given two functions g_1 and g_2, we set $g_1 \vee g_2 = \max\{g_1, g_2\}$ and $g_1 \wedge g_2 = \min\{g_1, g_2\}$. In the following result, whose simple proof we omit, the most elementary properties of f-rearrangements of functions are established.

Theorem 6. *Let f be a measurable function such that δ_f is not identically $+\infty$. If g, g_1 and g_2 are measurable functions, the following properties of superior f -rearrangements hold.*

i) *If $g_1 \leq g_2$, then $g_1^{*f} \leq g_2^{*f}$.*

ii) $(g_1 \wedge g_2)^{*f} \leq g_1^{*f} \wedge g_2^{*f}$ *and* $(g_1 \vee g_2)^{*f} \geq g_1^{*f} \vee g_2^{*f}$.

iii) $g^{*(f+c)} = g^{*f}, \ c \in \mathbb{R}$.

iv) $g^{*(tf)} = g^{*tf}, \ t > 0$.

Properties **i)-iv)** *also hold for inferior f-rearrangements. Furthermore, f-rearrangements enjoy the following additional properties:*

v) $f^{*f} = f_{*f} = f$.

vi) $(g^{*f})^{*f} = g^{*f}$, $(g_{*f})_{*f} = g_{*f}$, $(g^{*f})_{*f} = g_{*f}$ *and* $(g_{*f})^{*f} = g^{*f}$.

As it is stated by the following theorem, superior and inferior f-rearrangements coincide in the case in which f is a regular function.

Theorem 7. *If f is a regular function, then*

$$g^{*f}(x) = \sup\{\lambda \in \mathbf{R} : (\delta_f \circ f)(x) \le \delta_g(\lambda)\} = g_{*f}(x), \ x \in X. \tag{46}$$

Furthermore, if g is a regular function too, then

$$g^{*f}(x) = (\delta_g^{-1} \circ \delta_f \circ f)(x) = g_{*f}(x), \ x \in X. \tag{47}$$

Proof. By assuming that f is a regular function, for every $x \in X$ we can write

$$\begin{aligned} g^{*f}(x) &= \sup\{\lambda \in \mathbf{R} : f(x) \ge \sup_{\delta_f(\alpha) > \delta_g(\lambda)} \alpha\} \\ &= \sup\{\lambda \in \mathbf{R} : f(x) \ge (\delta_f^{-1} \circ \delta_g)(\lambda)\} \\ &= \sup\{\lambda \in \mathbf{R} : (\delta_f \circ f)(x) \le \delta_g(\lambda)\} \\ &= \sup\{\lambda \in \mathbf{R} : (\delta_f \circ f)(x) < \delta_g(\lambda)\} \\ &= \sup\{\lambda \in \mathbf{R} : f(x) > (\delta_f^{-1} \circ \delta_g)(\lambda)\} \\ &= \sup\{\lambda \in \mathbb{R} : f(x) > \inf_{\delta_f(\alpha) < \delta_g(\lambda)} \alpha\} \\ &= g_{*f}(x). \end{aligned}$$

This proves equality (46). Now, if function g is also regular, then from (46) we deduce

$$\begin{aligned} g^{*f}(x) &= \sup\{\lambda \in \mathbf{R} : (\delta_f \circ f)(x) \le \delta_g(\lambda)\} \\ &= \sup\{\lambda \in \mathbf{R} : (\delta_g^{-1} \circ \delta_f \circ f)(x) \ge \lambda\} \\ &= (\delta_g^{-1} \circ \delta_f \circ f)(x), \ x \in X, \end{aligned}$$

which completes the proof.

■

Classical symmetrizations of sets and functions (see [4], [6], [2], [5]) can be viewed as f-rearrangements for particular regular functions $f:\mathbb{R}^n\to\mathbb{R}$. Consider, for instance, a continuous function $f:\mathbb{R}^n\to\mathbb{R}$ such that $f(x)=\phi(|x|)$, $x\in R^n$, for a certain continuous and strictly decreasing function $\phi:\mathbb{R}_0^+\to\mathbb{R}$. Functions like f are known as *radially decreasing (continuous) functions.* Level sets of such functions are spheres centered at the origin given by $F_\lambda=B_{\phi^{-1}(\lambda)}(0)$, so that

$$\delta_f(\lambda)=\omega_n\left[\phi^{-1}(\lambda)\right]^n,\ \lambda\in\mathbb{R}, \tag{48}$$

where ω_n denotes the volume of the unitary n-sphere. Since f is a regular function, expression (46) for the f-rearrangement of a function g holds and in this way, f-rearrangements depend only on the composition $\delta_f\circ f$. Using (48), we obtain

$$(\delta_f\circ f)(x)=\omega_n\left[\phi^{-1}(\phi(|x|))\right]^n=\omega_n\,|x|^n,\ x\in\mathbb{R}^n,$$

whence we see that f-rearrangements with f a radially symmetric function do not depend on the particular choice of f and they are consequently expressed by

$$g^{*f}(x)=\sup\{\lambda\in\mathbb{R}:\omega_n\,|x|^n\le\delta_g(\lambda)\}=g_{*f}(x). \tag{49}$$

Rearrangements given by (49) are recognized to be the *Schwarz symmetrization* of function g ([2], [5]), so that we can say that Schwarz symmetrization corresponds to f-rearrangements for $X=\mathbb{R}^n$ and any radially decreasing continuous function f.

2.4 Simple functions

In this subsection we are concerned with the f-rearrangement of a simple function; i.e., a measurable function which assumes only a finite number of values. To begin with, we study f-rearrangements of the characteristic function χ_E of a measurable set $E\subseteq X$. We will show that

$$(\chi_E)^{*f}=\chi_{E^{*f}} \tag{50}$$

and

$$(\chi_E)_{*f}=\chi_{E_{*f}}. \tag{51}$$

Take for instance the equality (50). Since, for every $x \in X$,

$$(\chi_E)^{*f}(x) = \sup\{\lambda \in \mathbb{R} : x \in [(\Xi_E)_\lambda]^{*f}\}$$

and

$$\chi_{E^{*f}}(x) = \sup\{\lambda \in \mathbb{R} : x \in (\Xi_{E^{*f}})_\lambda\},$$

in order to establish equality (50) it is sufficient to prove that

$$[(\Xi_E)_\lambda]^{*f} = (\Xi_{E^{*f}})_\lambda$$

for every $\lambda \in \mathbb{R}$, which we make by means of a sequence of simple transformations:

$$\begin{aligned}
(\Xi_{E^{*f}})_\lambda &= \{x \in X : \chi_{E^{*f}}(x) \geq \lambda\} \\
&= \{x \in X : \chi_{F^+_{\sup_{\delta_f(\alpha)>\mu(E)}}}(x) \geq \lambda\} \\
&= \{x \in X : \chi_{\{y \in X : f(y) \geq \sup_{\delta_f(\alpha)>\mu(E)} \alpha\}}(x) \geq \lambda\} \\
&= \left| \begin{array}{ll} X, & \lambda \leq 0 \\ \{x \in X : f(x) \geq \sup_{\delta_f(\alpha)>\mu(E)} \alpha\}, & 0 < \lambda \leq 1 \\ \emptyset, & \lambda > 1 \end{array} \right. \\
&= \{x \in X : f(x) \geq \sup_{\delta_f(\alpha)>\delta^+_{\chi_E}(\lambda)} \alpha\} \\
&= F^+_{\sup_{\delta_f(\alpha)>\delta^+_{\chi_E}(\lambda)} \alpha} \\
&= [(\Xi_E)_\lambda]^{*f}.
\end{aligned}$$

The proof of equality (51) is similar.

Let g be a simple function and $a_1 < a_2 < \cdots < a_n$ be the non-zero values that g assumes. In terms of the characteristic functions corresponding to the sets $N_k = \{x \in X : g(x) = a_k\}$, $k = 1, 2, \ldots, n$, function g can be expressed in the form

$$g = \sum_{k=1}^{n} a_k \chi_{N_k}. \tag{52}$$

Characterization (52) of a simple function g is known as *canonical representation* of g (cf. [7], pg. 75).

In general, the distribution function δ_g of a simple function g, when restricted to $\mathfrak{F}(\delta_g)$, is again a simple function. After Lemma 2 -ii), we may confine ourselves tocompute δ_g for non-negative simple functions, so that we suppose g is given by (52) with $a_1 > 0$. A routine application of definitions give us in this case $\mathfrak{F}(\delta_g^+) = (0, +\infty)$, $\mathfrak{F}(\delta_g^-) = [0, +\infty)$ and

$$\delta_g^+|_{(0,+\infty)} = \sum_{k=1}^{n} \mu(N_k)\chi_{(0,a_k]}, \tag{53}$$

$$\delta_g^-|_{[0,+\infty)} = \sum_{k=1}^{n} \mu(N_k)\chi_{[0,a_k)}. \tag{54}$$

It should be noted that these representations of the distribution functions δ_g^+ and δ_g^- are not canonical whereas their canonical representations are respectively given by

$$\delta_g^+|_{(0,+\infty)} = \sum_{k=1}^{n} \delta_g^+(a_k)\chi_{(a_{k-1},a_k]} \tag{55}$$

and

$$\delta_g^-|_{[0,+\infty)} = \sum_{k=1}^{n} \delta_g^+(a_k)\chi_{[a_{k-1},a_k)}. \tag{56}$$

In (55) and (56) we have put $a_0 = 0$.

To represent a simple function f in a way different from (52) turns out to be useful in connection with rearrangements. Setting $\alpha_1 = a_1$, $\alpha_2 = a_2 - a_1, \ldots, \alpha_n = a_n - a_{n-1}$, we can write

$$g = \sum_{k=1}^{n} \alpha_k \chi_{G_k}, \tag{57}$$

where $G_k = \{x \in X : G(x) \geq a_k\} = G_{a_k}^+$, $k = 1, 2, \ldots, n$, (cf. [4], pg. 279, [5], pg. 24). In view of $\chi_{G_1} \geq \chi_{G_2} \geq \cdots \geq \chi_{G_k}$, we say that (57) is the *monotone representation* of the simple function g. For example, the second member of

$$\delta_g^+|_{\mathbb{R}+} = \sum_{k=1}^{n} \mu(N_k)\chi_{(0,a_k]}$$

corresponds to the monotone representation of $\delta_g^+|_{\mathbb{R}+}$.

Roughly speaking, next theorem says that the f-rearrangement operates in a linear way on the terms of a simple function g provided that g is given by its monotone representation.

Theorem 8. *Let f be a measurable function with δ_f non identically $+\infty$. If $\sum_{k=1}^{n} \alpha_k \chi_{G_{\alpha_k}}$ is the monotone representation of a non-negative simple function g, then*

$$g^{*f} = \sum_{k=1}^{n} \alpha_k (\chi_{G_{\alpha_k}})^{*f}, \tag{58}$$

and

$$g_{*f} = \sum_{k=1}^{n} \alpha_k (\chi_{G_{\alpha_k}})_{*f}. \tag{59}$$

In view of (50) and (51), we can equivalently write

$$g^{*f} = \sum_{k=1}^{n} \alpha_k \chi_{(G_{\alpha_k})^{*f}}, \tag{60}$$

and

$$g_{*f} = \sum_{k=1}^{n} \alpha_k \chi_{(G_{\alpha_k})_{*f}}. \tag{61}$$

Proof. Let $\sum_{k=1}^{n} \alpha_k \chi_{G_{\alpha_k}}$ be the monotone representation of a non-negative simple function g. We will prove that expression (60) holds for the superior f-rearrangement of g. From expression (43) for superior f-rearrangements we see that the computation of g^{*f} is reduced to compute the quantity $\sup_{\delta_f(\alpha)>\delta_g(\lambda)} \alpha$ as a function of λ. To this end, we set $a_0 = 0$ and $a_k = \sum_{i=1}^{k} \alpha_i$, $k = 1, 2, \dots, n$, and we consider the canonical representation (55) for δ_g^+. Thus, we obtain

$$\sup_{\delta_f(\alpha)>\delta_g(\lambda)} \alpha = \left| \begin{array}{ll} \sup_{\delta_f(\alpha)>\mu(X)} \alpha, & \lambda \leq 0 \\ \sup_{\delta_f(\alpha)>\delta_g^+(a_k)} \alpha, & a_{k-1} < \lambda \leq a_k \end{array} \right.$$

$$= \left| \begin{array}{ll} -\infty, & \lambda \leq 0 \\ \sup_{\delta_f(\alpha)>\delta_g^+(a_k)} \alpha, & a_{k-1} < \lambda \leq a_k \end{array} \right. ,$$

whence, for the values of x such that $\sup_{\delta_f(\alpha)>\delta_g^+(a_k)} \alpha \leq f(x) < \sup_{\delta_f(\alpha)>\delta_g^+(a_{k-1})}$, we deduce

$$g^{*f}(x) = a_k. \tag{62}$$

This can be compactly rewritten in the form

$$g^{*f} = \sum_{k=1}^{n} \alpha_k \chi_{F^+_{\sup_{\delta_f(\alpha)>\delta_g^+(a_k)} \alpha}}$$

which is seen to coincide with (60) as soon as we realize that $(G_{\alpha_k})^{*f} = F^+_{\sup_{\delta_f(\alpha)>\delta_g^+(a_k)} \alpha}$.

■

2.5 The Hardy-Littlewood inequality for f-rearrangements

Integral inequalities for classical rearrangements can be extended to f -rearrangements, the Hardy-Littlewood inequality for functions in $L^2(X, \mathcal{A}, \mu)$ (cf. [4], pg. 278, [5], pg. 23 and the appendix in [9]) being a good example of this fact. Let us now prove this extension.

Theorem 9. *Let f be a measurable function such that δ_f is not identically $+\infty$. Then the inequality*

$$\int_X gh\, d\mu(x) \leq \int_X g^{*f} h^{*f}\, d\mu(x) \tag{63}$$

holds for any two non-negative functions $g, h \in L^2(X, \mathcal{A}, \mu)$.

Proof. The idea of the proof used by Hardy and Littlewood (cf. [4] and also [5]) to prove inequality (63) for classical rearrangements also works here. In the first place, we show that (63) holds when $g = \chi_{E_1}$ and $h = \chi_{E_2}$, being $E_1, E_2 \subseteq X$ two measurable sets with $\mu(E_1), \mu(E_2) < +\infty$. In this case we have

$$\int_X \chi_{E_1}\chi_{E_2}\, d\mu(x) = \mu(E_1 \cap E_2) \leq \min\{\mu(E_1), \mu(E_2)\},$$

but, using inequality (37) and Theorem 5-**iii)**, we can write

$$\begin{aligned} \min\{\mu(E_1), \mu(E_2)\} &\leq \min\{\mu(E_1^{*f}), \mu(E_2^{*f})\} \\ &= \mu(E_1^{*f} \cap E_2^{*f}) = \int_X \chi_{E_1^{*f}} \chi_{E_2^{*f}}\, d\mu(x) \end{aligned}$$

and from (50), $\chi_{E_1^{*f}} = (\chi_{E_1})^{*f}$ and $\chi_{E_2^{*f}} = (\chi_{E_2})^{*f}$, so that

$$\int_X \chi_{E_1^{*f}} \chi_{E_2^{*f}} \, d\mu(x) = \int_X (\chi_{E_1})^{*f} (\chi_{E_2})^{*f} \, d\mu(x).$$

We then conclude that

$$\int_X \chi_{E_1} \chi_{E_2} \, d\mu(x) \leq \int_X (\chi_{E_1})^{*f} (\chi_{E_2})^{*f} \, d\mu(x). \tag{64}$$

Now, we extend inequality (64) to the product of non-negative simple functions g and h such that $g, h \in L^2(X, \mathcal{A}, \mu)$. To this end, let us consider their monotone representations $g = \sum_{k=1}^n \alpha_k \chi_{G_k}$ and $h = \sum_{k=1}^m \beta_k \chi_{H_k}$. In view of inequality (64) and Theorem 8, we have

$$\begin{aligned}
\int_X gh \, d\mu(x) &= \sum_{i,j=1}^{n,m} \alpha_i \beta_j \int_X \chi_{G_i} \chi_{H_j} \, d\mu(x) \\
&\leq \sum_{i,j=1}^{n,m} \alpha_i \beta_j \int_X (\chi_{G_i})^{*f} (\chi_{H_j})^{*f} \, d\mu(x) \\
&= \int_X \left(\sum_{i=1}^{n} \alpha_i (\chi_{G_{\alpha_i}})^{*f} \sum_{j=1}^{m} \beta_j (\chi_{H_{\beta_j}})^{*f} \right) d\mu(x) \\
&= \int_X g^{*f} h^{*f} \, d\mu(x).
\end{aligned}$$

Finally, let be given two non-negative functions $g, h \in L^2(X, \mathcal{A}, \mu)$. There exist two sequences $\{g_n\}$ and $\{h_n\}$ of non-negative simple functions such that $g_n \uparrow g$ and $h_n \uparrow h$ when $n \uparrow +\infty$. For these sequences we have just proved that

$$\int_X g_n h_n \, d\mu(x) \leq \int_X g_n^{*f} h_n^{*f} \, d\mu(x) \leq \int_X g^{*f} h^{*f} \, d\mu(x), \; n \in \mathbb{N}. \tag{65}$$

In the last inequality we have applied, Theorem 6 -**i)** and the monotonicity of the integral. The conclusion follows by making $n \uparrow +\infty$ in (65) and applying the monotone convergence theorem.

■

We remark that inequalities like (63) or its converse are not generally valid when superior f-rearrangements are replaced by inferior ones. In fact, suppose that δ_f has a point of discontinuity at least. Assume further that a measurable subset $E \subseteq X$ can be chosen so that $\mu(E)$ is any point in a gap of δ_f. In this way, the equation $\delta_f(\lambda) = \mu(E)$ is not solvable and therefore $\mu(E) > \mu(E_{*f})$. Then, by taking $g = h = \chi_E$; we have

$$\begin{aligned}\int_X gh\, d\mu(x) &= \int_X \chi_E\, d\mu(x) = \mu(E) > \mu(E_{*f}) \\ &= \int_X (\chi_E)_{*f}\, d\mu(x) = \int_X g_{*f} h_{*f}\, d\mu(x).\end{aligned}$$

On the other hand, if we choose an E such that the equation $\delta_f(\lambda) = \mu(E)$ admits a unique solution λ_0, then inferior f-rearrangement of g and h coincide with the corresponding superior f-rearrangements g^{*f} and h^{*f}. Theorem 9 shows that

$$\int_X gh\, d\mu(x) \leq \int_X g_{*f} h_{*f}\, d\mu(x)$$

in this case.

In the next section, a version of Theorem 9 for functions in $L^1(X, \mathcal{A}, \mu)$ is to be useful.

Theorem 10. *Let f be a measurable function such that δ_f is not identically $+\infty$. Then the inequality*

$$\int_X gh\, d\mu(x) \leq \int_X g^{*f} h^{*f}\, d\mu(x) \tag{66}$$

holds for any two non-negative functions $g, h \in L^1(X, \mathcal{A}, \mu)$.

The meaning of inequality (66) is the following: if the integral of the second member is finite, then so it is that appearing in the first member and the inequality holds.

Proof.

The proof does not practically differ from the proof of Theorem 9.

■

3 Coalescence of measures

Let $(X, \mathcal{A}, \mu)$ be a σ-finite measure space and consider another measure ν on X which is absolutely continuous with respect to μ. The Radon-Nikodym Theorem enable us to write

$$\nu(E) = \int_E f(x)\, d\mu(x), \ E \in \mathcal{A}, \tag{67}$$

where $f = d\nu/d\mu$, the so called Radon-Nikodym derivative of the measure ν with respect to μ, is a μ-unique non-negative measurable function defined on X. We define the *range* of the measure μ to be the set $\mu(\mathcal{A}) \subseteq [0, \mu(X)]$ so that, for any C that belongs to the range of μ there exists a measurable set E such that $\mu(E) = C$. Sets in $\{E \in \mathcal{A} : \mu(E) = C\}$ are to be named *admissible configurations* in what follows. For a C belonging to the range of μ such that $0 < C < \mu(X)$, we pose the optimization problem

$$\sup\{\nu(E) : E \in \mathcal{A}, \mu(E) = C\}. \tag{68}$$

Extremum values $C = 0$ and $C = \mu(X)$ are excluded in order to avoid trivial situations. Next theorem furnishes a characterization of optimal configurations corresponding to the optimization problem (68).

Theorem 11. *Assume that the distribution function δ_f corresponding to $f = d\nu/d\mu$ is not identically $+\infty$ and that $\sup_{\delta_f(\alpha)>C} \alpha$ is a point of continuity of δ_f. Then, the set E^{*f} with $E \in \mathcal{A}$ and $\mu(E) = C$ is an optimal configuration corresponding to problem (68). Furthermore, if $\sup_{\mu(E)=C} \nu(E) < +\infty$, then the optimal configuration E^{*f} is μ-unique; i.e., any other optimal configuration A differs from E^{*f} only in a μ-null set.*

Proof. Let $0 < C < \mu(X)$ belongs to the range of μ. In view of (67), for every $E \in \mathcal{A}$ with $\mu(E) = C$ we can write

$$\nu(E) = \int_E f(x)\, d\mu(x) = \int_X f(x)\chi_E(x)\, d\mu(x), \tag{69}$$

where $f = d\nu/d\mu \geq 0$ is measurable. Since we have supposed that δ_f is not identically $+\infty$, superior f-rearrangements can be taken and then, an application of Theorem 10 produces

$$\begin{aligned}\int_X f(x)\chi_E(x)\,d\mu(x) &\le \int_X f^{*f}(x)\chi_E^{*f}(x)\,d\mu(x)\\ &= \int_X f(x)\chi_{E^{*f}}(x)\,d\mu(x)\\ &= \nu(E^{*f}). \qquad (70)\end{aligned}$$

To get (70), both Theorem 6-**v)** and identity (50) have been used. From (69) and (70) we obtain

$$\sup_{\mu(E)=C} \nu(E) \le \nu(E^{*f})$$

and, taking into account that

$$E^{*f} = F^{+}_{\sup_{\delta_f(\alpha)>C}\alpha} \qquad (71)$$

and that $\sup_{\delta_f(\alpha)>C}\alpha$ is a continuity point of δ_f, Lemma 3 -**i)** shows that $\mu(E^{*f}) = \mu(E) = C$. This proves that E^{*f} is an optimal configuration for problem (68).

Let us now suppose that $\sup_{\mu(E)=C}\nu(E) < +\infty$. To prove that E^{*f} is μ-unique, we observe that, for every $A \in \mathcal{A}$, we can write

$$\nu(A) = \nu((A\setminus E^{*f})\cup(A\cap E^{*f})) = \int_{A\setminus E^{*f}} f(x)\,d\mu(x) + \int_{A\cap E^{*f}} f(x)\,d\mu(x) \qquad (72)$$

and

$$\nu(E^{*f}) = \nu((E^{*f}\setminus A)\cup(A\cap E^{*f})) = \int_{E^{*f}\setminus A} f(x)\,d\mu(x) + \int_{A\cap E^{*f}} f(x)\,d\mu(x). \qquad (73)$$

Now we realize that

$$f(x) < \sup_{\delta_f(\alpha)>C}\alpha,\ x\in A\setminus E^{*f}, \quad f(x) \ge \sup_{\delta_f(\alpha)>C}\alpha,\ x\in E^{*f}\setminus A \qquad (74)$$

and

$$0 < \sup_{\delta_f(\alpha)>C}\alpha < +\infty. \qquad (75)$$

In fact, inequalities (74) easily follows from expression (71) for E^{*f}. In respect to (75), we see that the non-negativity of f implies that

$\delta_f(\alpha) = \mu(X) > C$ for $\alpha < 0$. The inequality $\sup_{\delta_f(\alpha)>C} \alpha \geq 0$ follows from this observation. If $\sup_{\delta_f(\alpha)>C} \alpha = +\infty$, then $\delta_f(\alpha) > C$ for every α and therefore, $0 = \delta_f(+\infty) \geq C$, which is contrary to the assumption $C > 0$.

So we have proved that $0 \leq \sup_{\delta_f(\alpha)>C} \alpha < +\infty$. Furthermore, we have $\sup_{\delta_f(\alpha)>C} \alpha \neq 0$ because if, on the contrary, it would be $\sup_{\delta_f(\alpha)>C} \alpha = 0$, then $\delta_f(\alpha) \leq C$ for $\alpha > 0$ and thus $\delta_f(0^+) \leq C$. In view of $\sup_{\delta_f(\alpha)>C} \alpha$ is supposed to be a point of continuity of δ_f, we deduce that the inequality $\mu(X) = \delta_f(0^-) = \delta_f(0^+) \leq C$ holds, so violating again the assumption $C < \mu(X)$. Note that the possibility $f \equiv 0$ is excluded by the hypotheses: if $f \equiv 0$, then $\sup_{\delta_f(\alpha)>C} \alpha = 0$ for every $0 < C < \mu(X)$ and $\lambda = 0$ is not a point of continuity of δ_f.

If A is an optimal configuration for problem (68), then $\nu(A) = \nu(E^{*f})$ and equalities (72) and (73) imply that

$$\int_{A \setminus E^{*f}} f(x)\, d\mu(x) = \int_{E^{*f} \setminus A} f(x)\, d\mu(x). \tag{76}$$

Therefore, if we suppose that $\mu(A \setminus E^{*f}) > 0$, from (76) and (74) together we obtain

$$\begin{aligned} \mu(E^{*f} \setminus A) \sup_{\delta_f(\alpha)>C} \alpha &\leq \int_{E^{*f} \setminus A} f(x)\, d\mu(x) \\ &= \int_{A \setminus E^{*f}} f(x)\, d\mu(x) < \mu(A \setminus E^{*f}) \sup_{\delta_f(\alpha)>C} \alpha \end{aligned}$$

or, taking into account (75),

$$\mu(A) - \mu(A \cap E^{*f}) = \mu(A \setminus E^{*f}) < \mu(E^{*f} \setminus A) = \mu(E^{*f}) - \mu(A \cap E^{*f}),$$

which is in contradiction with the equality $\mu(A) = \mu(E^{*f}) = C$. Hence we conclude that $\mu(A \setminus E^{*f}) = 0$ and therefore, we can write

$$0 = \int_{A \setminus E^{*f}} f(x)\, d\mu(x) = \int_{E^{*f} \setminus A} f(x)\, d\mu(x) \geq \mu(E^{*f} \setminus A) \sup_{\delta_f(\alpha)>C} \alpha$$

which implies that $\mu(E^{*f} \setminus A) = 0$ as well. Thus we have shown that $\mu(A \triangle E^{*f}) = 0$ so finishing the proof.

■

We remark that some pathologies arise in the characterization of optimal configurations as level sets of $d\nu/d\mu$ when hypotheses of Theorem 11 fail. A few examples are now given attempting to clear this point. Beyond of the technical impossibility of defining f-rearrangements when δ_f is identically $+\infty$, the assumption $\mathfrak{F}(\delta_f) \neq \emptyset$ is needed in order that level sets might be admissible configurations for problem (68). Consider, for instance, the case $X = (0, +\infty)$, $\mu(\cdot) = |\cdot|$ and $f(x) \equiv x$. Since the measure of level sets is infinite, the optimum $\sup_{\mu(E)=C} \nu(E) = +\infty$ is realized by no one of them because they simply are not admissible configurations. Note, however, that $\nu(E^*) = \int_{E^*} x\, dx = +\infty$ holds for certain not bounded measurable set E^* with $|E^*| = C$.

Neither can the requirement of δ_f being continuous at the point $\sup_{\delta_f(\alpha)>C} \alpha$ be dropped from the statement of Theorem 11. For example, if $X = (0,1)$, $\mu(\cdot) = |\cdot|$ and $f = \chi_{(0,1/2)}$, we have $\delta_f^+ = \chi_{(-\infty,0]} + \frac{1}{2}\chi_{(0,1]}$ and $\sup_{\delta_f(\alpha)>1/4} \alpha = 1$. Thus, $F_1^+ = (0,1/2)$ satisfies $|F_1^+| = 1/2 > 1/4$ and therefore F_1^+ is not an admissible configuration for problem (68) with $C = 1/4$.

Our last example shows that, in general, μ-uniqueness of the optimal configuration E^{*f} breaks down when $\sup_{\mu(E)=C} \nu(E) = +\infty$. In fact, if we take $X = (0,1)$, $\mu(\cdot) = |\cdot|$ and $f(x) \equiv 1/x$, we see that, for every $0 < C < 1$,

$$+\infty = \sup_{\mu(E)=C} \nu(E) = \nu((0,C)) = \nu((0,C/2) \cup (1 - C/2, 1))$$

while

$$|(0,C) \,\Delta\, [(0,C/2) \cup (1 - C/2, 1)]| \neq 0.$$

Now consider a topological space X equipped with the σ-algebra of its Borel sets. As before, μ and ν denote two σ-finite Borel measures on X such that ν is absolutely continuous with respect to μ. A Borel subset $B \subsetneq X$ is said to be a *μ -connected subset of X* when there exists a μ-null set Z such that $B \cup Z$ is connected. When every level set F_λ of a measurable function $f : X \to \overline{\mathbb{R}}$ is μ-connected, we say that f is a *μ-connected function.* Thus, for every level set F_λ of a μ-connected function f, there exists a measurable set Z_λ with $\mu(Z_\lambda) = 0$ such that $F_\lambda \cup Z_\lambda$ is connected. The concept of coalesce of measures was anticipated in the introduction: the measure ν is coalescent with respect to μ whenever a connected optimal solution to problem (68) do exists for every C belonging to the range of μ, $0 < C < \mu(X)$. We will see that coalescence of a measure ν with respect to μ is closely related to the μ-connectedness of level sets of the Radon-Nikodym derivative $f = d\nu/d\mu$.

Theorem 12. *Suppose that δ_f, the distribution function of $f = d\nu/d\mu$, is continuous on its support. Then, the measure ν is coalescent with respect to μ provided that f is a μ-connected function. Furthermore, if the measure ν is coalescent with respect to μ and $\sup_{\mu(E)=C} \nu(E) < +\infty$ for every $0 < C < \mu(X)$ belonging to the range of μ; then $f = d\nu/d\mu$ is a μ-connected function.*

Proof. Assume that $d\nu/d\mu$ is a μ-connected function and let be given a $0 < C < \mu(X)$ belonging to the range of μ. Then, there exists a measurable set E with $\mu(E) = C$ and Theorem 11 shows that E^{*f} is an optimal configuration for problem (68). By realizing that E^{*f} is a level set of the μ-connectedfunction $f = d\nu/d\nu$, we conclude that there exists a μ-null set Z such that $E^{*f} \cup Z$ is connected. Since $\mu(E^{*f} \cup Z) = \mu(E^{*f}) = C$ and $\nu(E^{*f} \cup Z) = \nu(E^{*f})$, the set $E^{*f} \cup Z$ is a connected optimal configuration. From the arbitrariness of $0 < C < \mu(X)$ it follows that the measure ν is coalescent with respect to μ.

To prove the converse, assume that $\sup_{\mu(E)=C} \nu(E) < +\infty$ for every $0 < C < \mu(X)$ belonging to the range of μ. It will be sufficient to show that F_λ^+ is μ-connected for all $\lambda \in (0, \mathrm{ess\,sup} f)$. Now, if the measure ν is coalescent with respect to μ and $0 < C < \mu(X)$ belongs to the range of μ, then a connected configuration E_C such that $\sup_{\mu(E)=C} \nu(E) = \nu(E_C)$. From Theorem 11 we see that E_C differs from $E_C^{*f} = F^+_{\sup_{\delta_f(\alpha)>C} \alpha}$ only in a μ-null set and so, the level set $F^+_{\sup_{\delta_f(\alpha)>C} \alpha}$ is μ-connected whatever be $0 < C < \mu(X)$ belonging to the range of μ. If $\lambda \in (0, \mathrm{ess\,sup} f)$, then $C_\lambda = \delta_f(\lambda)$ obviously belongs to the range of μ; moreover, the inequalities $0 < C_\lambda < \mu(X)$ holds. We affirm that F_λ^+ differs from $F^+_{\sup_{\delta_f(\alpha)>C_\lambda} \alpha}$ in a μ-null set. In fact, we have $\sup_{\delta_f(\alpha)>C_\lambda} \alpha = \sup_{\delta_f(\alpha)>\delta_f(\lambda)} \alpha \le \lambda$ and, by the continuity of δ_f, $\delta_f(\lambda) = C_\lambda = \delta_f(\sup_{\delta_f(\alpha)>C_\lambda} \alpha)$, whence

$$F_\lambda^+ \subseteq F^+_{\sup_{\delta_f(\alpha)>C_\lambda} \alpha} \tag{77}$$

and

$$\mu(F_\lambda^+) = \mu(F^+_{\sup_{\delta_f(\alpha)>C_\lambda} \alpha}). \tag{78}$$

The assertion easily follows from (77) and (78), which finishes the proof.

■

Using a different argument, we proved in the Introduction that the harmonic measure $\omega((x_0, y_0), \mathbb{R}^2_+, \cdot)$ of the upper half-plane is coalescent with respect to the Lebesgue measure on the real axis. This fact directly follows from Theorem 12 by observing that the Poisson kernel (5) is a quasi-concave function of x satisfying $|N_\lambda| = 0$ for every λ.

To end this paper, we remark that the Poisson kernel for the circle $B_1(0)$, expressed by

$$P(r, \phi; \theta) = \frac{1}{2\pi} \frac{1 - r^2}{1 - 2r\cos(\phi - \theta) + r^2}, \quad 0 < r < 1, 0 \leq \theta < 2\pi,$$

is also a $|\cdot|$-connected function when considered as a function on $\partial B_1(0)$ or, what amounts the same, as a 2π-periodic function on $\mathbb{R}$. Since condition $|N_\lambda| = 0$, $\lambda \in \mathbb{R}$, is also satisfied by P, Theorem 12 applies and the harmonic measure $\omega((r, \phi), B_1(0), \cdot)$ corresponding to the circle turns out to be coalescent with respect to the Lebesgue measure on the boundary $\partial B_1(0)$.

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