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Instability of radial standing waves of Schrödinger equation on the exterior of a ball.

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Abstract

Under smoothness and growth assumptions on f we show that a standing wave $w(t,x) = e^{i\beta t}\phi(x)$ of the Schrödinger equation on the exterior Ω of a ball and Neumann boundary condition

$$w_t = i(\Delta w + f(|w|^2)w) \quad \frac{\partial w}{\partial n} = 0 \quad \text{on}\, \partial\Omega$$

where β is real and ϕ is real and radially symmetric, is always linearly unstable under perturbations in the space $H^1(\Omega)$ (it may be stable under perturbations in $H^1_{rad}(\Omega)$).

The instability is independent of ϕ having a fixed sign and of its Morse index.

The main tool is a theorem of linearized instability of M. Grillakis.

1 Introduction and statement of the result

In this paper we consider the Schrödinger equation

$$w_t = i(\Delta w + f(|w|^2)w) \quad \frac{\partial w}{\partial n} = 0 \quad \text{on} \quad \partial\Omega. \tag{1}$$

In (1) f is real valued function, $w(t,x)$ is a complex valued function defined for $t \in R$ and $x \in \Omega$ where Ω is the exterior of the ball centered at the origin and with radius $a > 0$.

A **standing wave** is a solution of (1) of the form

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$$w(t,x) = e^{i\beta t}\phi(x) \tag{2}$$

where β is real.

The stability and instability of waves have been studied by many authors (see [1], [2], [3], [4] and [5] among many others).

If we replace (2) in (1) we see that $\phi(x)$ has to satisfy the elliptic system

$$\Delta\phi + (f(|\phi|^2) - \beta)\phi = 0 \quad \frac{\partial\phi}{\partial n} = 0 \quad \text{on } \partial\Omega \tag{3}$$

If in (1) we replace $w(t,x)$ by $e^{-i\beta t}w(t,x)$ the new equation becomes

$$w_t = i(\Delta w + (f(|w|^2) - \beta)w \quad \frac{\partial w}{\partial n} = 0 \quad \text{on } \partial\Omega \tag{4}$$

and ϕ is an equilibrium solution for (4) and then studying stability properties of $e^{i\beta t}\phi(x)$ for (1) is equivalent to study stability of the equilibrium $\phi(x)$ for (4). So, if we let $w = u + iv$ we are led to the real system

$$\begin{aligned} u_t &= -\Delta v - (f(u^2+v^2) - c)v \quad \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega \\ v_t &= \Delta u + (f(u^2+v^2) - c)u \quad \frac{\partial v}{\partial n} = 0 \quad \text{on } \partial\Omega \end{aligned} \tag{5}$$

In (5) $u(t,x)$ and $v(t,x)$ are defined for $t \in R$ and $x \in \Omega$ where Ω is the exterior of the ball centered at the origin and with radius $a > 0$ and the constant c is introduced so that the normalization condition $f(0) = 0$ is met. Our goal is to show that radially symmetric equilibria of (5) are always linearly unstable.

Remark. The Cauchy problem for (1) (or (5)) is an open question but it is believed that it is well posed under the same assumptions on the function f that guarantees that it is well posed when Ω is the entire space R^N. So, instability will be proved modulo this technical point. The same attitude is taken in [6].

Notice that if a radial complex function $\phi(r)$ solves the equation

$$\Delta\phi + f(|\phi(r)|^2 - c)\phi = 0 \quad \phi'(a) = 0 \tag{6}$$

and we pretend that $f(|\phi(r)|^2 - c)$ is a known function of r, we see that the real and imaginary parts of ϕ solve the same ordinary differential equation and satisfy the same boundary condition at $r = a$. Hence they are linearly dependent and using the gauge invariance of (5) we conclude that $\phi(r)$ can be taken real.

So our assumptions are the following:

H_1) $f : R \to R$ is a C^1 function satisfying $f(0) = 0$ and $c > 0$ is a constant;

H_2) $\phi : [a, +\infty) \to R$, $a > 0$, is a nonconstant function that converges to zero exponentially as r goes to $+\infty$ and solves the problem

$$-\phi''(r) - \frac{N-1}{r}\phi'(r) - (f(|\phi(r)|^2) - c)\phi(r) = 0 \quad \phi'(a) = 0. \tag{7}$$

Assumption H_2 says, in particular that $(u, v) = (\phi, 0)$ is an equilibrium for (5) and we are going to show that it is always unstable for perturbation in $H^1(\Omega)$. Of course we cannot expect that such an equilibrium is unstable with respect to radially symmetric perturbation because if, for instance, such equilibrium is obtained through a minimization process inside that class it will be stable in that class. The fact that indicates the instability is that a radially symmetric function cannot be a local minimizer of the functional

$$V(u) = 1/2 \int_\Omega |\text{grad} u(x)|^2 \, dx + \int_\Omega F(u(x)) \, dx$$

subject to

$$\int_\Omega G(u(x)) \, dx = \lambda$$

for $u \in H^1(\Omega)$ (see [7] and [8]). We believe that it is important to emphasize that this break of symmetry is not, by itself, a proof of instability.

Together with system (5), we consider its linearization at $(\phi, 0)$:

$$\begin{aligned} u_t &= -\Delta v - (f(\phi^2(r)) - c)v \quad \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega \\ v_t &= \Delta u + (f(\phi^2(r) - c + 2\phi^2(r)f'(\phi^2(r)))u \quad \frac{\partial v}{\partial n} = 0 \quad \text{on } \partial\Omega \end{aligned} \tag{8}$$

If we define $q_1(x) = \dfrac{x_1}{r}$ then an elementary calculation shows that the set of the elements $(u(x), v(x))$ of the form $(u(x), v(x)) =$

$(q_1(x)U(r), q_1(x)V(r))$ is invariant under system (8), where, as indicated, U and V are radially symmetric. Moreover, since $\langle \text{grad} q_1(x), x\rangle = 0$ because $q_1(x)$ is constant on straight lines through the origin, the boundary condition for $U(r)$ and $V(r)$ becomes $U'(a) = 0 = V'(a)$. So we are led to the system:

$$\begin{aligned} U_t &= -V''(r) - \frac{N-1}{r}V'(r) + \frac{N-1}{r^2}V - (f(\phi^2(r) - c))V \quad U'(a) = 0 \\ V_t &= U''(r) + \frac{N-1}{r}U'(r) - \frac{N-1}{r^2}U + \\ &\quad (f(\phi^2(r)) - c + 2\phi^2(r)f'(\phi^2(r)))U \quad V'(a) = 0. \end{aligned} \tag{9}$$

So, system (9) governs the linearized equation (8) in the "mode" x_1/r.

As far the phase space is concerned, using the fact that $a > 0$, an elementary calculation shows that $u \in L^2(\Omega), H^1(\Omega), H^2(\Omega)$ if and only if $U \in L^2_n([a,+\infty), H^1_n([a,+\infty), H^2_n([a,+\infty)$, respectively, where $L^2_n([a,+\infty), H^1_n([a,+\infty)$ and $H^2_n([a,+\infty)$ are the spaces whose norms are defined by integrals with weight r^{N-1}.

If we define the operator $A(U,V) = (L(V), -M(U))$ by the right hand side of (9), $A : D(A) \subset L^2_n([a,+\infty) \times L^2_n([a,+\infty) \to L^2_n([a,+\infty) \times L^2_n([a,+\infty)$ where $D(A)$ is the set of the elements (U,V) belonging to $H^2_N([a,+\infty) \times H^2_N([a,+\infty)$ such that $U'(a) = 0 = V'(a)$ then our result is the following:

Theorem I. *Under assumptions H_1 and H_2, the spectrum of A has a strictly positive real element.*

Theorem I has been proved by M. Esteban and W. Strauss ([6]) assuming that f is a pure power and that ϕ is a ground stated. In particular, in their result ϕ is positive and has Morse index one in the space $H^1_{rad}(\Omega)$.

A more difficult question is to know whether a wave that changes sign may be stable in $H^1_{rad}(\Omega)$ or not (see also remark at the top of page 760 of [10]).

2 Proof of the theorem I

We start by stating a very useful result on spectral theory of a second order differential operator.

We consider the differential operator

$$(Tv)(r) = -v''(r) - \frac{N-1}{r}v'(r) + (p(r)+c)v(r) \tag{10}$$

in the interval $[a, +\infty), a > 0$, where $p : [a, +\infty) \to R$ is continuous and tends to zero at infinity. Together with (10) we consider one of the following boundary conditions:

- $v(a) = 0$
- $v'(a) = 0$.

We denote by $L^2_n([a, +\infty))$, $H^1_n([a, +\infty))$ and $H^2_n([a, +\infty))$ the Hilbert spaces we get using norms with weight r^{N-1}, that is, $\int_a^{+\infty} r^{N-1}v^2(r)\,dr$, and so on. In other words, those three spaces are the corresponding Sobolev spaces of the radially symmetric functions defined on the exterior of the ball with radius $a > 0$. We define $T : D(T) \subset L^2_n([a, +\infty)) \to L^2_n([a, +\infty))$ where $D(T)$ is the set of the elements $v \in H^2_n([a, +\infty))$ such that $v(r)$ satisfies the corresponding boundary conditions. Then the following result holds:

Theorem II.
1) T is a bounded below self-adjoint operator and the essential spectrum of T is the interval $[c, +\infty)$;
2) the rest of the spectrum $\sigma(T)$ consists of isolated simple eigenvalues lying below c ;
3) if $\sigma(r)$ is a nontrivial solution of $T(\sigma(r)) = 0$ satisfying the corresponding boundary condition at $r = a$ (this solution is determined up to a constant multiple), then the number of strictly negative eigenvalues of T is equal to the number of zeroes of $\sigma(r)$ in $(a, +\infty)$.

Theorem II can be proved using the change of variables $w(r) = r^{\frac{N-1}{2}}v(r)$ and theorem 53, page 1479 and corollary 54, page 1480 of [9].

The following result due to M. Grillaikis ([10]) will be very important for our argument.

Theorem III. *Let L and M be the self-adjoint operators defined in some Hilbert space X and suppose there is a strictly positive operator H such that $L - H$ and $M - H$ are relatively compact perturbations of H. Let Y be the kernel of M and P the orthogonal projection of X onto Y.*

If we denote by $N(M)$ the number of strictly negative eigenvalues of M, by $N_0(PLP)$ the number of nonpositive eigenvalues of the operator PLP and if $N(M) > N_0(PLP)$, then the operator $A(U,V) = (L(V), -M(U))$ has a strictly positive real eigenvalue.

Before passing to the proof of theorem I, we state a few lemmata about elementary properties of $\phi(r)$. Let us recall that $\phi(r)$ is a non-constant solution of the problem

$$-\phi''(r) - \frac{N-1}{r}\phi'(r) - (f(\phi(r)^2) - c)\phi(r) = 0 \quad \phi'(a) = 0. \tag{11}$$

and $\phi(r)$ tends to zero at infinity.

Lemma 1. *If for some point r_1 we have $\phi'(r_1) = 0$ then $\phi''(r_1) \neq 0$.*

Proof. In fact, otherwise, from (11) we see that $(f(\phi(r_1)^2) - c)\phi(r_1)$ would be zero and then the constant function equal to $\phi(r_1)$ would be a solution of the ordinary differential equation (11) and this is a contradiction because $\phi(r)$ would have the same initial data at $r = r_1$ as the constant function and the lemma is proved.

Lemma 2. *If for some point r_1 we have $\phi(r_1) = 0$ then $\phi'(r_1) \neq 0$.*

Proof. Follows immediately from the fact the the function identically zero solves (11) and the uniqueness of the Cauchy problem.

Lemma 3. *If for some point r_1 we have $\phi'(r_1) = 0$ then we cannot have $\phi(r_1) = \phi(r_2)$ for some $r_2 > r_1$.*

Proof. If we denote by F a primitive of f and we define $G(s) = F(s)+cs$ from (11) we see that

$$\frac{d}{dr}(\phi'^2(r) + G(\phi^2(r)) = -\frac{N-1}{r}\phi'^2(r)$$

and then for $r_1 < r_2$ we have

$$\phi'^2(r_2) + G(\phi^2(r_2) < \phi'^2(r_1) + G(\phi^2(r_1).$$

But this las inequality shows that we cannot have $\phi'(r_1) = 0$ and $\phi(r_1) = \phi(r_2)$ and this proves the lemma.

Lemma 4. *The number of zeroes of $\phi'(r)$ on $(a, +\infty)$ is equal to the number of zeroes of $\phi(r)$ on $(a, +\infty)$.*

Proof. It follows imediately from Lemmata 1,2 and 3 and the fact that $\phi(r)$ tends to zero at infinity.

Proof of theorem I. We start by defining the operators

$$L_0(V) = -V''(r) - \frac{N-1}{r}V'(r) - (f(\phi^2(r)) - c)V(r)$$

and

$$M_0(U) = -U''(r) - \frac{N-1}{r}U'(r) - (f(\phi^2(r)) - c + 2\phi^2(r)f'(\phi^2(r)))U(r)$$

and the domain of both is the set of the elements of $H^2_N([a,+\infty))$ whose derivative vanish at $r = a$. So we can write

$$L(V) = L_0(V) + \frac{N-1}{r^2}V$$

and

$$M(U) = M_0(U) + \frac{N-1}{r^2}U.$$

where, as before, L and M are defined by the right hand side of (9).

First of all if we denote by m the number of zeroes of ϕ in $(a,+\infty)$, we notice that (7) says that $L_0(\phi) = 0$ and $\phi'(a) = 0$ and then from Theorem II we conclude that 0 is an eigenvalue of L_0 and the number of strictly negative eigenvalues of L_0 is m. Moreover, due to the minimax characterization of the eigenvalues, we see that the addition of the term $\frac{N-1}{r^2}V$ to L_0 moves its eigenvalues strictly to the right. Hence, the spectrum of L has at most m nonpositive eigenvalues.

Now we pass to M. First we claim that 0 is not an eigenvalue of M. In fact, if we define $\psi = \phi'$ and we differentiate (7) with respect to r we see that

$$-\psi''(r) - \frac{N-1}{r}\psi'(r) + \frac{N-1}{r^2}\psi$$
$$-(f(\phi^2(r)) - c + 2\phi^2(r)f'(\phi^2(r)))\psi(r) = 0 \quad \psi(a) = 0. \quad (12)$$

Now suppose 0 is an eigenvalue of M. Then there is an element $\eta \in H^2_N([a,+\infty))$ such that

$$-\eta''(r) - \frac{N-1}{r}\eta'(r) + \frac{N-1}{r^2}\eta$$
$$-(f(\phi^2(r)) - c + 2\phi^2(r)f'(\phi^2(r)))\eta(r) = 0 \quad \eta'(a) = 0. \quad (13)$$

But the second order differential equation (13) has a saddle structure at the origin because $c > 0$. Actually, due to the presence of the nonintegrable term $\frac{N-1}{r}$ in front of η', (13) has two linearly independent solutions whose asymptotic behavior is $r^{\frac{N-1}{2}}e^{+\sqrt{c}r}$ and $r^{\frac{N-1}{2}}e^{-\sqrt{c}r}$. Anyway, the set of the solution of the linear equation (13) that tend to zero at infinity has dimension one. Hence ψ and η are linearly dependent and then ψ is a constant times η. So $\psi'(a) = 0$ and then, by uniqueness, ψ is identically zero, a contradiction. This shows that M is invertible and then the orthogonal projection P appearing in the statement of Theorem IV is the identity.

Moreover, (12) shows that if we consider the operator M with Dirichlet boundary condition at $r = a$ then it has zero as an eigenvalue and, according to theorem II and lemma 4, the number of strictly negative eigenvalues is exactly m. But when we pass from Dirichlet boundary condition to Newmann, we move the eigenvalues strictly to the right and so the operator M with Newmann boundary condition has at least $m+1$ strictly negative eigenvalues. Then $N(M) > N_0(PLP)$ and, in view of Theorem III, Theorem I is proved.

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