REVISTA MATEMÁTICA COMPLUTENSE
(2000) vol. XIII, num. 1, 85-109

ISNN 1139-1138

TREE STRUCTURE ON THE SET OF MULTIPLICATIVE SEMI-NORMS OF KRASNER ALGEBRAS $H(D)$

K. BOUSSAF, N. MAÏNETTI and M. HEMDAOUI

Abstract

Let $\mathbb{K}$ be an algebraically closed field, complete for an ultrametric absolute value, let D be an infinite subset of $\mathbb{K}$ and let $H(D)$ be the set of analytic elements on D [7]. We denote by $\mathrm{Mult}(H(D),\mathcal{U}_D)$ the set of semi-norms ψ of the $\mathbb{K}$-vector space $H(D)$ which are continuous with respect to the topology of uniform convergence on D and wich satisfy further $\psi(fg) = \psi(f)\psi(g)$ whenever $f, g \in H(D)$ such that $fg \in H(D)$. This set is provided with the topology of simple convergence. By the way of a metric topology thinner than the simple convergence, we establish the equivalence between the connectedness of $\mathrm{Mult}(H(D),\mathcal{U}_D)$, the arc-connectedness of $\mathrm{Mult}(H(D),\mathcal{U}_D)$ and the infraconnectedness of D. This generalizes a result of Berkovich given on affinoid algebras [2]. Next, we study the filter of neighbourhoods of an element of $\mathrm{Mult}(H(D),\mathcal{U}_D)$, and we give a condition on the field $\mathbb{K}$ such that this filter admits a countable basis. We also prove the local arc-connectedness of $\mathrm{Mult}(H(D),\mathcal{U}_D)$ when D is infraconnected. Finally, we study the metrizability of the topology of simple convergence on $\mathrm{Mult}(H(D),\mathcal{U}_D)$ and we give some conditions to have an equivalence with the metric topology defined above. The fundamental tool in this survey consists of circular filters.

Throughout this paper, $\mathbb{K}$ will denote an algebraically closed field which is complete for a non-trivial ultrametric absolute value denoted by $|\cdot|$. We also denote by $|\cdot|_\infty$ the classical absolute value of $\mathbb{R}$.

1 Preliminaries

Definitions and notation: Let $a \in \mathbb{K}$ and $r, r' > 0$ with $r < r'$. We

1991 Mathematics Subject Classification: Primary: 46S10, Secondary: 11Q25.
Servicio de Publicaciones. Universidad Complutense. Madrid, 2000

denote by $d(a, r)$ the *circumferenced disk* $\{x \in \mathbb{K} \mid |a - x| \leq r\}$, by $d(a, r^-)$ the *non-circumferenced disk* $\{x \in \mathbb{K} \mid |a - x| < r\}$, by $C(a, r)$ the circle $\{x \in \mathbb{K} \mid |a - x| = r\}$, by $\Gamma(a, r, r')$ the *non-circumferenced annulus* $\{x \in \mathbb{K} \mid r < |a - x| < r'\}$, and by $\Delta(a, r, r')$ the *circumferenced annulus* $\{x \in \mathbb{K} \mid r \leq |a - x| \leq r'\}$. We put $|\mathbb{K}| = \{|x| \mid x \in \mathbb{K}\}$ and we denote by $\Bbbk$ the *residue class field* $d(0,1)/d(0,1^-)$. The field $\mathbb{K}$ will be said to be *weakly valued* if both $|\mathbb{K}|$ and $\Bbbk$ are countable. Else $\mathbb{K}$ will be said to be *strongly valued.*

In any topological space E, the closure of a subset A is denoted by $\overline{A}$, and the interior is denoted by $\mathring{A}$.

Let D be an infinite subset of $\mathbb{K}$. We denote by $\widetilde{D}$ the smallest circumferenced disk which contains D. We call *holes* of D the maximal non-circumferenced disks of $\widetilde{D}\setminus\overline{D}$. The set of holes of D forms a partition of $\widetilde{D} \setminus \overline{D}$, [7]. We write $R(D)$ the $\mathbb{K}$-subalgebra of $\mathbb{K}^D$ of the rational functions with no poles in D. We denote by $H(D)$ the completion of $R(D)$ for the topology $\mathcal{U}_D$ of uniform convergence on D. The elements of $H(D)$ are called the *analytic elements on* D [4], [7].

We denote by $\mathcal{A}$ the set of the $D \subset \mathbb{K}$ such that $H(D)$ is a $\mathbb{K}$-algebra. It is known that $D \in \mathcal{A}$ if and only if $\overline{D} \setminus D \subset \mathring{\overline{D}}$ and $\widetilde{D} \setminus \overline{D}$ is bounded [5, Th. III.6]).

Let $D \subset \mathbb{K}$. Then D is said to be *infraconnected* if, for all $a \in D$, the set $\overline{\{|x - a|; x \in \mathbb{K}\}}$ is an interval of $\mathbb{R}$, [4], [5] and [7]. A closed bounded infraconnected set B in $\mathbb{K}$ is said to be *affinoid* if it only admits finitely many holes, if their diameters belong to $|\mathbb{K}|$ and if $\mathrm{diam}(B) \in |\mathbb{K}|$. More generaly, a bounded set D in $\mathbb{K}$ will be said to be *affinoid* if it is the union of finitely many closed infraconnected affinoids [8].

Remark. It is known that the intersection of two infraconnected affinoids is always an infraconnected affinoid [8]. But it is known that the intersection of two infraconnected sets may be a non-infraconnected subset of $\mathbb{K}$. However, we have the following lemma.

Lemma 1.1 *Let D be infraconnected and B be an infraconnected affinoid. Then $D \cap B$ is infraconnected.*

Proof. We suppose that $D \cap B$ is not infraconnected. Then, there exist $a, b \in D \cap B$ and $r_1, r_2 \in \mathbb{R}$ with $0 < r_1 < r_2 < |a - b|$ such that $\Gamma(a, r_1, r_2) \cap B \cap D = \emptyset$.

Since B is an infraconnected affinoid, there only exist finitely many

$\rho \in]0, |a-b|[$ such that the circle $C(a,\rho)$ contains holes of B. So, clearly there do exist ρ_1 and ρ_2 such that $r_1 < \rho_1 < \rho_2 < r_2$ and such that $\Gamma(a,\rho_1,\rho_2) \subset B$. Since D is infraconnected, then $\Gamma(a,\rho_1,\rho_2) \cap D \neq \emptyset$. This contradicts the hypothesis $\Gamma(a,r_1,r_2) \cap B \cap D = \emptyset$.

Definitions. A sequence $(a_n)_{n\in\mathbb{N}}$ in $\mathbb{K}$ is said to be *an increasing distances sequence* (resp. *a decreasing distances sequence*) if the sequence $|a_{n+1} - a_n|$ is strictly increasing (resp. decreasing) and has a limit $l \in {\mathbb{R}^*}_+$.

A sequence $(a_n)_{n\in\mathbb{N}}$ is said to be *a monotonous distances sequence* if it is either an increasing distances sequence or a decreasing distances sequence.

A sequence $(a_n)_{n\in\mathbb{N}}$ in $\mathbb{K}$ is said to be *an equal distances sequence* if $|a_n - a_m| = |a_m - a_q|$ whenever $n,m,q \in \mathbb{N}$ such that $n \neq m \neq q$.

We call a *decreasing filter of diameter* r on $\mathbb{K}$ a filter $\mathcal{G}$ on $\mathbb{K}$ that admits for basis a sequence $(D_n)_{n\in\mathbb{N}}$ in $\mathbb{K}$ of the form $D_n = d(a_n,r_n) \setminus (\bigcap_{m\in\mathbb{N}} d(a_m,r_m))$ with $d(a_{n+1},r_{n+1}) \subset d(a_n,r_n)$, $r_{n+1} < r_n$ and $\lim_{n\to\infty} r_n = r$. We call *center* of $\mathcal{G}$ each element of $\bigcap_{m\in\mathbb{N}} d(a_m,r_m)$. If $\bigcap_{m\in\mathbb{N}} d(a_m,r_m) = \emptyset$ then $\mathcal{G}$ is said to be *a decreasing filter with no center.* According to such a notation the sequence $(D_n)_{n\in\mathbb{N}}$ is called *a canonical basis of* $\mathcal{G}$.

Let $a \in \mathbb{K}$ and $r > 0$. We call *circular filter on* $\mathbb{K}$*, of center* a *and diameter* r, the filter $\mathcal{F}$ on $\mathbb{K}$ which admits as a generating system the family of the annuli $\Gamma(\alpha,r',r'')$ with $\alpha \in d(a,r)$ and $r' < r < r''$, i.e: $\mathcal{F}$ is the filter which admits for basis the family of sets of the form $\bigcap_{i=1}^{q} \Gamma(\alpha_i,r'_i,r''_i)$ with $\alpha_i \in d(a,r)$ and $r'_i < r < r''_i$ $(1 \le i \le q,\ q \in \mathbb{N})$. We call *circular filter on* $\mathbb{K}$ *with no center* any decreasing filter $\mathcal{G}$ with no center.

The filter of neighbourhoods of a point a in $\mathbb{K}$ is called *circular filter of center* a *and diameter* 0 *on* $\mathbb{K}$. It is also named *Cauchy circular filter* of center a on $\mathbb{K}$ and will be denoted by $\mathcal{F}_a$.

Finally we will call *circular filter on* $\mathbb{K}$ all filters of one of those three kind above. A circular filter on $\mathbb{K}$ will be said to be *large* if it has diameter different from 0. Given a circular filter $\mathcal{F}$ on $\mathbb{K}$, its diameter will be denoted by diam($\mathcal{F}$). As usual about filters, a filter $\mathcal{F}$ will be said to be *secant* with a subset D of $\mathbb{K}$ if every element A of $\mathcal{F}$ is such that $A \cap D \neq \emptyset$. Two filters $\mathcal{F}$ and $\mathcal{G}$ are said to be *secant* if for every $A \in \mathcal{F}$ and $B \in \mathcal{G}$, then $A \cap B \neq \emptyset$.

Let $\mathcal{G}$ be a decreasing filter of center a (resp. with no center) and diameter r. The circular filter $\mathcal{F}$ of center a (resp. $\mathcal{G}$) and diameter r is known to be the *unique circular filter less thin than* $\mathcal{G}$ (Proposition 3.13 [7]).

If two circular filters are secant, they are equal [7].

Remark. Every circular filter $\mathcal{F}$ on $\mathbb{K}$ admits a basis consisting of a family of affinoid sets. Indeed, if $\mathcal{F}$ is the circular filter on $\mathbb{K}$ of center a and diameter r, then we clearly obtain a basis of the form $\bigcap_{i=1}^{q} \Delta(\alpha_i, r'_i, r''_i)$ with $\alpha_i \in d(a,r)$, r'_i, $r''_i \in |\mathbb{K}|^*$ and $r'_i < r < r''_i$ $(1 \leq i \leq q,\ q \in \mathbb{N})$.

If $\mathcal{F}$ is a circular filter with no center, of canonical basis $(D_n)_{n\in\mathbb{N}}$, we can find a sequence of disks B_n, the diameter of which lie in $|\mathbb{K}|$, such that $D_n \subset B_n \subset D_{n-1}$.

If $\mathcal{F}$ is the Cauchy circular filter of center a, we just consider disks $d(a, r_n)$ with $r_n \in |\mathbb{K}|$ and $\lim_{n\to\infty} r_n = 0$.

Notation. We denote by Mult($\mathbb{K}[X]$) (resp. Mult($\mathbb{K}(X)$)) the set of multiplicative semi-norms on the $\mathbb{K}$-algebra $\mathbb{K}[X]$ (resp. $\mathbb{K}(X)$).

Given $D \subset \mathbb{K}$, we denote by Mult$(R(D), \mathcal{U}_D)$ the set of multiplicative semi-norms on the $\mathbb{K}$-algebra $R(D)$ that are continuous with respect to the topology $\mathcal{U}_D$. Furthermore, we denote by Mult$(H(D), \mathcal{U}_D)$ the set of continuous semi-norms ψ of the $\mathbb{K}$-vector space $H(D)$ satisfying $\psi(fg) = \psi(f)\psi(g)$ whenever $f, g \in H(D)$ such that $fg \in H(D)$. We notice that for defining Mult$(H(D), \mathcal{U}_D)$ we don't require $H(D)$ to be a $\mathbb{K}$-algebra.

2 Distance on circular filters

This chapter is aimed at defining a distance on the set of circular filters on $\mathbb{K}$, by the way of a partial order relation on this set.

Definitions and notation. Let $\mathcal{F}$ be a circular filter of center a and diameter r. We denote by $\mathcal{Q}(\mathcal{F})$ the set of the centers of $\mathcal{F}$. The set $\mathcal{Q}(\mathcal{F})$ will be called the *heart* of $\mathcal{F}$. Here we have $\mathcal{Q}(\mathcal{F}) = d(a,r)$. If $\mathcal{F}$ is a circular filter without centers, we put $\mathcal{Q}(\mathcal{F}) = \emptyset$.

Given two circular filters on $\mathbb{K}$, $\mathcal{F}$ and $\mathcal{G}$, we say that $\mathcal{G}$ *surrounds* $\mathcal{F}$ if $\mathcal{F}$ is secant with $\mathcal{Q}(\mathcal{G})$ or if $\mathcal{F} = \mathcal{G}$. We put $\mathcal{F} \preceq \mathcal{G}$ when $\mathcal{G}$ surrounds $\mathcal{F}$. We say that $\mathcal{G}$ *strictly surrounds* $\mathcal{F}$, if $\mathcal{F} \preceq \mathcal{G}$ and $\mathcal{F} \neq \mathcal{G}$; such a filter $\mathcal{G}$ clearly posseses centers and we note $\mathcal{F} \prec \mathcal{G}$.

Remark. If $\mathcal{F} \preceq \mathcal{G}$ and $\operatorname{diam}(\mathcal{F}) = \operatorname{diam}(\mathcal{G})$ then $\mathcal{F} = \mathcal{G}$.

It is clearly seen that " $\preceq$ " is a partial order relation on the set of circular filters on $\mathbb{K}$. Given a circular filter $\mathcal{F}$ on $\mathbb{K}$, we will call *wire* of $\mathcal{F}$ the set $\mathcal{W}(\mathcal{F})$ of circular filters $\mathcal{G}$ on $\mathbb{K}$ such that $\mathcal{F} \preceq \mathcal{G}$.

The following lemma is a direct adaptation of Lemma 41.2 of [7].

Lemma 2.1. *Let $\mathcal{F}$ be a circular filter on $\mathbb{K}$, of diameter $r > 0$. For all $s \in [r, +\infty[$, there exists a unique circular filter $\mathcal{G}$ of diameter s surrounding $\mathcal{F}$. Further, if $s > r$, then $\mathcal{Q}(\mathcal{G}) \neq \emptyset$.*

Proof. If $s = r$, we take $\mathcal{G} = \mathcal{F}$ and the uniqueness is obvious. Now, suppose $s > r$ and let $d(a,s)$ be a disk which belongs to $\mathcal{F}$. Then, the circular filter $\mathcal{G}$ of center a and diameter s surrounds $\mathcal{F}$. Suppose that an other circular filter $\mathcal{G}'$ of center b and diameter s also surrounds $\mathcal{F}$. Since $\mathcal{F}$ is secant with both $d(a,s)$ and $d(b,s)$ and since $r < s$, we have $|a-b| \leq s$, and therefore $\mathcal{G} = \mathcal{G}'$.

Lemma 2.2 is obvious.

Lemma 2.2. *Let $\mathcal{F}$, $\mathcal{G}$ be two circular filters with centers such that $\mathcal{Q}(\mathcal{F}) \subset \mathcal{Q}(\mathcal{G})$. Then $\mathcal{G}$ surrounds $\mathcal{F}$.*

Lemma 2.3. *Given any circular filter $\mathcal{F}$ on $\mathbb{K}$, then $\mathcal{W}(\mathcal{F})$ is totally ordered by $\preceq$.*

Proof. Let $\mathcal{G}$ and $\mathcal{H}$ belong to $\mathcal{W}(\mathcal{F}) \setminus \{\mathcal{F}\}$. By Lemma 2.1, both $\mathcal{Q}(\mathcal{G})$ and $\mathcal{Q}(\mathcal{H})$ are not empty. So $\mathcal{F}$ is secant with both $\mathcal{Q}(\mathcal{G})$ and $\mathcal{Q}(\mathcal{H})$. Let $d(a,r) \in \mathcal{F}$ such that $d(a,r) \subset \mathcal{Q}(\mathcal{G})$. Then, as $d(a,r) \cap \mathcal{Q}(\mathcal{H}) \neq \emptyset$, we have $\mathcal{Q}(\mathcal{H}) \cap \mathcal{Q}(\mathcal{G}) \neq \emptyset$. Hence $\mathcal{Q}(\mathcal{H})$ and $\mathcal{Q}(\mathcal{G})$ are comparable for the relation $\subset$ and therefore $\mathcal{H}$ and $\mathcal{G}$ are comparable for $\preceq$.

Definition. *A family of circular filters on $\mathbb{K}$ will be said to be* on the same wire *if their set is all ordered for $\preceq$.*

Remark and definitions. Given a circular filter $\mathcal{F}$ on $\mathbb{K}$, we may define a distance δ' on $\mathcal{W}(\mathcal{F})$ in this way: given $\mathcal{G}, \mathcal{H} \in \mathcal{W}$, we put $\delta'(\mathcal{G}, \mathcal{H}) = |\text{diam}(\mathcal{G}) - \text{diam}(\mathcal{H})|_\infty$.

The elements of $\mathcal{W}(\mathcal{F})$ are just characterized by their diameters and then $\mathcal{W}(\mathcal{F})$, topologized with δ', is clearly isometrically homeomorphic to the real interval $[\text{diam}(\mathcal{F}), +\infty[$. Moreover this homeomorphism does respect the order. Given $\mathcal{G}$, $\mathcal{H} \in \mathcal{W}(\mathcal{F})$ with $\mathcal{G} \preceq \mathcal{H}$, we will denote by $[\mathcal{G}, \mathcal{H}]$ the set of the circular filters $\mathcal{X}$ such that $\mathcal{G} \preceq \mathcal{X} \preceq \mathcal{H}$. Then $[\mathcal{G}, \mathcal{H}]$ is isometrically homeomorphic to the real interval $[\text{diam}(\mathcal{G}), \text{diam}(\mathcal{H})]$.

We shall now generalize this distance to the set of circular filters.

Lemma 2.4. *Let $\mathcal{F}$ and $\mathcal{G}$ be non comparable circular filters on $\mathbb{K}$. There exist disks $d(a,\rho) \in \mathcal{F}$, $d(b,\sigma) \in \mathcal{G}$ such that $d(a,\rho) \cap d(b,\sigma) = \emptyset$.*

Proof. Suppose one can't find $d(a,\rho) \in \mathcal{F}$, $d(b,\sigma) \in \mathcal{G}$ such that $d(a,\rho) \cap d(b,\sigma) = \emptyset$. Then the family S of circumferenced disks which belong to $\mathcal{F}$ and $\mathcal{G}$ is totally ordered. Let $\Lambda = \cap_{A \in S} A$ and let $\mathcal{H}$ be the decreasing filter admitting for basis the family $\{A \setminus \Lambda; A \in S\}$.

If $\text{diam}(\mathcal{F}) = \text{diam}(\mathcal{G})$, we see that $\mathcal{F} = \mathcal{G}$.

Now let $r = \text{diam}(\mathcal{F})$, let $s = \text{diam}(\mathcal{G})$, and suppose $r < s$. Then $\mathcal{F}$ contains a disk $d(\alpha, \lambda)$ with $r < \lambda < s$. Such a disk is included in all disks $d(\beta, \mu) \in \mathcal{G}$, because $\mu > s$. Hence $\mathcal{F}$ is secant with $\mathcal{Q}(\mathcal{G})$ and therefore $\mathcal{G}$ surrounds $\mathcal{F}$, a contradiction to the hypothesis.

Theorem 2.1. *Let $\mathcal{F}$, $\mathcal{G}$ be circular filters on $\mathbb{K}$. Let $(D_i)_{i \in I}$ be the family of circumferenced disks that belong to both $\mathcal{F}$ and $\mathcal{G}$, and let $\Lambda = \cap_{i \in I} D_i$. Let $\mathcal{H}$ be the decreasing filter admitting for basis the family $\{D_i \setminus \Lambda;\ i \in I\}$ and let $\mathcal{M}$ be the circular filter less thin than $\mathcal{H}$. Then $\mathcal{M} = \sup(\mathcal{F}, \mathcal{G})$ and $\mathcal{W}(\mathcal{M}) = \mathcal{W}(\mathcal{F}) \cap \mathcal{W}(\mathcal{G})$.*

Proof. As the claims are immediate if $\mathcal{F} \preceq \mathcal{G}$, we may suppose that $\mathcal{F}$ and $\mathcal{G}$ are not comparable. By Lemma 2.4 there exist $d(a,\rho) \in \mathcal{F}$, $d(b,\sigma) \in \mathcal{G}$ such that $d(a,\rho) \cap d(b,\sigma) = \emptyset$. Let $t = |a-b|$. Both $\mathcal{F}$, $\mathcal{G}$ are secant with $d(a,t)$. Therefore, the circular filter $\mathcal{N}$ of center a and diameter t surrounds $\mathcal{F}$ and $\mathcal{G}$. We will show that $\mathcal{N} = \inf(\mathcal{W}(\mathcal{F}) \cap \mathcal{W}(\mathcal{G}))$. Indeed, let $\mathcal{E} \in \mathcal{W}(\mathcal{F}) \cap \mathcal{W}(\mathcal{G})$ and let $u = \text{diam}(\mathcal{E})$. Let $l = \max(\rho, \sigma, u)$ and suppose $u < t$. Then we have $l < t$ and $d(a,l) \cap d(b,l) = \emptyset$. Let $\mathcal{L}$ be the circular filter of diameter l, surrounding $\mathcal{F}$. Then $\mathcal{L}$ and $\mathcal{E}$ lie in the wire of $\mathcal{F}$. But since $\text{diam}(\mathcal{L}) \geq \text{diam}(\mathcal{E})$, then $\mathcal{L}$ surrounds $\mathcal{E}$. As a consequence $\mathcal{L} \in \mathcal{W}(\mathcal{G})$. So, $\mathcal{F}$ is secant with $d(a,l)$

and $\mathcal{G}$ is secant with $d(b,l)$. Hence a and b lie in $\mathcal{Q}(\mathcal{L})$, and therefore $|a-b| \leq l$, which contradicts $l < t$. Thus $u \geq t$. As a consequence, $\mathcal{N}$ and $\mathcal{E}$ are two elements of $\mathcal{W}(\mathcal{F})$ such that $\mathrm{diam}(\mathcal{N}) \leq \mathrm{diam}(\mathcal{E})$. Hence $\mathcal{N} \preceq \mathcal{E}$. This proves $\mathcal{N} = \inf(\mathcal{W}(\mathcal{F}) \cap \mathcal{W}(\mathcal{G}))$. Consequently we have $\mathcal{N} = \sup(\mathcal{F}, \mathcal{G})$ and therefore $\mathcal{W}(\mathcal{N}) = \mathcal{W}(\mathcal{F}) \cap \mathcal{W}(\mathcal{G})$.

Finally, as $d(a,\rho) \in \mathcal{F}$, $d(b,\sigma) \in \mathcal{G}$ and $d(a,\rho) \cap d(b,\sigma) = \emptyset$ we check that $\Lambda = d(a,t)$. Then, clearly $\mathcal{N}$ is equal to $\mathcal{M}$.

Notation. For any two circular filters $\mathcal{F}$ and $\mathcal{G}$ on $\mathbb{K}$, we will denote by $\mathcal{M}_{\mathcal{F},\mathcal{G}}$ the circular filter $\sup(\mathcal{F}, \mathcal{G})$ whose existence has been shown in the previous theorem, and by $r_{\mathcal{F},\mathcal{G}}$ its diameter.

Remark 1. If $\mathcal{F} \neq \mathcal{G}$ then $\mathcal{M}_{\mathcal{F},\mathcal{G}}$ owns centers.

Remark 2. Let $\mathcal{F}$ and $\mathcal{G}$ be two circular filters on $\mathbb{K}$ such that $\mathcal{F} \preceq \mathcal{G}$. Then $\mathcal{M}_{\mathcal{F},\mathcal{G}} = \mathcal{G}$.

Lemma 2.5. *Let $\mathcal{F}$ and $\mathcal{G}$ be two circular filters on $\mathbb{K}$, let $\mathcal{H} \in \mathcal{W}(\mathcal{F}) \setminus \mathcal{W}(\mathcal{G})$ and $\mathcal{I} \in \mathcal{W}(\mathcal{G}) \setminus \mathcal{W}(\mathcal{F})$. Then we have $\mathcal{M}_{\mathcal{F},\mathcal{G}} = \mathcal{M}_{\mathcal{H},\mathcal{I}}$.*

Proof. We have $\mathcal{M}_{\mathcal{F},\mathcal{G}} \in \mathcal{W}(\mathcal{F}) \cap \mathcal{W}(\mathcal{G})$. Since $\mathcal{H} \in \mathcal{W}(\mathcal{F}) \setminus \mathcal{W}(\mathcal{G})$ and $\mathcal{I} \in \mathcal{W}(\mathcal{G}) \setminus \mathcal{W}(\mathcal{F})$, then $\mathcal{M}_{\mathcal{F},\mathcal{G}} \in \mathcal{W}(\mathcal{H}) \cap \mathcal{W}(\mathcal{I})$. Suppose that there exists $\mathcal{M}' \in \mathcal{W}(\mathcal{H}) \cap \mathcal{W}(\mathcal{I})$ such that $\mathcal{M}' \preceq \mathcal{M}_{\mathcal{F},\mathcal{G}}$. As $\mathcal{M}' \in \mathcal{W}(\mathcal{H})$, then $\mathcal{M}' \in \mathcal{W}(\mathcal{F})$ and as $\mathcal{M}' \in \mathcal{W}(\mathcal{I})$, then $\mathcal{M}' \in \mathcal{W}(\mathcal{G})$. Hence $\mathcal{M}' \in \mathcal{W}(\mathcal{F}) \cap \mathcal{W}(\mathcal{G})$, and then we have $\mathcal{M}' = \mathcal{M}_{\mathcal{F},\mathcal{G}}$. So $\mathcal{M}_{\mathcal{F},\mathcal{G}}$ is the lower bound of $\mathcal{W}(\mathcal{H}) \cap \mathcal{W}(\mathcal{I})$, hence $\mathcal{M}_{\mathcal{F},\mathcal{G}} = \mathcal{M}_{\mathcal{H},\mathcal{I}}$.

Definition and notation. We are now able to extend δ' to a distance δ defined on all circular filters on $\mathbb{K}$. Let $\mathcal{F}, \mathcal{G}$ be two circular filters on $\mathbb{K}$. We put $\delta(\mathcal{F},\mathcal{G}) = \delta'(\mathcal{F}, \mathcal{M}_{\mathcal{F},\mathcal{G}}) + \delta'(G, \mathcal{M}_{\mathcal{F},\mathcal{G}}) = 2r_{\mathcal{F},\mathcal{G}} - \mathrm{diam}(\mathcal{F}) - \mathrm{diam}(\mathcal{G})$.

Theorem 2.2. *The mapping δ is a distance on the set of circular filters on $\mathbb{K}$, satisfying further $\delta(\mathcal{F},\mathcal{G}) = \delta'(\mathcal{F},\mathcal{G})$ when $\mathcal{F}$ and $\mathcal{G}$ are comparable for $\preceq$.*

Proof. We first notice that if $\mathcal{F} \preceq \mathcal{G}$, then $\delta(\mathcal{F},\mathcal{G}) = 2r_{\mathcal{F},\mathcal{G}} - \mathrm{diam}(\mathcal{F}) - \mathrm{diam}(\mathcal{G})$. But since $\delta'(\mathcal{F},\mathcal{G}) = \mathrm{diam}(\mathcal{G}) - \mathrm{diam}(\mathcal{F})$ and $r_{\mathcal{F},\mathcal{G}} = \mathrm{diam}(\mathcal{G})$, we obviously have $\delta(\mathcal{F},\mathcal{G}) = \delta'(\mathcal{F},\mathcal{G})$.

It is clearly seen that $\delta(\mathcal{F},\mathcal{G}) = 0$ if and only if $\mathcal{F} = \mathcal{G}$ and that $\delta(\mathcal{F},\mathcal{G}) = \delta(\mathcal{G},\mathcal{F})$ for all circular filters $\mathcal{F}$ and $\mathcal{G}$.

We now have to check the triangle inequality. Let $\mathcal{F}$, $\mathcal{G}$, $\mathcal{H}$ be circular filters on $\mathbb{K}$ whose diameters are respectivly λ, μ and ν. It is clearly seen that, if $\mathcal{F}$ and $\mathcal{G}$ are on the same wire, then the inequality is satisfied. Suppose that $\mathcal{F}$ and $\mathcal{G}$ are not on the same wire.

If $\mathcal{H} \in \mathcal{W}(\mathcal{F}) \cap \mathcal{W}(\mathcal{G})$ then $\mathcal{M}_{\mathcal{F},\mathcal{G}} \preceq \mathcal{H}$, hence $r_{\mathcal{F},\mathcal{G}} \leq \nu$. So we have $\delta(\mathcal{F},\mathcal{G}) = 2r_{\mathcal{F},\mathcal{G}} - \lambda - \mu \leq (\nu - \lambda) + (\nu - \mu) = \delta(\mathcal{F},\mathcal{H}) + \delta(\mathcal{H},\mathcal{G})$.

If $\mathcal{H} \in \mathcal{W}(\mathcal{F}) \backslash \mathcal{W}(\mathcal{G})$ then by Lemma 2.5, we have $\mathcal{M}_{\mathcal{F},\mathcal{G}} = \mathcal{M}_{\mathcal{H},\mathcal{G}}$ and then $r_{\mathcal{F},\mathcal{G}} = r_{\mathcal{H},\mathcal{G}}$. Hence $\delta(\mathcal{F},\mathcal{H}) = \nu - \lambda$ and $\delta(\mathcal{G},\mathcal{H}) = 2r_{\mathcal{F},\mathcal{G}} - \nu - \mu$. So we have $\delta(\mathcal{F},\mathcal{G}) = 2r_{\mathcal{F},\mathcal{G}} - \lambda - \mu = \delta(\mathcal{F},\mathcal{H}) + \delta(\mathcal{G},\mathcal{H})$.

If $\mathcal{H} \preceq \mathcal{F}$, then $\nu \leq \lambda$, so $-\lambda \leq -2\nu + \lambda$. Moreover, by Lemma 2.5 we have $\mathcal{M}_{\mathcal{F},\mathcal{G}} = \mathcal{M}_{\mathcal{H},\mathcal{G}}$. So $\delta(\mathcal{F},\mathcal{G}) = 2r_{\mathcal{F},\mathcal{G}} - \lambda - \mu \leq (\lambda - \nu) + 2r_{\mathcal{F},\mathcal{G}} - \nu - \mu = \delta(\mathcal{F},\mathcal{H}) + \delta(\mathcal{G},\mathcal{H})$.

Finally, suppose $\mathcal{H} \notin \mathcal{W}(\mathcal{F}) \cup \mathcal{W}(\mathcal{G})$. Of course $\mathcal{M}_{\mathcal{F},\mathcal{G}}$ and $\mathcal{M}_{\mathcal{F},\mathcal{H}}$ are on the wire of $\mathcal{F}$. Put $\mathcal{E} = \mathcal{M}_{\mathcal{F},\mathcal{H}}$. First suppose $\mathcal{M}_{\mathcal{F},\mathcal{H}} \prec \mathcal{M}_{\mathcal{F},\mathcal{G}}$, then we have $\mathcal{M}_{\mathcal{F},\mathcal{H}} \in \mathcal{W}(\mathcal{F}) \setminus \mathcal{W}(\mathcal{G})$, then by Lemma 2.5 $\mathcal{M}_{\mathcal{E},\mathcal{G}} = \mathcal{M}_{\mathcal{F},\mathcal{G}}$. In the same way, as $\mathcal{M}_{\mathcal{F},\mathcal{H}} \in \mathcal{W}(\mathcal{H}) \setminus \mathcal{W}(\mathcal{G})$, we have $\mathcal{M}_{\mathcal{E},\mathcal{G}} = \mathcal{M}_{\mathcal{H},\mathcal{G}}$, and then $\mathcal{M}_{\mathcal{F},\mathcal{G}} = \mathcal{M}_{\mathcal{H},\mathcal{G}}$. So, we have $\delta(\mathcal{F},\mathcal{G}) = 2r_{\mathcal{F},\mathcal{G}} - \lambda - \mu = 2r_{\mathcal{H},\mathcal{G}} - \lambda - \mu \leq 2r_{\mathcal{H},\mathcal{G}} - \lambda - \mu + 2r_{\mathcal{F},\mathcal{H}} - 2\nu = \delta(\mathcal{F},\mathcal{H}) + \delta(\mathcal{G},\mathcal{H})$ (as $\mathcal{H} \preceq \mathcal{M}_{\mathcal{F},\mathcal{H}}$ we have $2r_{\mathcal{F},\mathcal{H}} - 2\nu \geq 0$). Finally, if $\mathcal{M}_{\mathcal{F},\mathcal{G}} \preceq \mathcal{M}_{\mathcal{F},\mathcal{H}}$, we have $\delta(\mathcal{F},\mathcal{G}) = 2r_{\mathcal{F},\mathcal{G}} - \lambda - \mu \leq 2r_{\mathcal{F},\mathcal{H}} - \lambda - \mu \leq 2r_{\mathcal{F},\mathcal{H}} - \lambda - \mu + 2r_{\mathcal{G},\mathcal{H}} - 2\nu = \delta(\mathcal{F},\mathcal{H}) + \delta(\mathcal{G},\mathcal{H})$ (as $\mathcal{H} \preceq \mathcal{M}_{\mathcal{G},\mathcal{H}}$ we have $2r_{\mathcal{G},\mathcal{H}} - 2\nu \geq 0$). This ends the proof.

Remark. Cauchy circular filters on $\mathbb{K}$ are canonically identified with the points of $\mathbb{K}$. For $a, b \in \mathbb{K}$, let $\mathcal{F}$ and $\mathcal{G}$ be the Cauchy circular filters whose centers are respectively a and b. We have $\delta(\mathcal{F},\mathcal{G}) = 2|a - b|$. Thus the usual distance on $\mathbb{K}$, defined by the absolute value and the restriction of δ to $\mathbb{K}$, are equivalent on $\mathbb{K}$.

3 Topologies on Mult($\mathbb{K}[X]$)

Notation and definitions. We will denote by Φ the mapping from the set of circular filters on $\mathbb{K}$ onto Mult($\mathbb{K}[X]$), defined as $\Phi(\mathcal{F}) = \varphi_{\mathcal{F}}$ where $\varphi_{\mathcal{F}}$ is the multiplicative semi-norm on $\mathbb{K}[X]$ defined by $\varphi_{\mathcal{F}}(h) = \lim_{\mathcal{F}} |h(x)|, \quad \forall h \in \mathbb{K}[X]$. We know that Φ is a bijection, [9] and [10].

This bijection allows us to consider an order relation and a distance on Mult($\mathbb{K}[X]$), also respectively denoted by $\preceq$ and δ, and defined in a natural way by $\varphi_{\mathcal{F}} \preceq \varphi_{\mathcal{G}}$ if and only if $\mathcal{F} \preceq \mathcal{G}$ and by $\delta(\varphi_{\mathcal{F}}, \varphi_{\mathcal{G}}) = \delta(\mathcal{F},\mathcal{G})$. So, we may consider Mult($\mathbb{K}[X]$) as a metric space.

We will denote by $\mathcal{S}$ the topology of simple convergence on $\mathrm{Mult}(\mathbb{K}[X])$ and by $\mathfrak{T}_\delta$ the metric topology defined by δ.

Given $\psi \in \mathrm{Mult}(\mathbb{K}[X])$, $h \in \mathbb{K}[X]$, $\varepsilon > 0$, we denote by $V(\psi, h, \varepsilon)$ the set of the $\varphi \in \mathrm{Mult}(\mathbb{K}[X])$ such that $|\varphi(h) - \psi(h)|_\infty < \varepsilon$.

Remark. We obtain a basis of neighbourhoods for the topology $\mathcal{S}$ of any $\psi \in \mathrm{Mult}(\mathbb{K}[X])$ by taking the sets of the form $\bigcap_{j=1}^{q} V(\psi, h_j, \varepsilon_j)$, $q \in \mathbb{N}^*$.

Proposition 3.1. *On* $\mathrm{Mult}(\mathbb{K}[X])$, *the topology* $\mathfrak{T}_\delta$ *is strictly thinner than the topology* $\mathcal{S}$.

Proof. For $h \in \mathbb{K}[X]$, let ξ_h be the mapping from $\mathrm{Mult}(\mathbb{K}[X])$ onto $\mathbb{R}$ such that $\xi_h(\varphi_{\mathcal{F}}) = \varphi_{\mathcal{F}}(h) = \lim_{\mathcal{F}} |h(x)|$. It is known that $\mathcal{S}$ is the least thin topology on $\mathrm{Mult}(\mathbb{K}[X])$ such that ξ_h is continuous for all $h \in \mathbb{K}[X]$. So, by proving that ξ_h is continuous for $\mathfrak{T}_\delta$, we will show that $\mathfrak{T}_\delta$ is thinner than $\mathcal{S}$.

We denote by $B(\varphi_{\mathcal{F}}, \beta)$ the open ball in $\mathrm{Mult}(\mathbb{K}[X])$ of center $\varphi_{\mathcal{F}}$ and radius β with respect to the distance δ. Given $\varepsilon > 0$, by definition of $\varphi_{\mathcal{F}}(h)$, there exists an element $A \subset \mathbb{K}$ of the canonical basis of $\mathcal{F}$ such that

$$|\varphi_{\mathcal{F}}(h) - |h(x)||_\infty < \varepsilon, \ \forall x \in A. \tag{1}$$

If $\mathcal{F}$ is large and admits a center (resp. $\mathcal{F}$ has no center or $\mathcal{F}$ is a Cauchy circular filter), A is of the form $\cap_{i \in I}\Gamma(a_i, r_i, r)$ (resp. $d(a, r)$) with $r > \mathrm{diam}(\mathcal{F})$ and $|a_i - a_j| = \mathrm{diam}(\mathcal{F})$ if $i \neq j$ (resp. $r > \mathrm{diam}(\mathcal{F})$).

Let $\lambda = \sup_{i \in I}(r_i)$, $\alpha = \inf(r - \mathrm{diam}(\mathcal{F}), \mathrm{diam}(\mathcal{F}) - \lambda)$ (resp. $\alpha = r - \mathrm{diam}(\mathcal{F})$). For all $\psi \in B(\varphi_{\mathcal{F}}, \alpha)$, the circular filter on $\mathbb{K}$ associated to ψ is secant with A. Hence by (1), we have $|\psi(h) - \varphi_{\mathcal{F}}(h)|_\infty < \varepsilon$. As $|\xi_h(\psi) - \xi_h(\varphi_{\mathcal{F}})|_\infty = |\psi(h) - \varphi_{\mathcal{F}}(h)|_\infty$, for all $\psi \in B(\varphi_{\mathcal{F}}, \alpha)$, we have $|\xi_h(\psi) - \xi_h(\varphi_{\mathcal{F}})|_\infty < \varepsilon$. Hence ξ_h is continuous for $\mathfrak{T}_\delta$ and so, $\mathfrak{T}_\delta$ is thinner than $\mathcal{S}$. Now, it rests to show that $\mathcal{S}$ is not thinner than $\mathfrak{T}_\delta$.

For this, let $\mathcal{F}$ be a large circular filter on $\mathbb{K}$ of center $a \in \mathbb{K}$ and let $\beta \in]0, \mathrm{diam}(\mathcal{F})[$. Now, the filter of neighbourhoods of $\varphi_{\mathcal{F}}$, with respect to $\mathcal{S}$, admits a basis of the form $\cap_{j=1}^{q} V(\varphi_{\mathcal{F}}, h_j, \varepsilon_j)$ with $q \in \mathbb{N}^*$, $h_j \in \mathbb{K}[X]$. In particular, we consider such a neighbourhood $W = \cap_{j=1}^{q} V(\varphi_{\mathcal{F}}, h_j, \varepsilon_j)$. We put $\varepsilon = \inf_{j=1,..,q}(\varepsilon_j)$. For any $j \in \{1, ..., q\}$, there exists an element A_j of $\mathcal{F}$ such that $|\varphi_{\mathcal{F}}(h_j) - |h_j(x)||_\infty < \varepsilon$, $\forall x \in A_j$. We put $A = \cap_{j=1}^{q} A_j$ and then, we have $\forall j \in \{1, ..., q\}$, $\forall x \in A$, $|\varphi_{\mathcal{F}}(h_j) - |h_j(x)||_\infty < \varepsilon$. Of course A is not empty because $\mathcal{G}$ is a filter. Let $\mathcal{G}$ be a circular filter on $\mathbb{K}$ of center $b \in d(a, \mathrm{diam}(\mathcal{F})) \cap A$ and of diameter $\gamma \in]0, \mathrm{diam}(\mathcal{F}) - \beta[$ (which is obviously secant with A). Such a circular filter exists because A is open. We have $|\varphi_{\mathcal{F}}(h_j) - \varphi_{\mathcal{G}}(h_j)|_\infty < \epsilon$, $\forall j \in \{1, ..., q\}$. Then $\varphi_{\mathcal{G}} \in W$. But we clearly have $\delta(\varphi_{\mathcal{F}}, \varphi_{\mathcal{G}}) = \mathrm{diam}(\mathcal{F}) - \gamma > \beta$. Hence $\varphi_{\mathcal{G}} \notin B(\varphi_{\mathcal{F}}, \beta)$. And then, $B(\varphi_{\mathcal{F}}, \beta)$ may not be a neighbourhood of $\varphi_{\mathcal{F}}$ with respect to the topology $\mathcal{S}$. In particular, $B(\varphi_{\mathcal{F}}, \beta)$ does not contain images by Φ of Cauchy filters on $\mathbb{K}$ i.e. it only contains absolute values on $\mathbb{K}[X]$, [9]. This ends the proof.

Definitions. Given $\mathcal{F}$ and $\mathcal{G}$ two circular filters on $\mathbb{K}$ such that $\mathcal{F} \preceq \mathcal{G}$, we call *segment* $[\varphi_{\mathcal{F}}, \varphi_{\mathcal{G}}]$ of $\mathrm{Mult}(\mathbb{K}[X])$ the image by Φ of the interval $[\mathcal{F}, \mathcal{G}]$, i.e. $[\varphi_{\mathcal{F}}, \varphi_{\mathcal{G}}] = \{\varphi_{\mathcal{H}} \in \mathrm{Mult}(\mathbb{K}[X]) \mid \varphi_{\mathcal{F}} \preceq \varphi_{\mathcal{H}} \preceq \varphi_{\mathcal{G}}\}$.

A continuous function γ from an interval $[a, b]$ of $\mathbb{R}$ into a topological space E is called *a path* of E. A subset S of a Hausdorff topological space E is said to be arc-connected if for every $A, B \in S$, there exists a path γ from $[0, 1]$ into S such that $\gamma(0) = A$ and $\gamma(1) = B$.

Proposition 3.2. *Every segment of* $\mathrm{Mult}(\mathbb{K}[X])$ *is an arc-connected set with respect to the topology* $\mathfrak{T}_\delta$.

Proof. Given $\mathcal{F}$ and $\mathcal{G}$ two circular filters on $\mathbb{K}$ such that $\mathcal{F} \preceq \mathcal{G}$, we respectively denote by λ and μ their diameters and we consider the segment $[\varphi_{\mathcal{F}}, \varphi_{\mathcal{G}}]$ of $\mathrm{Mult}(\mathbb{K}[X])$.

For every $t \in [\lambda, \mu]$, we denote by $\mathcal{F}_t$ the circular filter in $\mathcal{W}(\mathcal{F})$ of diameter t, so $\mathcal{F}_t \in [\mathcal{F}, \mathcal{G}]$. Let f be the path on $\mathrm{Mult}(\mathbb{K}[X])$ defined from $[\lambda, \mu]$ into $\mathrm{Mult}(\mathbb{K}[X])$ by $f(t) = \varphi_{\mathcal{F}_t}$. Given $\varepsilon > 0$ and $t_0 \in [\lambda, \mu]$, for all $t \in [\lambda, \mu]$ such that $|t - t_0|_\infty < \varepsilon$, we have $\delta(\varphi_{\mathcal{F}_{t_0}}, \varphi_{\mathcal{F}_t}) < \varepsilon$. Hence, the path f is continuous with respect to the topology $\mathfrak{T}_\delta$ on $\mathrm{Mult}(\mathbb{K}[X])$ and this ends the proof.

Theorem 3.1. $\mathrm{Mult}(\mathbb{K}[X])$ *is an arc-connected space with respect to the topology* $\mathfrak{T}_\delta$.

Proof. Let $\varphi_{\mathcal{F}}$ and $\varphi_{\mathcal{G}}$ be two elements of $\mathrm{Mult}(\mathbb{K}[X])$ associated to the circular filters $\mathcal{F}$ and $\mathcal{G}$. By Proposition 3.2, both segments $[\varphi_{\mathcal{F}}, \varphi_{\mathcal{M}_{\mathcal{F},\mathcal{G}}}]$ and $[\varphi_{\mathcal{G}}, \varphi_{\mathcal{M}_{\mathcal{F},\mathcal{G}}}]$ are arc-connected. Hence there exists a path f from $[0, 1]$ into $\mathrm{Mult}(\mathbb{K}[X])$ such that $f(0) = \mathcal{F}$, $f(1) = \mathcal{G}$ and $f(\frac{1}{2}) = \mathcal{M}_{\mathcal{F},\mathcal{G}}$.

Corollary 3.1. $\text{Mult}(\mathbb{K}[X])$ *is an arc-connected space with respect to the topology* $\mathcal{S}$.

Definitions and Notation. We denote by Φ^* the restriction of Φ to the set of large circular filters on $\mathbb{K}$. Then, given a large circular filter $\mathcal{F}$ on $\mathbb{K}$, we may extend $\Phi^*(\mathcal{F}) = \varphi_{\mathcal{F}}$ to $\mathbb{K}(X)$. The mapping Φ^* is a bijection from the set of large circular filters on $\mathbb{K}$ onto $\text{Mult}(\mathbb{K}(X))$, [9]. This bijection allows us to define the distance δ on $\text{Mult}(\mathbb{K}(X))$ by putting again $\delta(\varphi_{\mathcal{F}}, \varphi_{\mathcal{G}}) = \delta(\mathcal{F}, \mathcal{G})$, for all large circular filters $\mathcal{F}$ and $\mathcal{G}$ on $\mathbb{K}$. We also denote by $\mathcal{S}$ the topology of simple convergence on $\text{Mult}(\mathbb{K}(X))$ and by $\mathfrak{T}_\delta$ the metric one associated to the distance δ.

The same proof of the one of Proposition 3.1 holds on $\text{Mult}(\mathbb{K}(X))$, then we have the following proposition.

Proposition 3.3. *On* $\text{Mult}(\mathbb{K}(X))$, $\mathfrak{T}_\delta$ *is strictly thinner than* $\mathcal{S}$.

Theorem 3.2. $\text{Mult}(\mathbb{K}(X))$ *is an arc-connected space with respect to both topologies.*

Proof. Let $\varphi_{\mathcal{F}}$, $\varphi_{\mathcal{G}} \in \text{Mult}(\mathbb{K}(X))$. Then $\mathcal{F}$, $\mathcal{G}$ are large circular filter on $\mathbb{K}$ and so is each element of $[\mathcal{F}, \mathcal{M}_{\mathcal{F},\mathcal{G}}]$ (resp. $[\mathcal{G}, \mathcal{M}_{\mathcal{F},\mathcal{G}}]$). Put $\mathcal{E} = \mathcal{M}_{\mathcal{F},\mathcal{G}}$. Therefore $[\varphi_{\mathcal{F}}, \varphi_{\mathcal{E}}]$ (resp. $[\varphi_{\mathcal{G}}, \varphi_{\mathcal{E}}]$) is included in $\text{Mult}(\mathbb{K}(X))$, so the conclusion comes from Theorem 3.1 and Corollary 3.1.

4 Topologies on $\text{Mult}(H(D), \mathcal{U}_D)$.

Remark. If two circular filters $\mathcal{F}$, $\mathcal{G}$ on $\mathbb{K}$ are secant with a set D and satisfy ${}_D\mathcal{F} = {}_D\mathcal{G}$ then $\mathcal{F} = \mathcal{G}$ because $\mathcal{F}$ and $\mathcal{G}$ are secant.

Definitions and notation. Let $D \subset \mathbb{K}$ and let $\mathcal{F}$ be a large circular filter on $\mathbb{K}$ secant with D. We denote by ${}_D\mathcal{F}$ the filter $\mathcal{F} \cap D$ which is called *circular filter on* D. The filter of neighbourhoods, in D, of a point $a \in D$ is also called *circular filter on* D. This filter is the filter $\mathcal{F}_a \cap D$ that we also call *Cauchy circular filter on* D, [7] and [9]. The set of circular filters on D will be denoted by $\Theta(D)$.

Remark. Let $a \in \overline{D} \setminus D$. The Cauchy filter $\mathcal{F}_a$ is secant with D but it is not a circular filter on D. If D is closed, then each circular filter on $\mathbb{K}$ secant with D, large or not, induces on D a circular filter on D, [7] and [9].

By properties of the intersection, we may obviously define on $\Theta(D)$ a partial order relation, also denoted by $\preceq$ i.e.: ${}_D\mathcal{F} \preceq {}_D\mathcal{G}$ if $\mathcal{F} \preceq \mathcal{G}$. In the same way, we may also define a distance on $\Theta(D)$, denoted by δ again, as $\delta({}_D\mathcal{F}, {}_D\mathcal{G}) = \delta(\mathcal{F}, \mathcal{G})$.

Lemma 4.1. *Let D be an infraconnected subset of $\mathbb{K}$ and let $\mathcal{F}$ and $\mathcal{G}$ be two circular filters on $\mathbb{K}$ secant with D such that $\mathcal{F} \preceq \mathcal{G}$. Then for all $\mathcal{H} \in [\mathcal{F}, \mathcal{G}]$, $\mathcal{H}$ is secant with D.*

Proof. Let $\mathcal{H} \in [\mathcal{F}, \mathcal{G}]$ and $\lambda = \text{diam}(\mathcal{H})$. Since $\lambda \in [\text{diam}(\mathcal{F}), \text{diam}(\mathcal{G})]$, by Lemma 41.2 of [7] there exists a unique circular filter ${}_D\mathcal{X}$ on D of diameter λ satisfying ${}_D\mathcal{F} \preceq {}_D\mathcal{X}$. But by Lemma 2.1, $\mathcal{H}$ is the unique circular filter of diameter λ surrounding $\mathcal{F}$. So, we have $\mathcal{H} = \mathcal{X}$, hence $\mathcal{H}$ is secant with D.

Lemma 4.2. *Let D be an infraconnected subset of $\mathbb{K}$ and let $\mathcal{F}$ and $\mathcal{G}$ be two circular filters on $\mathbb{K}$ secant with D. Then $\mathcal{M}_{\mathcal{F},\mathcal{G}}$ is secant with D.*

Proof. If $\mathcal{F} \preceq \mathcal{G}$ or $\mathcal{G} \preceq \mathcal{F}$, Lemma 4.2 is obvious by Remark 2 of section 2. Else, by Lemma 2.4 there exist disks $d(a,r) \in \mathcal{F}$ and $d(b,s) \in \mathcal{G}$ such that $d(a,r) \cap d(b,r') = \emptyset$. Since $\mathcal{F}$ and $\mathcal{G}$ are secant with D, without loss of generality we may suppose $a, b \in D$. Let $\mathcal{H}$ be the circular filter of center a and diameter $|a-b|$. Since D is infraconnected, by Proposition 3.14 [7], $\mathcal{H}$ is secant with D and then we have $\mathcal{F} \preceq \mathcal{H}$ and $\mathcal{G} \preceq \mathcal{H}$, so $\mathcal{M}_{\mathcal{F},\mathcal{G}} \preceq \mathcal{H}$. Hence, by Lemma 4.1, $\mathcal{M}_{\mathcal{F},\mathcal{G}}$ is secant with D.

Definitions and notation. Let $D \subset \mathbb{K}$. Circular filters on D are known to characterize the elements of $\text{Mult}(H(D), \mathcal{U}_D)$ in the following way. To each circular filter ${}_D\mathcal{F}$ on D, we can associate an element ${}_D\varphi_{\mathcal{F}}$ of $\text{Mult}(H(D), \mathcal{U}_D)$ such that $\forall f \in H(D)$, ${}_D\varphi_{\mathcal{F}}(f) = \lim_{{}_D\mathcal{F}} |f(x)|$. The mapping ${}_D\Phi \ : \ {}_D\mathcal{F} \mapsto {}_D\varphi_{\mathcal{F}}$ is a bijection from $\Theta(D)$ onto $\text{Mult}(H(D), \mathcal{U}_D)$ (Theorem 4.14 [7]).

Then, as in $\text{Mult}(\mathbb{K}[X])$, this bijection defines an order relation and a distance on $\text{Mult}(H(D), \mathcal{U}_D)$, also respectively denoted by $\preceq$ and δ; they are defined in a natural way as: ${}_D\varphi_{\mathcal{F}} \preceq {}_D\varphi_{\mathcal{G}}$ if ${}_D\mathcal{F} \preceq {}_D\mathcal{G}$ and $\delta({}_D\varphi_{\mathcal{F}}, {}_D\varphi_{\mathcal{G}}) = \delta({}_D\mathcal{F}, {}_D\mathcal{G})$. Given two circular filters ${}_D\mathcal{F}$ and ${}_D\mathcal{G}$ on D, we define in a natural way the *segment* $[{}_D\varphi_{\mathcal{F}}, {}_D\varphi_{\mathcal{G}}]$ of $\text{Mult}(H(D), \mathcal{U}_D)$ as $[{}_D\varphi_{\mathcal{F}}, {}_D\varphi_{\mathcal{G}}] = \{{}_D\varphi_{\mathcal{H}} \in \text{Mult}(H(D), \mathcal{U}_D) \mid {}_D\varphi_{\mathcal{F}} \preceq {}_D\varphi_{\mathcal{H}} \preceq {}_D\varphi_{\mathcal{G}}\}$.

As we did on Mult($\mathbb{K}[X]$), we will denote by $\mathcal{S}$ the topology of simple convergence on Mult($H(D),\mathcal{U}_D$) and by $\mathfrak{T}_\delta$ the metric one (defined by δ).

Proposition 4.1. *On* Mult($H(D),\mathcal{U}_D$)*, the topology* $\mathfrak{T}_\delta$ *is thinner than the topology* $\mathcal{S}$.

Proof. The proof is similar to this of Proposition 3.1. For $h \in H(D)$, let ξ_h be the mapping from Mult($H(D),\mathcal{U}_D$) onto $\mathbb{R}$ such that $\xi_h({}_D\varphi_{\mathcal{F}}) = {}_D\varphi_{\mathcal{F}}(h) = \lim_{{}_D\mathcal{F}}|h(x)|$. It is known that $\mathcal{S}$ is the least thin topology on Mult($H(D),\mathcal{U}_D$) such that ξ_h is continuous for all $h \in H(D)$. So, by proving that ξ_h is continuous for $\mathfrak{T}_\delta$, we will show that $\mathfrak{T}_\delta$ is thinner than $\mathcal{S}$.

We denote by $B({}_D\varphi_{\mathcal{F}}, \beta)$ the open ball in Mult($H(D),\mathcal{U}_D$) of center ${}_D\varphi_{\mathcal{F}}$ and radius β with respect to the distance δ. Given $\varepsilon > 0$, by definition of ${}_D\varphi_{\mathcal{F}}(h)$, there exists an element $A \subset \mathbb{K}$ of a canonical basis of $\mathcal{F}$ such that

$$(1) \qquad |\varphi_{\mathcal{F}}(h) - |h(x)|\,|_\infty < \varepsilon,\ \forall x \in A \cap D.$$

If $\mathcal{F}$ is large and admits a center (resp. $\mathcal{F}$ has no center or $\mathcal{F}$ is a Cauchy circular filter), A is of the form $\cap_{i\in I}\Gamma(a_i, r_i, r)$ (resp. $d(a,r)$) with $r > \text{diam}(\mathcal{F})$ and $|a_i - a_j| = \text{diam}(\mathcal{F})$ if $i \neq j$ (resp. $r > \text{diam}(\mathcal{F})$).

Let $\lambda = \sup_{i\in I}(r_i)$, $\alpha = \inf(r - \text{diam}(\mathcal{F}), \text{diam}(\mathcal{F}) - \lambda)$ (resp. $\alpha = r - \text{diam}(\mathcal{F})$). For ${}_D\varphi_{\mathcal{G}} \in B({}_D\varphi_{\mathcal{F}}, \alpha)$, the circular filter ${}_D\mathcal{G}$ is secant with $A \cap D$. Hence by (1), we have $|{}_D\varphi_{\mathcal{G}}(h) - {}_D\varphi_{\mathcal{F}}(h)\,|_\infty < \varepsilon$. As $|\xi_h({}_D\varphi_{\mathcal{G}}) - \xi_h({}_D\varphi_{\mathcal{F}})\,|_\infty = |{}_D\varphi_{\mathcal{G}}(h) - {}_D\varphi_{\mathcal{F}}(h)\,|_\infty$, for all ${}_D\varphi_{\mathcal{G}} \in B({}_D\varphi_{\mathcal{F}}, \alpha)$, we have $|\xi_h({}_D\varphi_{\mathcal{G}}) - \xi_h({}_D\varphi_{\mathcal{F}})\,|_\infty < \varepsilon$. Hence ξ_h is continuous for $\mathfrak{T}_\delta$ and so, $\mathfrak{T}_\delta$ is thinner than $\mathcal{S}$.

Remark. Take care that, here, topologies $\mathcal{S}$ and $\mathfrak{T}_\delta$ may be equivalent in certain particular cases. See explanations and examples in Chapter IV.

Notation and definitions. As for Mult($\mathbb{K}[X]$), given ${}_D\varphi_{\mathcal{F}} \in$ Mult($H(D),\mathcal{U}_D$), $f \in H(D)$, $\varepsilon > 0$ we will denote by $V({}_D\varphi_{\mathcal{F}}, f, \varepsilon)$ the set of the ${}_D\varphi_{\mathcal{G}} \in$ Mult($H(D),\mathcal{U}_D$) such that $|{}_D\varphi_{\mathcal{F}}(f) - {}_D\varphi_{\mathcal{G}}(f)\,|_\infty < \varepsilon$. So, we have a basis of neighbourhoods of any ${}_D\varphi_{\mathcal{F}} \in$ Mult($H(D),\mathcal{U}_D$)

for the topology $\mathcal{S}$ by taking the sets of the form $\bigcap_{j=1}^{q} V({}_D\varphi_{\mathcal{F}}, f_j, \varepsilon_j)$, $q \in \mathbb{N}^*$ that we call *canonical neighbourhoods of* ${}_D\varphi_{\mathcal{F}}$.

We will denote by $W({}_D\varphi_{\mathcal{F}}, f, \varepsilon)$ the set of the ${}_D\varphi_{\mathcal{G}} \in \text{Mult}(H(D), \mathcal{U}_D)$ such that $|{}_D\varphi_{\mathcal{F}}(f) - {}_D\varphi_{\mathcal{G}}(f)|_\infty \leq \varepsilon$. Thus, we also have a basis of neighbourhoods of ${}_D\varphi_{\mathcal{F}}$ with respect to $\mathcal{S}$ by taking the sets of the form $\bigcap_{j=1}^{q} W({}_D\varphi_{\mathcal{F}}, f_j, \varepsilon_j)$, $q \in \mathbb{N}^*$.

Proposition 4.2. *Let D be infraconnected. Then every segment in* $\text{Mult}(H(D), \mathcal{U}_D)$ *is arc-connected with respect to both topologies.*

Proof. Let ${}_D\varphi_{\mathcal{F}}$, ${}_D\varphi_{\mathcal{G}} \in \text{Mult}(H(D), \mathcal{U}_D)$ such that ${}_D\varphi_{\mathcal{F}} \preceq {}_D\varphi_{\mathcal{G}}$. As $\mathcal{F}$ and G are secant with D, by Lemma 4.1, every circular filter of $[\mathcal{F}, \mathcal{G}]$ is secant with D. Further, as every circular filter $\mathcal{H} \in [\mathcal{F}, \mathcal{G}]$ such that $\mathcal{F} \prec \mathcal{H}$ is large, we see that every circular filter in $[\mathcal{F}, \mathcal{G}]$ induces a circular filter on D. Hence we may consider the segment $[\varphi_{\mathcal{F}}, \varphi_{\mathcal{G}}]$ in $\text{Mult}(\mathbb{K}[X])$ as a subset of $\text{Mult}(H(D), \mathcal{U}_D)$. Then, by Theorem 3.1, $[\varphi_{\mathcal{F}}, \varphi_{\mathcal{G}}]$ is arc-connected with respect to $\mathfrak{T}_\delta$ and therefore by Proposition 4.1, it is arc-connected with respect to $\mathcal{S}$.

Definitions. An element $u \in H(D)$ will be called *idempotent* if $u(x) = 0$ or $u(x) = 1$ for every $x \in D$. (This definition holds even when $D \notin \mathcal{A}$).

An idempotent u is said to be *trivial* if $u = 0$ or $u = 1$.

Now we can prove the following theorem.

Theorem 4.1. *Given $D \subset \mathbb{K}$, the following properties are equivalent:*

i) There does not exist non-trivial idempotents on $H(D)$.

ii) D is infraconnected.

iii) $\text{Mult}(H(D), \mathcal{U}_D)$ *is arc-connected with respect to the topology $\mathcal{S}$.*

iv) $\text{Mult}(H(D), \mathcal{U}_D)$ *is connected with respect to the topology $\mathcal{S}$.*

Proof. Since it is known that *i)* $\Leftrightarrow$ *ii)* ([5] and [7]) and since trivially *iii)* $\Rightarrow$ *iv)*, we only have to prove that *ii)* $\Rightarrow$ *iii)* and that *iv)* $\Rightarrow$ *i)*.

We first show that *ii)* $\Rightarrow$ *iii)*. The proof is similar to this of Proposition 3.2. Let ${}_D\varphi_{\mathcal{F}}$, ${}_D\varphi_{\mathcal{G}} \in \text{Mult}(H(D), \mathcal{U}_D)$. Then $\mathcal{F}$ and $\mathcal{G}$ are two circular filters on $\mathbb{K}$ secant with D. By Proposition 4.2, the circular filter $\mathcal{M}_{\mathcal{F},\mathcal{G}}$ is secant with D. Hence by Proposition 4.2, both $[{}_D\varphi_{\mathcal{F}}, {}_D\varphi_{\mathcal{M}_{\mathcal{F},\mathcal{G}}}]$ and $[{}_D\varphi_{\mathcal{G}}, {}_D\varphi_{\mathcal{M}_{\mathcal{F},\mathcal{G}}}]$ are arc-connected subsets of $\mathrm{Mult}(H(D),\mathcal{U}_D)$ with respect to $\mathcal{S}$. Hence, we may obviously construct a continuous path f from $[0,1]$ into $\mathrm{Mult}(H(D),\mathcal{U}_D)$, provided with $\mathcal{S}$, such that $f(0) = {}_D\varphi_{\mathcal{F}}$ and $f(1) = {}_D\varphi_{\mathcal{G}}$.

Now we prove that $iv) \Rightarrow i)$. Suppose that there exists a non-trivial idempotent f in $H(D)$. Then, for all ${}_D\varphi_{\mathcal{F}} \in \mathrm{Mult}(H(D),\mathcal{U}_D)$, we have either ${}_D\varphi_{\mathcal{F}}(f) = 0$ or ${}_D\varphi_{\mathcal{F}}(f) = 1$. Let A and B be the subsets of $\mathrm{Mult}(H(D),\mathcal{U}_D)$ defined as $A = \{{}_D\varphi_{\mathcal{F}} \in \mathrm{Mult}(H(D),\mathcal{U}_D) \mid {}_D\varphi_{\mathcal{F}}(f) = 0\}$ and $B = \{{}_D\varphi_{\mathcal{F}} \in \mathrm{Mult}(H(D),\mathcal{U}_D) \mid {}_D\varphi_{\mathcal{F}}(f) = 1\}$. We have $A \cup B = \mathrm{Mult}(H(D),\mathcal{U}_D)$. Both A and B are not empty because for $a \in D$ such that $f(a) = 0$, we have ${}_D\Phi(\mathcal{F}_a) \in A$, and for $b \in D$ such that $f(b) = 1$, we have ${}_D\Phi(\mathcal{F}_b) \in B$. So, we just have to check that A is closed. Let ${}_D\varphi_{\mathcal{F}} \in \overline{A}$, and let ${}_D\varphi_{\mathcal{G}} \in V({}_D\varphi_{\mathcal{F}}, f, \frac{1}{2}) \cap A$. Then $|{}_D\varphi_{\mathcal{F}}(f) - {}_D\varphi_{\mathcal{G}}(f)|_\infty = |{}_D\varphi_{\mathcal{G}}(f)|_\infty \le \frac{1}{2}$ and therefore ${}_D\varphi_{\mathcal{F}}(f) = 0$. In the same way, so is B. This ends the proof.

Remark 1. In general, in [10] B. Guennebaud proved that given a $\mathbb{K}$-Banach algebra, then $\mathrm{Mult}(A, \|\cdot\|)$ is connected if and only if A has no non trivial idempotents. Here we get a link between this property, arc-connectedness, and infraconnected sets.

Remark 2. According to [2], given an affinoid $\mathbb{K}$-algebra A, if $\mathrm{Mult}(A, \|\cdot\|)$ is connected then it is arc-connected.

According to [7, Th. 12.1], every element of $\mathrm{Mult}(R(D),\mathcal{U}_D)$ uniquely extends to $H(D)$ to an element of $\mathrm{Mult}(H(D),\mathcal{U}_D)$. Conversely, every element of $\mathrm{Mult}(H(D),\mathcal{U}_D)$ defines by restriction to $R(D)$, an element of $\mathrm{Mult}(R(D),\mathcal{U}_D)$. Hence, since $R(D)$ is dense in $H(D)$ with respect to $\mathcal{U}_D$, we clearly see that $\mathrm{Mult}(H(D),\mathcal{U}_D)$ and $\mathrm{Mult}(R(D),\mathcal{U}_D)$ are homeomorphic with respect to the topology $\mathcal{S}$. So, we have the following theorem.

Theorem 4.2. *Given $D \subset \mathbb{K}$, the following properties are equivalent:*

i) D is infraconnected.

ii) $\mathrm{Mult}(R(D),\mathcal{U}_D)$ is arc-connected with respect to the topology $\mathcal{S}$.

iii) $\mathrm{Mult}(R(D),\mathcal{U}_D)$ is connected with respect to the topology $\mathcal{S}$.

Notation. As noticed in the Remark following Theorem 2.2, there exists a natural injection from $\mathbb{K}$ into Mult($\mathbb{K}[X]$) that, to each point $a \in \mathbb{K}$, associates φ_a. In the same way, there exists a natural injection Ψ from D into Mult($H(D),\mathcal{U}_D$) that, to each point $a \in D$, associates ${}_D\varphi_a$. So, every subset A of D may be considered as a subset of Mult($H(D),\mathcal{U}_D$) and we denote by $\underline{A}$ the closure of A in Mult($H(D),\mathcal{U}_D$) with respect to $\mathcal{S}$.

If A is a subset of $\mathbb{K}$, we denote by U_A the set of the $\varphi_{\mathcal{F}} \in$ Mult($\mathbb{K}[X]$) such that the associated circular filter $\mathcal{F}$ on $\mathbb{K}$ is secant with A.

In the same way, if A is a subset of D, we denote by ${}_DU_A$ the set of the ${}_D\varphi_{\mathcal{F}} \in$ Mult($H(D),\mathcal{U}_D$) such that ${}_D\mathcal{F}$ is secant with A.

Remark. Given two subsets A and B of D, ${}_DU_{A\cap B}$ is included in ${}_DU_A \cap {}_DU_B$.

Proposition 4.3. *Let $D \subset \mathbb{K}$. For every subset A of D, we have* $\underline{A} = {}_DU_A$.

Proof. We first show that ${}_DU_A \subset \underline{A}$. Let ${}_D\varphi_{\mathcal{F}} \in {}_DU_A$. As $\mathcal{F}$ is secant with A, there exists a sequence $(x_n)_{n\in\mathbb{N}}$ in A thinner than $\mathcal{F}$. Then, for all $f \in H(D)$, we have ${}_D\varphi_{\mathcal{F}}(f) = \lim_{n\to\infty} |f(x_n)| = \lim_{n\to\infty} {}_D\varphi_{x_n}(f)$. Hence the sequence $({}_D\varphi_{x_n})_{n\in\mathbb{N}}$ converges in Mult($H(D),\mathcal{U}_D$) to ${}_D\varphi_{\mathcal{F}}$ with respect to $\mathcal{S}$. Since for all $n \in \mathbb{N}$, ${}_D\varphi_{x_n}$ lies in A, then ${}_D\varphi_{\mathcal{F}}$ lies in $\underline{A}$.

Now, we will show that $\underline{A} \subset {}_DU_A$. Let ${}_D\varphi_{\mathcal{F}} \in \underline{A}$ and suppose that $\mathcal{F}$ is not secant with A.

If $\mathcal{F}$ has no center, we denote by $(D_n)_{n\in\mathbb{N}} = d(a_n, r_n))_{n\in\mathbb{N}}$ a canonical basis of $\mathcal{F}$. So, there exists a disk D_i in this basis such that $A \cap D_i = \emptyset$. Hence, for all $c \in A$, we have $|c - a_{i+1}| > r_i > r_{i+1}$ and therefore $|\varphi_{\mathcal{F}}(x - a_{i+1}) - \varphi_{\mathcal{F}_c}(x - a_{i+1})|_\infty = |r_{i+1} - |c - a_{i+1}||_\infty > |r_i - r_{i+1}|_\infty$. Hence, we have $V({}_D\varphi_{\mathcal{F}}, x - a_{i+1}, r_i - r_{i+1}) \cap A = \emptyset$ and therefore ${}_D\varphi_{\mathcal{F}} \notin \underline{A}$, which contradicts the hypothesis.

If $\mathcal{F}$ has a center and is large, then, there exists an infraconnected affinoid B, element of the canonical basis of $\mathcal{F}$, whose holes are denoted by $T_i = d(a_i, r_i^-)$, $i = 1, ..., n$, such that $|a_i - a_j| = \text{diam}(\mathcal{F})$ for $i \neq j$ and $B \cap A = \emptyset$. Since $r_i < \text{diam}(\mathcal{F}) < \text{diam}(B)$ for all $i = 1, ..., n$, there exists $\varepsilon > 0$ such that $\varepsilon < \text{diam}(B) - \text{diam}(\mathcal{F})$ and $\varepsilon < \inf_{i=1,..,n} (\text{diam}(\mathcal{F}) - r_i)$. Let $b \in A$. Then, since $B \cap A = \emptyset$, for all $i \in \{1, ..., n\}$: either $|b - a_i| < r_i$

or $|b-a_i| > \operatorname{diam}(B)$, and therefore, we have either $|\operatorname{diam}(\mathcal{F}) - |b - a_i||_\infty > \operatorname{diam}(\mathcal{F}) - r_i$, or $|\operatorname{diam}(\mathcal{F}) - |b-a_i||_\infty > |\operatorname{diam}(B) - \operatorname{diam}(\mathcal{F})|_\infty$. In both cases, we have $|\operatorname{diam}(\mathcal{F}) - |b-a_i||_\infty > \varepsilon$, hence, $|{}_D\varphi_b(x-a_i) - \varphi_{\mathcal{F}}(x-a_i)|_\infty > \varepsilon$. This last inequality is obtained for all $b \in A$, hence, we have $\bigcap_{i=1}^{n} V({}_D\varphi_{\mathcal{F}}, x-a_i, \varepsilon) \cap A = \emptyset$, and then ${}_D\varphi_{\mathcal{F}} \notin \underline{A}$. This contradicts the hypothesis.

Finally suppose that $\mathcal{F}$ is a Cauchy circular filter of center a. So, there exists a disk $d(a,r)$ in $\mathcal{F}$ such that $d(a,r) \cap A = \emptyset$. Hence, for $r' \in]0,r[$ we have $|a-b| > r - r'$ for all $b \in A$. Hence we have $V({}_D\varphi_{\mathcal{F}}, x-a, r-r') \cap A = \emptyset$, which contradicts the hypothesis "${}_D\varphi_{\mathcal{F}} \in \underline{A}$" and completes the proof.

The two following lemmas are useful when proving Theorem 4.3.

Lemma 4.3. *Let $\mathcal{F}$ be a circular filter on $\mathbb{K}$, let $a \in \mathbb{K}$ and let $r = \varphi_{\mathcal{F}}(x-a)$.*

If $r > 0$ then for all $\varepsilon \in]0,r[$ we have $W(\varphi_{\mathcal{F}}, x-a, \varepsilon) = U_{\Delta(a,r-\varepsilon,r+\varepsilon)}$.

If $r = 0$ then, for all $\varepsilon > 0$, $W(\varphi_{\mathcal{F}}, x-a, \varepsilon) = U_{d(a,\varepsilon)}$.

Proof. We notice that if $r = 0$ then $\mathcal{F}$ is the Cauchy circular filter of center a. Let $\mathcal{G}$ be a circular filter on $\mathbb{K}$ secant with $\Delta(a, r-\varepsilon, r+\varepsilon)$ (resp. $d(a,\varepsilon)$). There exists a sequence $(\alpha_n)_{n\in\mathbb{N}}$ in $\Delta(a, r-\varepsilon, r+\varepsilon)$ (resp. $d(a,\varepsilon)$) thinner than $\mathcal{G}$. So, we have $||\alpha_n - a| - r|_\infty \leq \varepsilon \ \forall n \in \mathbb{N}$. But, since $\varphi_{\mathcal{G}}(x-a) = \lim_{n\to+\infty} |\alpha_n - a|$, we have $|\varphi_{\mathcal{G}}(x-a) - r|_\infty \leq \varepsilon$. Hence, $\varphi_{\mathcal{G}} \in W(\varphi_{\mathcal{F}}, x-a, \varepsilon)$.

Conversely, let $\varphi_{\mathcal{G}} \in W(\varphi_{\mathcal{F}}, x-a, \varepsilon)$ (where r may be equal to 0). Then, we have $|\varphi_{\mathcal{G}}(x-a) - r|_\infty \leq \varepsilon$. We first suppose that $a \in \mathcal{Q}(\mathcal{G})$. If $\varphi_{\mathcal{G}}(x-a) > r$ (resp. $\varphi_{\mathcal{G}}(x-a) < r$, resp. $\varphi_{\mathcal{G}}(x-a) = r$), we consider an increasing (resp. decreasing, resp. monotonuous) distances sequence $(\alpha_n)_{n\in\mathbb{N}} \subset d(a, \operatorname{diam}(\mathcal{G})^-)$ (resp. $(\alpha_n)_{n\in\mathbb{N}} \subset \mathbb{K}$) thinner than $\mathcal{G}$ ([9, 7]). Since $\varphi_{\mathcal{G}}(x-a) = \lim_{n\to+\infty} |\alpha_n - a|$, there exists $N_1 \in \mathbb{N}$ such that for all $n \geq N_1$, we have $\varphi_{\mathcal{G}}(x-a) > |\alpha_n - a| > r$ (resp. $\varphi_{\mathcal{G}}(x-a) < |\alpha_n - a| < r$, resp. $|r - |\alpha_n - a||_\infty < \varepsilon$). Now, for every $B \in \mathcal{G}$, there exists $N_2 \in \mathbb{N}$ such that $\alpha_n \in B$ whenever $n \geq N_2$. So, $\mathcal{G}$ is secant with $\Delta(a, r-\varepsilon, r+\varepsilon)$ (resp. $d(a,\varepsilon)$). If $a \notin \mathcal{Q}(\mathcal{G})$, it is easly seen that there exists $B \in \mathcal{G}$ such that $\varphi_{\mathcal{G}}(x-a) = |y-a|$, whenever $y \in B$. Hence $B \subset \Delta(a, r-\varepsilon, r+\varepsilon)$ (resp. $B \subset d(a,\varepsilon)$) and then it follows that $\mathcal{G}$ is secant with $\Delta(a, r-\varepsilon, r+\varepsilon)$ (resp. $d(a,\varepsilon)$).

Proposition 4.4. *For $i \in \{1,..,n\}$, let $a_i \in \mathbb{K}$ and let $r'_i > r_i > 0$. Let $E = \bigcap_{i=1}^{n} \Delta(a_i, r_i, r'_i)$. Then $\cap_{i=1}^{n} U_{\Delta(a_i,r_i,r'_i)} = U_E$.*

Proof. It is clear that $U_E \subset \bigcap_{i=1}^{n} U_{\Delta(a_i,r_i,r'_i)}$. Let $\mathcal{F}$ be a circular filter on $\mathbb{K}$ such that $\varphi_{\mathcal{F}} \in \bigcap_{i=1}^{n} U_{\Delta(a_i,r_i,r'_i)}$. If $E = \emptyset$, then the claim is trivial. So we suppose $E \neq \emptyset$. Then $\bigcap_{i=1}^{n} d(a_i, r'_i) \neq \emptyset$, hence we may assume $a_1 \in \bigcap_{i=1}^{n} d(a_i, r'_i)$. Let $\rho = \inf_{1 \leq i \leq n} (r'_i)$. Then $\mathbb{K} \setminus E = (\mathbb{K} \setminus d(a_1, \rho)) \bigcup (\bigcup_{i=1}^{n} d(a_i, r_i^-))$. More precisely, There exists a set $I \subset \{1,..,n\}$ such that $\mathbb{K} \setminus E = (\mathbb{K} \setminus d(a_1, \rho)) \bigcup (\bigcup_{i \in I} d(a_i, r_i^-))$ and $d(a_i, r_i^-) \cap d(a_j, r_j^-) = \emptyset$ if $i, j \in I$ and $i \neq j$. Suppose that $\mathcal{F}$ is not secant with E. Then either it is secant with $\mathbb{K} \setminus d(a_1, \rho)$ or it is secant with one of the $d(a, r_i^-)$ $(i \in I)$ which are the holes of E.

Suppose first $\mathcal{F}$ is secant with $\mathbb{K} \setminus d(a_1, \rho)$. Since it is not secant with E, more precisely there does exist $\rho' > \rho$ such that $\mathcal{F}$ is not secant with $d(a_1, \rho')$. And therefore $\mathcal{F}$ is not secant with $\Delta(a_1, r_1, r'_1)$, which contradicts the hypothesis $\varphi_{\mathcal{F}} \in U_{\Delta(a_1,r_1,r'_1)}$.

Now suppose that $\mathcal{F}$ is secant with a certain $d(a_i, r_i^-)$ $(i \in I)$. Since $\mathcal{F}$ is not secant with E, we have $\mathrm{diam}(\mathcal{F}) < r_i$ and therefore $\mathcal{F}$ is not secant with $\Delta(a_i, r_i, r'_i)$. A contradiction with the hypothesis. As a consequence $\mathcal{F}$ is secant with E, and therefore $\varphi_{\mathcal{F}} \in U_E$. This finishes proving that $\bigcap_{i=1}^{n} U_{\Delta(a_i,r_i,r'_i)} \subset U_E$.

Theorem 4.3. *Let D be infraconnected and let ${}_D\varphi_{\mathcal{F}} \in \mathrm{Mult}(H(D), \mathcal{U}_D)$. Then the set $\{{}_DU_A \mid A \in {}_D\mathcal{F}\}$ is a basis of the filter $\mathfrak{F}$ of neighbourhoods of ${}_D\varphi_{\mathcal{F}}$ with respect to $\mathcal{S}$.*

Proof. It is clearly seen that $\{{}_DU_A \mid A \in {}_D\mathcal{F}\}$ is a basis of a filter, since $\emptyset \notin \{{}_DU_A \mid A \in {}_D\mathcal{F}\}$ and ${}_DU_{A \cap B} \subset {}_DU_A \cap {}_DU_B$ for any A, $B \in {}_D\mathcal{F}$.

Let $\bigcap_{j=1}^{q} V({}_D\varphi_{\mathcal{F}}, f_j, \varepsilon_j)$ be a canonical neighbourhood of ${}_D\varphi_{\mathcal{F}}$ and let $\varepsilon = \inf_{i=1,..,q}(\varepsilon_i)$. As ${}_D\varphi_{\mathcal{F}}(f_i) = \lim_{D\mathcal{F}} |f_i(x)|$, for all $i = 1, ..., q$, there exists an infraconnected affinoid element B_i of the canonical basis of $\mathcal{F}$ (in $\mathbb{K}$) such that $|{}_D\varphi_{\mathcal{F}}(f_i) - |f_i(x)||_\infty < \varepsilon$, $\forall x \in B_i \cap D$. Let $E = \cap_{j=1}^{q} B_i$. Given ${}_D\varphi_{\mathcal{G}} \in \mathrm{Mult}(H(D), \mathcal{U}_D)$ such that the circular filter ${}_D\mathcal{G}$ on D is secant with E, we have $|{}_D\varphi_{\mathcal{F}}(f_i) - {}_D\varphi_{\mathcal{G}}(f_i)|_\infty < \varepsilon$, $\forall i = 1, ..., q$. Then ${}_DU_E \subset \cap_{j=1}^{q} V({}_D\varphi_{\mathcal{F}}, f_j, \varepsilon_j)$. Hence $\cap_{j=1}^{q} V({}_D\varphi_{\mathcal{F}}, f_j, \varepsilon_j)$ belongs to $\mathfrak{F}$ since $\mathfrak{F}$ is a filter.

Now let $A \in {}_D\mathcal{F}$. We first suppose that $\mathcal{F}$ is large and has a center. So, there exists an infraconnected affinoid set B of the canonical basis of $\mathcal{F}$ in $\mathbb{K}$ such that $B \cap D \subset A$ and such that the holes $T_i = d(a_i, r_i^-)$ of B satisfy $|a_i - a_j| = \mathrm{diam}(\mathcal{F})$, $\forall i \neq j$, $i = 1, ..., n$. Let $r = \sup_{i=1,..,n}(r_i)$. It is clear that $r < \mathrm{diam}(\mathcal{F}) < \mathrm{diam}(B)$. Let $\lambda > 0$ be such that $\lambda < \inf(\mathrm{diam}(\mathcal{F}) - r, \mathrm{diam}(B) - \mathrm{diam}(\mathcal{F}))$. For all $i \in \{1, ..., n\}$ we have ${}_D\varphi_{\mathcal{F}}(x - a_i) = \mathrm{diam}(\mathcal{F})$. Put $F = \cap_{i=1}^{n} \Delta(a_i, \mathrm{diam}(\mathcal{F}) - \lambda, \mathrm{diam}(\mathcal{F}) + \lambda) \cap D$ and $F_i = \Delta(a_i, \mathrm{diam}(\mathcal{F}) - \lambda, \mathrm{diam}(\mathcal{F}) + \lambda) \cap D$ for $i = 1, .., n$. So, by Lemma 4.3 and Proposition 4.4, we have $\cap_{i=1}^{n} V({}_D\varphi_{\mathcal{F}}, x - a_i, \lambda) = \cap_{i=1}^{n} V(\varphi_{\mathcal{F}}, x - a_i, \lambda) \cap \mathrm{Mult}(H(D), \mathcal{U}_D) \subset \cap_{i=1}^{n} W(\varphi_{\mathcal{F}}, x - a_i, \lambda) \cap \mathrm{Mult}(H(D), \mathcal{U}_D) = \cap_{i=1}^{n} {}_DU_{F_i} \cap \mathrm{Mult}(H(D), \mathcal{U}_D) \subset {}_DU_F \cap \mathrm{Mult}(H(D), \mathcal{U}_D) \subset {}_DU_{B \cap D} \cap \mathrm{Mult}(H(D), \mathcal{U}_D) \subset {}_DU_A$.

Now, suppose that $\mathcal{F}$ is a Cauchy circular filter of center a. So, there exists a disk d(a,r) such that $d(a, r) \cap D \subset A$. By Lemma 4.3 we see that $W({}_D\varphi_{\mathcal{F}}, x - a, r) = W(\varphi_{\mathcal{F}}, x - a, r) \cap \mathrm{Mult}(H(D), \mathcal{U}_D) = U_{d(a,r)} \cap \mathrm{Mult}(H(D), \mathcal{U}_D) \subset {}_DU_A$.

Finally we suppose that $\mathcal{F}$ has no center. We denote by $(D_n)_{n \in \mathbb{N}}$ a canonical basis $(d(a_n, r_n))_{n \in \mathbb{N}}$ of $\mathcal{F}$ in $\mathbb{K}$. There exists a disk $D_i = d(a_i, r_i)$ of this basis such that $D_i \cap D \in A$. We may clearly suppose that $a_i \notin D_{i+1} = d(a_{i+1}, r_{i+1})$. Let $\lambda > 0$ be such that $\lambda < |a_{i+1} - a_i|$. For all $i \in \mathbb{N}$, we put $F_i = \Delta(a_{i+1}, \varphi_{\mathcal{F}}(x - a_{i+1}) - \lambda, \varphi_{\mathcal{F}}(x - a_{i+1}) + \lambda)$, then by Lemma 4.3, we have $V({}_D\varphi_{\mathcal{F}}, x - a_{i+1}, \lambda) = V(\varphi_{\mathcal{F}}, x - a_{i+1}, \lambda) \cap \mathrm{Mult}(H(D), \mathcal{U}_D) \subset W(\varphi_{\mathcal{F}}, x - a_{i+1}, \lambda) \cap \mathrm{Mult}(H(D), \mathcal{U}_D) = {}_DU_{F_i} \cap \mathrm{Mult}(H(D), \mathcal{U}_D) \subset {}_DU_{D_i \cap D} \cap \mathrm{Mult}(H(D), \mathcal{U}_D) \subset {}_DU_A$.

So, in any case, ${}_DU_A$ is a neighbourhood of ${}_D\varphi_{\mathcal{F}}$ and this ends the proof.

Corollary 4.1. *Let D be infraconnected and let $\mathcal{F}$ be a circular filter on $\mathbb{K}$ secant with D. Let $\mathcal{B}(\mathcal{F})$ be a basis of $\mathcal{F}$. Then, the set $\{{}_DU_{B\cap D} \mid B \in \mathcal{B}(\mathcal{F})\}$ is a basis of the filter of neighbourhoods of ${}_D\varphi_{\mathcal{F}}$ in $\mathrm{Mult}(H(D),\mathcal{U}_D)$ with respect to $\mathcal{S}$.*

Corollary 4.2. *Let D be infraconnected. If $\mathbb{K}$ is weakly valued, then the filter of neighbourhoods of any ${}_D\varphi_{\mathcal{F}} \in \mathrm{Mult}(H(D),\mathcal{U}_D)$ admits a countable basis.*

Proof. This is a direct consequence of Corollary 4.1, since a circular filter on $\mathbb{K}$ admits a countable basis when $\mathbb{K}$ is weakly valued, [7].

Proposition 4.5. *Let $D \subset \mathbb{K}$ and let A be a closed subset of $\mathbb{K}$ such that $A\cap D \neq \emptyset$. Then the mapping which to ${}_{A\cap D}\varphi_{\mathcal{F}} \in \mathrm{Mult}(H(A\cap D),\mathcal{U}_{A\cap D})$, associates its restriction ${}_D\varphi_{\mathcal{F}}$ to $H(D)$ is a continuous bijection from $\mathrm{Mult}(H(A\cap D),\mathcal{U}_{A\cap D})$ into ${}_DU_{A\cap D}$, both provided with the topology of simple convergence.*

Proof. This mapping is denoted ϕ. By Theorem 4.14 [7], ϕ is injective. Now, let ${}_D\varphi_{\mathcal{F}} \in {}_DU_{A\cap D}$. So, ${}_D\mathcal{F}$ is secant with $A\cap D$. First, suppose that ${}_D\mathcal{F}$ is large, then it defines a circular filter on $A\cap D$ and consequently, ${}_{A\cap D}\varphi_{\mathcal{F}} \in \mathrm{Mult}(H(A\cap D),\mathcal{U}_{A\cap D})$ and $\phi({}_{A\cap D}\varphi_{\mathcal{F}}) = {}_D\varphi_{\mathcal{F}}$. On the other hand, if ${}_D\mathcal{F}$ is a Cauchy circular filter on D of center a, then by definition $a \in D$. Further, as A is closed in $\mathbb{K}$ and ${}_D\mathcal{F}$ is secant with A, we see that $a \in A$. Therefore $a \in A\cap D$ and then ${}_{A\cap D}\varphi_a \in \mathrm{Mult}(H(A\cap D),\mathcal{U}_{A\cap D})$ and $\phi({}_{A\cap D}\varphi_a) = {}_D\varphi_{\mathcal{F}} = {}_D\varphi_a$. So, ϕ is bijective.

Now, we will show that ϕ is continuous. Let ${}_D\varphi_{\mathcal{F}} \in {}_DU_{A\cap D}$ and let $V = \bigcap_{j=1}^{q} V({}_D\varphi_{\mathcal{F}}, f_j, \varepsilon_j)$ ($f_j \in H(D)$, $\varepsilon_j > 0$ for all $j \in \{1,..,q\}$ and $q \in \mathbb{N}^*$) be a canonical neighbourhood of ${}_D\varphi_{\mathcal{F}}$ with respect to topology of simple convergence on ${}_DU_{A\cap D}$. Then, obviously we see that

$\phi^{-1}(V) = \bigcap_{j=1}^{q} V({}_{A\cap D}\varphi_{\mathcal{F}}, f_j/{}_{A\cap D}, \varepsilon_j)$ which is a canonical neighbourhood of ${}_{A\cap D}\varphi_{\mathcal{F}}$ with respect to topology of simple convergence on $\mathrm{Mult}(H(A\cap D),\mathcal{U}_{A\cap D})$. This proves that ϕ is continuous.

Theorem 4.4. *Let D be infraconnected. Then $\mathrm{Mult}(H(D),\mathcal{U}_D)$ is a local arc-connected space with respect to $\mathcal{S}$.*

Proof. We have to prove that, given any ${}_D\varphi_{\mathcal{F}} \in \mathrm{Mult}(H(D),\mathcal{U}_D)$, there exists a basis of neighbourhoods of ${}_D\varphi_{\mathcal{F}}$ whose elements are arc-connected. In chapter 0, we have shown that a such circular filter $\mathcal{F}$ on $\mathbb{K}$ admits a basis $\mathcal{B}(\mathcal{F})$ which consists of infraconnected affinoid sets. Given $B \in \mathcal{B}(\mathcal{F})$ secant with D, by Lemma 1.1, $B \cap D$ is infraconnected. Hence, by Theorem 4.1 $\mathrm{Mult}(H(B \cap D),\mathcal{U}_{B\cap D})$ is arc-connected and then by Proposition 4.5, ${}_DU_{B\cap D}$ is arc-connected too. This ends the proof.

Remark. It is well known that a topological space which is connected and locally arc-connected is arc-connected. Here, conversely, we have shown that when $\mathrm{Mult}(H(D),\mathcal{U}_D)$ is connected, then it is locally arc-connected. However, we notice that the proof is just based on Theorem 4.1. So, it does not seem easy to prove first the local arc-connectedness.

5 Metrizability of $(\mathrm{Mult}(H(D),\mathcal{U}_D),\mathcal{S})$.

In this chapter, we give some conditions for metrizability of the topology $\mathcal{S}$ on $\mathrm{Mult}(H(D),\mathcal{U}_D)$ and we look for equivalence between topologies $\mathcal{S}$ and $\mathfrak{T}_\delta$. We need the following basic lemma in topology (see, for example ex. 16A4 [13]).

Notation. Given any topological space E, countable intersection of open sets is usually named G_δ-*set.* Here, in order to avoid any confusion with the distance δ already defined, we will denote such a set a G_τ-*set.*

Lemma 5.1. *Let (E,T) be a compact topological space and let $x \in E$. If $\{x\}$ is a G_τ-set, then x admits a countable system of neighbourhoods.*

Proof. Since $\{x\}$ is a G_τ-set, there exists a decreasing sequence of open sets $(U_n)_{n\in\mathbb{N}}$ such that $\{x\} = \cap_{n\in\mathbb{N}}U_n$. Since E is a regular space, as it is compact, there exists a decreasing sequence of open sets $(V_n)_{n\in\mathbb{N}}$ such that, for all $n \in \mathbb{N}$, $x \in V_n \subset \overline{V}_n \subset U_n$. Let W be an open neighbourhood of x, and suppose that, for all $n \in \mathbb{N}$, $\overline{V}_n$ is not included in W. Then, the sequence $(\overline{V}_n \setminus W)_{n\in\mathbb{N}}$ is a decreasing sequence of compact subsets of E. So, their intersection is not empty. This contradicts the fact that $\{x\} = \cap_{n\in\mathbb{N}}U_n$. Hence, there exists $N \in \mathbb{N}$, such that $\overline{V}_n \subset W$ and therefore, the sequence $(V_n)_{n\in\mathbb{N}}$ is a countable system of neighbourhoods of x. This ends the proof.

Theorem 5.1. *Let $D \subset \mathbb{K}$ be closed and bounded. If* $\mathrm{Mult}(H(D),\mathcal{U}_D)$ *is countable, then the topology $\mathcal{S}$ is metrizable.*

Proof. By Tykhonov's theorem, it is known that when D is closed and bounded, then $\mathrm{Mult}(H(D),\mathcal{U}_D)$ is compact with respect to $\mathcal{S}$, Theorem 1.11 [7]. Suppose that

$\mathrm{Mult}(H(D),\mathcal{U}_D)$ is countable. Given any $\varphi \in \mathrm{Mult}(H(D),\mathcal{U}_D)$, it is clearly seen that $\{\varphi\}$ is a G_τ-set because it is the intersection of complementaries of a countable family of finite subsets of $\mathrm{Mult}(H(D),\mathcal{U}_D)$ which do not contain φ. Then, by Lemma 5.1, every $\varphi \in \mathrm{Mult}(H(D),\mathcal{U}_D)$ admits a countable system of neighbourhoods. Hence, since $\mathrm{Mult}(H(D),\mathcal{U}_D)$ is countable, there exists a countable basis of open sets for the topology $\mathcal{S}$. Then, by the Nagata-Smirnov Theorem [3], $\mathcal{S}$ is metrizable.

Recall that Ψ denotes the injection from D into $\mathrm{Mult}(H(D),\mathcal{U}_D)$ that, to each point $a \in D$, associates ${}_D\varphi_a$.

Definition. *D will be said* simple *if there is no large circular filter on D. i.e. if Ψ is a bijection onto* $\mathrm{Mult}(H(D),\mathcal{U}_D)$.

Remark. If a closed simple set D lies in $\mathcal{A}$, then it is bounded. In order to simplify notation, when D is simple, we will identify every $a \in D$ with $\Psi(a)$.

Simplicity is not equivalent to countability as it will be shown in Example 2.

Theorem 5.2. *Let $D \in \mathcal{A}$ be closed. The following propositions are equivalent:*

i) D is simple.

ii) D is compact.

iii) Ψ is a bijection.

iv) Topologies $\mathcal{S}$ and $\mathfrak{T}_\delta$ on $\mathrm{Mult}(H(D),\mathcal{U}_D)$ *are equivalent.*

Proof. For convenience we identify D with $\Psi(D)$. The equivalence between *i)* and *iii)* is obvious. We first show that *i)* $\Leftrightarrow$ *iv)*. Given $\varepsilon > 0$ and ${}_D\varphi_{\mathcal{F}} \in \mathrm{Mult}(H(D),\mathcal{U}_D)$, we denote by $B({}_D\varphi_{\mathcal{F}}, \varepsilon)$ the open ball in

$\mathrm{Mult}(H(D),\mathcal{U}_D)$ of center ${}_D\varphi_{\mathcal{F}}$ and radius ε with respect to the distance δ.

i) $\Rightarrow$ *iv)*. Suppose that D is simple. Given $a \in D$ and $\varepsilon > 0$, by definition of the distance δ it is seen that $B(a,\varepsilon) = \{y \in D \mid |y-a| < \frac{\varepsilon}{2}\}$. For any $x, y \in D$, we define $P_y \in H(D)$ by $P_y(x) = x - y$. Then we see that $B(a,\varepsilon) = V(a, P_a, \frac{\varepsilon}{2})$, and then $B(a,\varepsilon)$ is an open set with respect to $\mathcal{S}$. This shows that $\mathcal{S}$ is thinner than $\mathfrak{T}_\delta$, and then, by Proposition 4.1, topologies $\mathcal{S}$ and $\mathfrak{T}_\delta$ are equivalent.

iv) $\Rightarrow$ *i)*. We suppose that D is not simple. Hence, there exists a large circular filter ${}_D\mathcal{F}$ on D. By Lemma 3.2 [7], there exists a sequence $(x_n)_{n\in\mathbb{N}}$ thinner than ${}_D\mathcal{F}$. Let $\beta > 0$ be such that $\beta < \mathrm{diam}\,({}_D\mathcal{F})$. For all $a \in \mathbb{K}$, we clearly have $\delta({}_D\varphi_{\mathcal{F}}, {}_D\varphi_a) \geq \mathrm{diam}\,({}_D\mathcal{F})$ and then $B({}_D\varphi_{\mathcal{F}}, \beta)$ does not contain images by ${}_D\Phi$ of Cauchy filters on D, i.e. $B({}_D\varphi_{\mathcal{F}}, \beta)$ does not contain images by Ψ of points of D.

Let us take a basic open set W of the topology $\mathcal{S}$. It is of the form $\bigcap_{j=1}^{q} V({}_D\varphi_{\mathcal{F}}, h_j, \varepsilon_j)$, $q \in \mathbb{N}^*$. We put $\varepsilon = \inf_{j=1,..,q} \varepsilon_j$. Since the sequence $(x_n)_{n\in\mathbb{N}}$ is thinner than ${}_D\mathcal{F}$, there exists $N \in \mathbb{N}$ such that, for all $n \geq N$ and for all $j = 1, ..., q$, we have $|{}_D\varphi_{\mathcal{F}}(h_j) - |h_j(x_n)||_\infty < \varepsilon$. Hence, W contains all images by ${}_D\Phi$ of Cauchy filters on D associated to the x_n, $n \geq N$. So, $B({}_D\varphi_{\mathcal{F}}, \beta)$ may not be an open set for the topology $\mathcal{S}$, and therefore $\mathcal{S}$ and $\mathfrak{T}_\delta$ are not equivalent.

iv) $\Rightarrow$ *ii)*. We have seen that if topologies $\mathcal{S}$ and $\mathfrak{T}_\delta$ on $\mathrm{Mult}(H(D),\mathcal{U}_D)$ are equivalent, then D is simple. Since D is closed, by the previous remark, it is bounded too. Hence, $\mathrm{Mult}(H(D),\mathcal{U}_D)$ is compact with respect to $\mathcal{S}$ ([7, Th 1.11]). The mapping Ψ, which is a bijection, is here an homeomorphism because the distance δ extends that of D. Hence, D is compact.

Finally we show that *ii)* $\Rightarrow$ *i)*. Suppose that D is not simple. There exists a large circular filter ${}_D\mathcal{G}$ on D. It is known that there exists a monotonous distances sequence $(x_n)_{n\in\mathbb{N}} \subset D$, thinner than ${}_D\mathcal{G}$. But such a sequence does not admit accumulation point with respect to the metric topology of $\mathbb{K}$. As a consequence, D is not compact. This shows *ii)* $\Rightarrow$ *i)* and completes the proof.

Example 1. In this example, we construct a set D closed, bounded and not simple such that $\mathrm{Mult}(H(D),\mathcal{U}_D)$ is countable. By Theorem 5.1, $\mathcal{S}$ is metrizable, but by Theorem 5.2, topologies $\mathcal{S}$ and $\mathfrak{T}_\delta$ are not equivalent. However, we are not able to construct a distance giving $\mathcal{S}$.

Let $(a_n)_{n\in\mathbb{N}}$ be an injective sequence in $d(0,1)$ such that, $\forall p,q \in \mathbb{N}$, $p \neq q$, $|a_p - a_q| = 1$ (each a_n lies in a different class of $d(0,1)$). We put $D = \cup_{n\in\mathbb{N}}\{a_n\}$. The only one large circular filter on $\mathbb{K}$ secant with D is the circular filter $\mathcal{G}$ of center 0 and diameter 1. Then, $\text{Mult}(H(D),\mathcal{U}_D) = (\cup_{n\in\mathbb{N}}\, {}_D\varphi_{a_n}) \cup {}_D\varphi_{\mathcal{G}}$ is countable.

Example 2. In this example, we show a set D closed, bounded and simple but not countable. Hence, by Theorem 5.2, this shows that topologies $\mathcal{S}$ and $\mathfrak{T}_\delta$ are equivalent on $\text{Mult}(H(D),\mathcal{U}_D)$ although D is not countable.

Let p be a prime number. We put $\mathbb{K} = \mathbb{C}_p$ and $D = \mathbb{Z}_p$. It is well known that $\mathbb{Z}_p$ is not countable, but since $\mathbb{Z}_p$ is compact, then it is simple. In particular, there is no large circular filter on $\mathbb{C}_p$ secant with $\mathbb{Z}_p$.

Remark. We have seen that countability of $\text{Mult}(H(D),\mathcal{U}_D)$ is not a necessary condition for metrizability of the topology $\mathcal{S}$ and that simplicity of D is not sufficient. It seems difficult to find a convenient necessary and sufficient condition for metrizability.

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Boussaf Kamal and Maïnetti Nicolas
Laboratoire de Mathématiques Pures
Université Blaise Pascal
(Clermont-Ferrand)
Complexe Scientifique des Cézeaux
F 63177 AUBIERE CEDEX
FRANCE

Hemdaoui Mohamed
Département de Mathématiques Pures
Université Mohammed I
OUJDA
MAROC

E-mail address:
boussaf@ucfma.univ-bpclermont.fr
mainetti@ucfma.univ-bpclermont.fr
hemdaoui@sciences.univ-oujda.ac.ma

Recibido: 12 de Abril de 1999
Revisado: 21 de Enero de 2000