REVISTA MATEMÁTICA COMPLUTENSE
(2001) vol. XIV, num. 1, 127-143

ISSN 1139-1138

REAL INTERPOLATION FOR NON-DISTANT MARCINKIEWICZ SPACES*

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Abstract

We describe the real interpolation spaces between given Marcinkiewicz spaces that have fundamental functions of the form $t^{1/q}\left(\ln\frac{e}{t}\right)^{\alpha}$ with the same exponent q. The spaces thus obtained are used for the proof of optimal interpolation theorem from [7], concerning spaces $L_{\infty,\alpha,E}$.

1 Introduction and Preliminaries

The Marcinkiewicz spaces play an important rôle in so-called *weak type interpolation*, i.e. interpolation of (quasi)linear operators acting from Lorentz to Marcinkiewicz spaces. Recall that, for arbitrary positive concave function $\varphi(t)$, the corresponding Lorentz space Λ_φ and Marcinkiewicz space M_φ on the interval $(0,1)$ are defined as rearrangement invariant spaces with the norms

$$\|f\|_{\Lambda_\varphi} = \int_0^1 f^*(t)\, d\,\varphi(t), \qquad \|f\|_{M_\varphi} = \sup_{0<t<1} \varphi(t)\, f^{**}(t),$$

where $f^*(t)$ is the nonincreasing rearrangement of $f(t)$ and $f^{**}(t) = \frac{1}{t}\int_0^t f^*(s)\, ds$. Various properties of Lorentz and Marcinkiewicz spaces can be found, e.g., in the monograph [4].

Conditions of the type $T : \Lambda_\varphi \to M_\psi$ are particularly simple for checking if T is an integral operator. If we have two actions of such a type, say, $T : \Lambda_{\varphi_i} \to M_{\psi_i},\ i = 0, 1$, then we can apply arbitrary interpolation functor $\mathcal{F}$ and thus obtain that $T : \mathcal{F}(\Lambda_{\varphi_0}, \Lambda_{\varphi_1}) \to \mathcal{F}(M_{\psi_0}, M_{\psi_1})$

*The research was supported by the Center of Scientific Absorption of the Ministry of Absorption of the State of Israel.

2000 Mathematics Subject Classification: 46B70, 46E30.
Servicio de Publicaciones. Universidad Complutense. Madrid, 2001

(for basic definitions of interpolation theory see, e.g., [2]). Just the description of spaces $\mathcal{F}(\Lambda_{\varphi_0}, \Lambda_{\varphi_1})$ and $\mathcal{F}(M_{\psi_0}, M_{\psi_1})$ is the main subject of weak type interpolation theory.

This theory is rather developed for the cases when the spaces $\Lambda_{\varphi_0}, \Lambda_{\varphi_1}$ (as well as the spaces $M_{\varphi_0}, M_{\varphi_1}$) are sufficiently "distant" from each other. Such a "distance" can be estimated via comparison of fundamental functions φ_0, φ_1 (correspondingly, ψ_0, ψ_1) with the help of their extension indices

$$p_\varphi = \lim_{t\to 0} \frac{\ln m_\varphi(t)}{\ln t}, \qquad q_\varphi = \lim_{t\to\infty} \frac{\ln m_\varphi(t)}{\ln t}, \qquad m_\varphi(t) = \sup_{0<s<1} \frac{\varphi(st)}{\varphi(s)}.$$

We say φ_0 and φ_1 are "distant" from each other if either $q_{\varphi_0} < p_{\varphi_1}$ or $q_{\varphi_1} < p_{\varphi_0}$. For example, $p_\varphi = q_\varphi = 1/p$ for any function $\varphi(t) = t^{1/p}(ln\frac{e}{t})^\alpha$ with $\alpha \in \mathbb{R}$, and two functions $\varphi_0 = t^{1/p_0}(ln\frac{e}{t})^\alpha$, $\varphi_1 = t^{1/p_1}(ln\frac{e}{t})^\beta$ are "distant" from each other if and only if $p_0 \neq p_1$. Notice, by the way, that all interpolation spaces between Lorentz or Marcinkiewicz spaces with such fundamental functions can be obtained solely by the use of real interpolation functors, i.e. functors of the form

$$\begin{aligned} &\mathcal{F}(A,B) = (A,B)^K_G = \{f(t) :\ K(u,f,A,B) \\ &= \inf_{f=f_0+f_1} (\|f_0\|_A + u\|f_1\|_B) \in G\}, \end{aligned}$$

where the parameter G is an arbitrary Banach function space on $(0,\infty)$ satisfying the condition $\min(1,u) \in G$.

Real interpolation between "distant" Lorentz or Marcinkiewicz spaces is now well-investigated (see, e.g., [5], [6], [4] etc.). It turns out that in the case, when the parameter space G itself is "distant" from L_1 and L_∞, the corresponding functor $\mathcal{F}$ gives one and the same result for any two rearrangement invariant spaces A, B with the fundamental functions φ_0, φ_1 respectively, i.e. we obtain no difference between Lorentz, Marcinkiewicz and other spaces.

Interpolation between "non-distant" spaces is studied much less, mainly for those spaces with fundamental functions $\varphi(t) = t^{\frac{1}{p}}\left(\ln\frac{e}{t}\right)^\alpha$, in which only the second parameter α is varying. This study was initiated by the famous work [1], considering weak type interpolation in the so-called Lorentz-Zygmund spaces that, in general, are different from Lorentz and Marcinkiewicz spaces. In the contrast to "distant" case, such a difference appeared to be essential: as shown in [3], real interpolation in the Marcinkiewicz and in the Lorentz spaces themselves gives some other kinds of spaces which alone provide optimality of weak type interpolation with respect to Lorentz-Zygmund (and some similar) spaces.

A detailed study of such new spaces was made in the paper [7]. The following three types of spaces were introduced and investigated there:

1. The spaces of *Lorentz-Zygmund* (briefly LZ) *type* with the quasinorm

$$\|f\|_{L_{p,\alpha,E}} = \left\|t^{\frac{1}{p}}\left(\ln\frac{e}{t}\right)^{\alpha} f^*(t)\right\|_{\widetilde{E}}.$$

2. The spaces of A-type with the norm

$$\|f\|_{A_{p,\alpha,E}} = \Big\|\left(\ln\frac{e}{t}\right)^{\alpha-1}\int_t^1 s^{\frac{1}{p}-1} f^{**}(s)\,ds\Big\|_{\widetilde{E}}.$$

3. The spaces of B-type with the norm

$$\|f\|_{B_{p,\alpha,E}} = \Big\|\sup_{0<s<t} s^{\frac{1}{p}}\left(\ln\frac{e}{s}\right)^{\alpha-1} f^{**}(s)\Big\|_{\widetilde{E}}.$$

As a parameter space $\widetilde{E}$, we take here an arbitrary Banach function space on the interval $(0,1)$ which is rearrangement invariant with respect to the measure dt/t, i.e. an interpolation space for the Banach couple $(\widetilde{L}_1, \widetilde{L}_\infty)$, where

$$\|f\|_{\widetilde{L}_1} = \int_0^1 |f(t)|\frac{dt}{t}, \qquad \|f\|_{\widetilde{L}_\infty} = \sup_{0<t<1}|f(t)|.$$

By the same letter E (without tilde) we denote the space of functions $g:(0,\infty)\to\mathbb{R}$ connected with $\widetilde{E}$ by the formula $\|f\|_{\widetilde{E}} = \|g\|_E$ for $f(t) = g\left(\ln\frac{1}{t}\right)$ (or, in another form, $g(u) = f(e^{-u})$). Clearly, these spaces are interpolation in the corresponding couple (L_1, L_∞). We shall also need the Boyd indices of such spaces, that are defined similarly to extension indices of fundamental functions:

$$p_E = \lim_{t\to 0}\frac{\ln d_E(t)}{\ln t}, \qquad q_E = \lim_{t\to\infty}\frac{\ln d_E(t)}{\ln t}, \qquad d_E(t) = \sup_{f\in E}\frac{\|f(s/t)\|_E}{\|f(s)\|_E}.$$

Notice, by the way, that, for many rearrangement invariant spaces, both types of indices coincide; for instance, this happens for all Lorentz and Marcinkiewicz spaces.

The spaces of LZ-type include the classical Lorentz-Zygmund spaces from [1] which would be obtained if to take $E = L_r,\ r \geq 1$. Taking $E = L_1$, we obtain a Lorentz space $\Lambda_{p.\alpha}$ with the fundamental function $\varphi(t) = t^{\frac{1}{p}} \left(\ln \frac{e}{t}\right)^{\alpha}$; the choice $E = L_\infty$ gives a Marcinkiewicz space $M_{p,\alpha}$ with the same fundamental function. It is usual to omit the second index α when it equals zero and to say that a (quasi)linear operator $T : \Lambda_p \to M_q$ is *of weak type* (p, q).

The rôle of all above mentioned spaces can be cleared up by the following main theorem proved in [7].

Theorem 1.1. *Let a (quasi)linear operator T be of two weak types (a, b) and (p, q), where $1 \leq a < p,\ 1 \leq b < q$. Let $\alpha \in \mathbb{R}$ and let the upper Boyd index $q_E < 1 - \alpha$. Then $T : A_{p,\alpha,E} \to B_{q,\alpha,E}$. At the same time $B_{q,\alpha,E} \subset L_{q,\alpha-1,E}$, $A_{p,\alpha,E} \supset L_{p,\alpha,E}$, thus $T : A_{p,\alpha,E} \to L_{q,\alpha-1,E}$ and $T : L_{p,\alpha,E} \to B_{q,\alpha,E}$. Moreover, the space $A_{p,\alpha,E}$ is the largest one which admits such interpolation with respect to the space $L_{q,\alpha-1,E}$, and the space $B_{q,\alpha,E}$ is the smallest one which admits such interpolation with respect to the space $L_{p,\alpha,E}$.*

The spaces of A and B-types were computed directly as real interpolation spaces between Lorentz and between Marcinkiewicz spaces respectively, and this fact has led to their optimality. (The expressions of norm in these spaces even keep some features of generating spaces: integral in A-norm, supremum in B-norm.) More exactly we can write that

$$B_{q,\alpha,E} = (M_q, M_{q,\alpha-1})^K_G, \qquad A_{p,\alpha,E} = (\Lambda_p, A)^K_G, \qquad A = (\Lambda_a, \Lambda_p)^K_F,$$

where

$$\|f(t)\|_F = \sup_{0<t<1} \frac{1}{t} \left(\ln \frac{e}{t}\right)^{\alpha-1} |f(t)|,$$
$$\|g(u)\|_G = \|(1+u)^{\alpha-1} g\big((1+u)^{1-\alpha}\big)\|_E .$$

The definitions of A and B-type spaces apply also to the cases of infinite p and q. It turns out that, for such values, these spaces coincide with the corresponding spaces of LZ-type, namely, $A_{\infty,\alpha,E} = L_{\infty,\alpha,E}$, $B_{\infty,\alpha,E} = L_{\infty,\alpha-1,E}$. Moreover, Theorem 1.1 remains true for $q = \infty$, giving optimality of LZ-type space itself as a range space of interpolation in this exceptional case. The case of $p = \infty$ is still more exceptional, since the standard consideration of two separate weak types

(a, b) and (p, q) becomes very restrictive for infinite p and (following to [1]) should be replaced by that of "joint weak type $(a, b; \infty, q)$". The last name is given to quasilinear operators T, satisfying inequality

$$\begin{aligned} &(Tf)^*(t) \le C\left(t^{-\frac{1}{b}} \int_0^{t^m} s^{\frac{1}{a}} f^*(s) \frac{ds}{s} + t^{-\frac{1}{q}} \int_{t^m}^1 f^*(s) \frac{ds}{s}\right), \\ &m = a\left(\frac{1}{b} - \frac{1}{q}\right), \end{aligned} \tag{1.2}$$

for some constant C and any measurable function f. Consequently, optimal weak type interpolation for infinite p has to be considered on the set of all quasilinear (not only linear) operators. Such an optimality was claimed in [7] (as part of Theorem 7.3), but the proof given there is sufficient only for finite p and should be corrected in infinite case.

Careful analysis shows that the optimal interpolation range space for the domain space $L_{\infty,\alpha,E}$ can be constructed only after indentification of interpolation spaces for more general couples of "non-distant" Marcinkiewicz spaces, such as $M_{q,\alpha}, M_{q,\beta}$ with arbitrary $\alpha, \beta \le 0$. The last problem is also of own interest, enlarging our knowledge about "non-distant" interpolation. It turns out (see Theorem 2.1 below) that we again obtain spaces of B-type with some intermediate indices like the well-known reiteration formulas. However, this is not a proper reiteration, since the spaces $M_{q,\alpha}, M_{q,\beta}$, in general, do not belong to the family of spaces $B_{q,\alpha,E}$; apparently, we should treat this fact as some new kind of *stability* for "non-distant" spaces.

2 Interpolation for the couple $(M_{q,\alpha}, M_{q,\beta})$

The proofs of this section mostly are similar to corresponding proofs from [7], however various (even small) distinctions are important and do not allow us to shorten our exposition by means of permanent refering to [7] (of course, we do refer to those results from [7], which are taken unchanged). We will use some rather standard notations, such as $a \lesssim b$ ($a \gtrsim b$) if $a \le Cb$ ($a \ge Cb$) for some constant C, or $a \approx b$ (equivalence relation) if $a \lesssim b$ and $a \gtrsim b$ simultaneously (possibly with different C). We say "a function f is almost increasing (decreasing)" if it is equivalent to an increasing (decreasing) function g.

The main assertion of this paper is as follows.

Theorem 2.1. *Let $\alpha < \beta \le 0$ and let $q_E < \beta - \alpha$. Then for any fixed $q \in (1, \infty]$ the space $B_{q,\alpha+1,E}$ is real interpolation in the couple $(M_{q,\beta}, M_{q,\alpha})$, namely,*

$$B_{q,\alpha+1,E} = (M_{q,\beta}, M_{q,\alpha})^K_H,$$

where

$$\|f(u)\|_H = \|(1+u)^\varepsilon f\big((1+u)^{-\varepsilon}\big)\|_E \tag{2.1}$$

with $\varepsilon = \alpha - \beta$.

Proof. It is known (see, e.g., [2] or [6]) that, for any two function indices φ_0, φ_1, the space $M_{\varphi_0} + M_{\varphi_1}$ is equal (with equivalent norms) to the space $M_{\min(\varphi_0,\varphi_1)}$. Since $K(u, f, M_{q,\beta}, M_{q,\alpha})$ is defined as norm in $M_{q,\beta} + uM_{q,\alpha}$, we obtain immediately that

$$K(u, f, M_{q,\beta}, M_{q,\alpha}) \approx \|f\|_{M_{\varphi_u}},$$

where

$$\varphi_u(s) = \min\left(s^{\frac{1}{q}}\left(\ln\frac{e}{s}\right)^\beta, us^{\frac{1}{q}}\left(\ln\frac{e}{s}\right)^\alpha\right) = s^{\frac{1}{q}}\left(\ln\frac{e}{s}\right)^\beta \min\left(1, u\left(\ln\frac{e}{s}\right)^\varepsilon\right).$$

Denote temporarily $(M_{q,\beta}, M_{q,\alpha})^K_H$ as B_1, then we can write that

$$\begin{aligned}
\|f\|_{B_1} &= \|K(u, f, M_{q,\beta}, M_{q,\alpha})\|_H \\
&= \|\sup_{0<s<1} \min\left((1+u)^\varepsilon, \left(\ln\frac{e}{s}\right)^\varepsilon\right) s^{\frac{1}{q}}\left(\ln\frac{e}{s}\right)^\beta f^{**}(s)\|_E \\
&= \|\sup_{0<s<1} \min\left(\left(\ln\frac{e}{t}\right)^\varepsilon, \left(\ln\frac{e}{s}\right)^\varepsilon\right) s^{\frac{1}{q}}\left(\ln\frac{e}{s}\right)^\beta f^{**}(s)\|_{\widetilde{E}}.
\end{aligned}$$

Taking supremum only over $s < t$, we will have there that $\left(\ln\frac{e}{s}\right)^\varepsilon \le \left(\ln\frac{e}{t}\right)^\varepsilon$, because $\varepsilon < 0$. Thus we obtain an inequality

$$\|f\|_{B_1} \ge \Big\|\sup_{0<s<t} s^{\frac{1}{q}}\left(\ln\frac{e}{s}\right)^\alpha f^{**}(s)\Big\|_{\widetilde{E}} = \|f\|_{B_{q,\alpha+1,E}},$$

which means the embedding $B_{q,\alpha+1,E} \supset (M_{q,\beta}, M_{q,\alpha})^K_H$.

The proof of the inverse embedding is more difficult and requires some preliminary auxiliary assertions. Let us take some numbers a, b, p

such that $1 \leq a < p < \infty,\ 1 \leq b < q$. The first assertion, we need, was proved in [7].

Lemma 2.1. *Let a (quasi)linear operator T is of two weak types (a, b) and (p, q). Then $T : A \to M_{q,\alpha}$ for the space A defined by the norm*

$$\|f\|_A = \sup_{0<t<1} \left(\ln\frac{e}{t}\right)^{\alpha} \int_t^1 s^{\frac{1}{p}-1} f^{**}(s)\, ds.$$

General properties of Lorentz and Marcinkiewicz spaces (see, e.g., [5]) imply that, under conditions of this lemma, also $T : \Lambda_{p,\beta} \to M_{q,\beta}$. As a consequence, we obtain that

$$T : (\Lambda_{p,\beta}, A)^K_H \longrightarrow (M_{q,\beta}, M_{q,\alpha})^K_H. \tag{2.2}$$

Our next purpose is to estimate the first space of this relation.

Lemma 2.2. *Let $f \in \Lambda_{p,\beta} + A$. Then, for any $z > 1$, the following inequality holds*

$$K(z^{-\varepsilon}, f, \Lambda_{p,\beta}, A) \lesssim \sup_{0<t<e^{1-z}} \left(\frac{z}{\ln\frac{e}{t}}\right)^{-\varepsilon} \int_t^1 s^{\frac{1}{p}-1} \left(\ln\frac{e}{s}\right)^{\beta} f^{**}(s)\, ds. \tag{2.3}$$

Proof. For a given $f(s)$ and fixed z, we define two functions

$$f_0(s) = \min\{|f(s)|,\ f^*(e^{1-z})\} \operatorname{sign} f(s),$$

$$f_1(s) = [\max\{|f(s)|,\ f^*(e^{1-z})\} - f^*(e^{1-z})] \operatorname{sign} f(s).$$

Then $f = f_0 + f_1$ and, from the definition of K-functional, we have

$$K(z^{-\varepsilon}, f, \Lambda_{p,\beta}, A) \leq \|f_0\|_{\Lambda_{p,\beta}} + z^{-\varepsilon}\|f_1\|_A. \tag{2.4}$$

Thus to prove (2.3) it suffices to compare $\|f_0\|_{\Lambda_{p,\beta}}$ and $z^{-\varepsilon}\|f_1\|_A$ separately with the right-hand side of (2.3), namely, with $\sup_{0<t<x} R(t)$, where $x = e^{1-z}$ and

$$R(t) = \left(\frac{z}{\ln\frac{e}{t}}\right)^{-\varepsilon} \int_t^1 S^{\frac{1}{p}-1} \left(\ln\frac{e}{s}\right)^{\beta} f^{**}(s)\, ds.$$

Consider first the norm

$$\|f_0\|_{\Lambda_{p,\beta}} = \int_0^x s^{\frac{1}{p}-1}\left(\ln\frac{e}{s}\right)^\beta f^*(x)\,ds + \int_x^1 s^{\frac{1}{p}-1}\left(\ln\frac{e}{s}\right)^\beta f^*(s)\,ds = I_1 + I_2.$$

Since $f^*(s) \le f^{**}(s)$, we obtain that $I_2 \le R(x) \le \sup_{0<t<x} R(t)$. For the first integral, we have

$$I_1 = f^*(x)\int_0^x s^{\frac{1}{p}-1}\left(\ln\frac{e}{s}\right)^\beta ds \le p x^{\frac{1}{p}}\left(\ln\frac{e}{x}\right)^\beta f^{**}(x).$$

At the same time, since the functions $sf^{**}(s)$ and $\left(\ln\frac{e}{s}\right)^\beta$ are increasing, we have, for $x \le 1/e$,

$$\begin{aligned} \sup_{0<t<x} R(t) \ge R(x) &= \int_x^1 s^{\frac{1}{p}-2}\left(\ln\frac{e}{s}\right)^\beta \left(sf^{**}(s)\right) ds \\ &\ge x f^{**}(x)\left(\ln\frac{e}{x}\right)^\beta \int_x^1 s^{\frac{1}{p}-2} ds \\ &= \frac{p}{p-1}\left(1 - x^{1-\frac{1}{p}}\right) x^{\frac{1}{p}} f^{**}(x)\left(\ln\frac{e}{x}\right)^\beta \gtrsim I_1, \end{aligned} \tag{2.5}$$

because $1 - x^{1-1/p} \ge 1 - (1/e)^{1-1/p} > 0$. On the other hand, for $x \ge 1/e$,

$$\begin{aligned} \sup_{0<t<x} R(t) \ge R\left(\frac{1}{e}\right) &\ge \left(\frac{1}{2}\right)^{-\varepsilon} \int_{1/e}^1 s^{\frac{1}{p}-2}\left(\ln\frac{e}{s}\right)^\beta \left(sf^{**}(s)\right) ds \\ &\ge \left(\frac{1}{2}\right)^{-\varepsilon} \frac{1}{e} f^{**}\left(\frac{1}{e}\right) \int_{1/e}^1 s^{\frac{1}{p}-2}\left(\ln\frac{e}{s}\right)^\beta ds \\ &\gtrsim f^{**}\left(\frac{1}{e}\right) \ge f^{**}(x). \end{aligned}$$

Since $x^{1/p}$ and $1/\ln(e/x)$ are less than 1, we obtain in this case too that

$$\sup_{0<t<x} R(t) \gtrsim x^{\frac{1}{p}}\left(\ln\frac{e}{x}\right)^\beta f^{**}(x) \gtrsim I_1. \tag{2.6}$$

Let us now proceed to study the second norm

$$\|f_1\|_A = \sup_{0<t<1}\left(\ln\frac{e}{t}\right)^\alpha \int_t^1 s^{\frac{1}{p}-1} f_1^{**}(s)\,ds = \max(\xi_1, \xi_2),$$

where we denote by ξ_1 the supremum over $t \in (0, x)$ and by ξ_2 the supremum over $t \in (x, 1)$. Since $f_1^*(s) \leq f^*(s)$ for all $s \in (0,1)$, we obtain immediately that

$$\xi_1 \leq \sup_{0<t<x} \left(\ln\frac{e}{t}\right)^\alpha \int_t^1 s^{\frac{1}{p}-1} f^{**}(s)\, ds$$
$$\leq \sup_{0<t<x} \left(\ln\frac{e}{t}\right)^\varepsilon \int_t^1 s^{\frac{1}{p}-1} \left(\ln\frac{e}{s}\right)^\beta f^{**}(s)\, ds = z^\varepsilon \sup_{0<t<x} R(t),$$

hence we need to estimate only the second supremum. But for all $t \in (x, 1)$, the function $f_1^*(t) = 0$, thus

$$\xi_2 = \sup_{x<t<1} \left(\ln\frac{e}{t}\right)^\alpha \int_t^1 s^{\frac{1}{p}-2} \left(\int_0^x f_1^*(\tau) d\tau\right) ds \leq \int_0^x f^*(\tau) d\tau \cdot \sup_{x<t<1} g(t),$$

where

$$g(t) = \left(\ln\frac{e}{t}\right)^\alpha \int_t^1 s^{\frac{1}{p}-2} ds \lesssim t^{\frac{1}{p}-1} \left(\ln\frac{e}{t}\right)^\alpha.$$

It is easy to check that the right-hand side of the last inequality is an almost decreasing function, which implies that

$$\sup_{x<t<1} g(t) \lesssim x^{\frac{1}{p}-1} \left(\ln\frac{e}{x}\right)^\alpha$$

and

$$\xi_2 \lesssim x^{\frac{1}{p}-1} \left(\ln\tfrac{e}{x}\right)^\alpha \int_0^x f^*(\tau) d\tau =$$
$$x^{\frac{1}{p}} \left(\ln\tfrac{e}{x}\right)^\alpha f^{**}(x) = z^\varepsilon x^{\frac{1}{p}} \left(\ln\tfrac{e}{x}\right)^\beta f^{**}(x).$$

Using again the inequalities (2.5) (for $x \leq 1/e$) or (2.6) (for $x > 1/e$), we obtain that $\xi_2 \lesssim z^\varepsilon \sup_{0<t<x} R(t)$ like ξ_1 before. Consequently, all summands of the right-hand side of inequality (2.4) are almost less than the right-hand side of (2.3), and the lemma is proved.

■

Having an estimate for the K-functional, we are now able to estimate the norm in the first space from the relation (2.2), taking the same parameter space H as in (2.1). We obtain that

$$\begin{aligned}
\|f\|_{(\Lambda_{p,\beta},A)^K_H} &= \|K(u,f,\Lambda_{p,\beta},A)\|_H \\
&= \|(1+u)^\varepsilon K\big((1+u)^{-\varepsilon},f,\Lambda_{p,\beta},A\big)\|_E \\
&\lesssim \left\| \sup_{0<t<e^{-u}} \left(\frac{1}{\ln\frac{e}{t}}\right)^{-\varepsilon} \int_t^1 s^{\frac{1}{p}-1}\left(\ln\frac{e}{s}\right)^{\beta} f^{**}(s)\,ds\right\|_E .
\end{aligned}$$

After replacing t by e^{-v}, the last norm takes a form

$$\left\|\sup_{v>u}(1+v)^\varepsilon \int_{e^{-v}}^1 s^{\frac{1}{p}-1}\left(\ln\frac{e}{s}\right)^{\beta} f^{**}(s)\,ds\right\|_E = \|\sup_{v>u} g(v)h(v)\|_E,$$

where the functions

$$g(v)=(1+v)^\varepsilon, \quad h(v)=\int_{e^{-v}}^1 s^{\frac{1}{p}-1}\left(\ln\frac{e}{s}\right)^{\beta} f^{**}(s)\,ds$$

satisfy all conditions of Theorem 4.2 from [7]. Applying this theorem, we have the right to omit the supremum:

$$\|\sup_{v>u} g(v)h(v)\|_E \approx \|g(u)h(u)\|_E.$$

As a consequence, we obtain that $(\Lambda_{p,\beta},A)^K_H \supset A_1$, where A_1 is a rearrangement invariant space with the norm

$$\|f\|_{A_1} = \left\|\left(\ln\frac{e}{t}\right)^{\varepsilon}\int_t^1 s^{\frac{1}{p}-1}\left(\ln\frac{e}{s}\right)^{\beta} f^{**}(s)\,ds\right\|_{\widetilde{E}} .$$

On the next step we will compare the space A_1 with spaces of Lorentz-Zygmund type. Consider first a linear operator

$$Pg(u)=(1+u)^\varepsilon\int_0^u (1+v)^{-\varepsilon-1}g(v)\,dv, \qquad 0<u<\infty, \tag{2.7}$$

Lemma 2.3 *Operator (2.7) is bounded on any rearrangement invariant space E such that $q_E<-\varepsilon$.*

Proof. It is easy to check that P is always bounded on L_∞:

$$|Pg(u)| \le \sup_{0<v<u}|g(v)|\cdot(1+u)^\varepsilon\int_0^u(1+v)^{-\varepsilon-1}dv \le \frac{1}{|\varepsilon|}\,\|g\|_{L_\infty}.$$

The following proof is divided into three special cases relative to the value of ε.

If $\varepsilon < -1$ then we have no restrictions for spaces E. In this case

$$\begin{aligned}
\|Pg\|_{L_1} &\leq \int_0^\infty (1+u)^\varepsilon \left(\int_0^u (1+v)^{-\varepsilon-1} |g(v)|\, dv \right) du \\
&= \int_0^\infty (1+v)^{-\varepsilon-1} |g(v)| \left(\int_v^\infty (1+u)^\varepsilon du \right) dv \\
&= \frac{1}{|\varepsilon+1|} \int_0^\infty |g(v)|\, dv,
\end{aligned}$$

which means that P is bounded on L_1. Since we admit as parameters only those spaces E which are interpolation between L_1 and L_∞, the lemma is proved for all considered ε.

If $\varepsilon = -1$ then

$$|Pg(u)| = \left| \frac{1}{1+u} \int_0^u g(v)\, dv \right| \leq \frac{1}{u} \int_0^u |g(v)| dv,$$

i.e. the operator P is dominated by the standard Hardy operator which is bounded on any rearrangement invariant space E with $q_E < 1$ (see, e.g., [4], p. 138). Thus P again is proved to be bounded on all spaces, claimed in the lemma for such an ε.

Let now $\varepsilon \in (-1, 0)$. We set $r = -1/\varepsilon$ and show that the operator (2.7) acts boundedly from $\Lambda_r = \Lambda_r(0,\infty)$ into $M_r = M_r(0,\infty)$. Indeed,

$$|Pg(u)| \leq (1+u)^\varepsilon \|(1+v)^{-\varepsilon-1}\|_{M_{r'}} \|g\|_{\Lambda_r}, \qquad r' = \frac{1}{1+\varepsilon},$$

owing to duality of the spaces Λ_r and $M_{r'}$. Thus

$$\|Pg\|_{M_r} \leq \|(1+u)^\varepsilon\|_{M_r} \|(1+v)^{-\varepsilon-1}\|_{M_{r'}} \|g\|_{\Lambda_r} = \frac{1}{|\varepsilon(1+\varepsilon)|} \|g\|_{\Lambda_r}.$$

Together with boundedness of P on L_∞, this gives that P is bounded on any space E such that the triple (Λ_r, L_∞, E) is interpolation with respect to the triple (M_r, L_∞, E). As follows from [4], p. 133, the condition $q_E < 1/r = -\varepsilon$ is sufficient for this. So the lemma is proved in the last case as well. ∎

Let us take now a function $f \in L_{p,\alpha+1,E}$ and define $g(u) = (1+u)^{\alpha+1} f^{**}(e^{-u}) e^{-u/p}$. Then

$$\|g(u)\|_E = \big\|g\big(\ln\frac{1}{t}\big)\big\|_{\widetilde{E}} = \big\|t^{\frac{1}{p}}\left(\ln\frac{e}{t}\right)^{\alpha+1} f^{**}(t)\big\|_{\widetilde{E}} \approx \|f\|_{L_{p,\alpha+1,E}},$$

i.e. $g \in E$. Lemma 2.3 gives then that $Pg \in E$ too. But

$$\|Pg(u)\|_E = \big\|Pg\big(\ln\frac{1}{t}\big)\big\|_{\widetilde{E}} =$$
$$\|\left(\ln\frac{e}{t}\right)^{\varepsilon}\int_0^{\ln(1/t)}(1+v)^{\beta} f^{**}(e^{-v})e^{-v/p}dv\|_{\widetilde{E}}.$$

Change of variable $e^{-v} = s$ in this integral gives immediately that $\|Pg\|_E = \|f\|_{A_1}$ and thus $f \in A_1$. Since f was taken arbitrarily, this implies that $L_{p,\alpha+1,E} \subset A_1$.

We proceed now to the final part of the proof of Theorem 2.1. Recall that $A_1 \subset (\Lambda_{p,\beta}, A)^K_H$ thus also $L_{p,\alpha+1,E} \subset (\Lambda_{p,\beta}, A)^K_H$. From (2.2) we obtain then that, for any (quasi)linear operator T which is of two weak types (a,b) and (p,q), the following relation holds

$$T:\ L_{p,\alpha+1,E} \longrightarrow (M_{q,\beta}, M_{q,\alpha})^K_H.$$

But the last assertion of Theorem 1.1 says that the smallest space, into which all such operators can act, is the space $B_{q,\alpha+1,E}$ (all needed relations between α and E are valid here). Thus $B_{q,\alpha+1,E} \subset (M_{q,\beta}, M_{q,\alpha})^K_H$, and this is the required inverse embedding that proves the theorem.

■

3 Optimal interpolation for infinite first parameter

As was mentioned in Introduction, the last assertion of Theorem 1.1 was claimed (but not proved correctly) in [7] for the case of $p = \infty$ too. It turns out that this case is very special and requires additional arguments, such as obtained in the previous section.

The quasinorm in the space $L_{\infty,\alpha,E}$ is defined as

$$\|f\|_{L_{\infty,\alpha,E}} = \|\left(\ln\frac{e}{t}\right)^{\alpha} f^*(t)\|_{\widetilde{E}} = \|(1+u)^{\alpha} f^*(e^{-u})\|_E.$$

Being rearrangement invariant on the interval $(0, 1)$, this space should contain all constant functions, thus α must be nonpositive and, moreover, $(1 + u)^\alpha \in E$. For the proof of the theorem below, we need a slightly more restrictive condition $q_E < -\alpha$ (this automatically excludes the case of $\alpha = 0$).

Theorem 3.1. *Let W be the set of all quasilinear operators, which are of "joint weak type $(a, b; \infty, q)$" for some numbers $a, b \in [1, \infty)$ and $q > b$ (as it was defined in Section 1). Let $\alpha < 0$ and $q_E < -\alpha$. Then $T : L_{\infty,\alpha,E} \to B_{q,\alpha,E}$ for any $T \in W$, and $B_{q,\alpha,E}$ is the smallest possible space in such an assertion.*

Proof. The first assertion of the theorem was proved in [7], Theorem 7.1, without any additional difficulties for $p = \infty$; particular problems appear only when proving optimality of the space $B_{q,\alpha,E}$. Notice that if even the smallest possible space in this interpolation is different from $B_{q,\alpha,E}$, it should be rearrangement invariant. Indeed, such a space is necessarily interpolation in its couple (M_b, M_q) (see, e.g., [2]). But, as shown in [4], any space, which is interpolation in a couple of rearrangement invariant spaces, must be of the same kind. Thus the theorem will be proved if we show that, for any function $g \in B_{q,\alpha,E}$, there exist an operator $T \in W$ and a function $f \in L_{\infty,\alpha,E}$ such that $|Tf(t)| \geq g^*(t)$ for all $t \in (0, 1)$.

As in [7], we can solve this problem, using only one operator for all functions g, namely,

$$Tf(t) = t^{-1/q} \int_{t^m}^1 f^*(s) \frac{ds}{s}, \qquad \text{where } \ m = a\left(\frac{1}{b} - \frac{1}{q}\right). \tag{3.1}$$

This operator is the second term of the right-hand side in (1.2), hence it is of "joint weak type $(a, b; \infty, q)$" and thus admissible for our consideration.

For the further proof we need a special "averaging" operator

$$Ug(t) = t^{-1/q} \ln\frac{e}{t} \cdot \sup_{t<s<1} s^{1/q} \left(\ln\frac{e}{s}\right)^{-1} g^{**}(s).$$

Evidently, the operator U is quasilinear and $Ug(t)$ is a continuous decreasing function for every g.

Lemma 3.1. *The operator U is bounded on the space $B_{q,\alpha,E}$.*

Proof. Let us consider this operator on the spaces $M_{q,-1}$ and $M_{q,\alpha-1}$. On the first of them, we have

$$\|Ug\|_{M_{q,-1}} \approx \sup_{0<t<1} t^{1/q} \left(\ln\frac{e}{t}\right)^{-1} (Ug)^*(t)$$
$$= \sup_{0<t<1} \sup_{t<s<1} s^{1/q} \left(\ln\frac{e}{s}\right)^{-1} g^{**}(s).$$

The second supremum here is dominated by the first one and may be omitted, thus $\|Ug\|_{M_{q,-1}} \approx \|g\|_{M_{q,-1}}$, i.e., the operator U is bounded on $M_{q,-1}$. In the same vein,

$$\|Ug\|_{M_{q,\alpha-1}} \approx \sup_{0<t<1} t^{1/q} \left(\ln\frac{e}{t}\right)^{\alpha-1} (Ug)^*(t)$$
$$= \sup_{0<t<1} \left(\ln\frac{e}{t}\right)^{\alpha} \sup_{t<s<1} s^{1/q} \left(\ln\frac{e}{s}\right)^{-1} g^{**}(s)$$
$$\leq \sup_{0<t<1} \sup_{t<s<1} s^{1/q} \left(\ln\frac{e}{s}\right)^{\alpha-1} g^{**}(s) = \|g\|_{M_{q,\alpha-1}},$$

since α is negative. Thus U is bounded on the both spaces $M_{q,-1}$ and $M_{q,\alpha-1}$, which allows us to apply Theorem 2.1, taking $\beta = -1$ and $\alpha - 1$ instead of α. As a consequence, U is bounded on the space $B_{q,\alpha,E}$.

■

Let us return to the proof of Theorem 3.1. For arbitrary given function g, we define a function

$$h(t) = \sup_{0<s<t} s^{1/q} \left(\ln\frac{e}{s}\right)^{\alpha-1} Ug(s), \tag{3.2}$$

which is monotone increasing and continuous for all $t \in (0,1)$. Recall that $q > 1$, hence we may use a result from [7], p. 131, which states that, for any such q, the norm in the space $B_{q,\alpha,E}$ is equivalent to the quasinorm obtained via replacing f^{**} by f^*. Consequently,

$$\|Ug\|_{B_{q,\alpha,E}} \approx \| \sup_{0<s<t} s^{1/q} \left(\ln\frac{e}{s}\right)^{\alpha-1} (Ug)^*(s)\|_{\widetilde{E}} = \|h\|_{\widetilde{E}}.$$

Applying Lemma 3.1, we derive that $h \in \widetilde{E}$ for every $g \in B_{q,\alpha,E}$.

After substitution of the operator U into (3.2), we obtain that

$$h(t) = \sup_{0<s<t} \left(\ln\frac{e}{s}\right)^{\alpha} \widetilde{g}(s), \qquad \text{where } \widetilde{g}(s) = \sup_{s<\tau<1} \tau^{1/q} \left(\ln\frac{e}{\tau}\right)^{-1} g^{**}(\tau),$$

so that $\widetilde{g}(s)$ is a decreasing function. This implies that $h(t)$ can increase only "on account" of the first (logarithmic) factor and, for any $t_2 > t_1$,

$$h(t_2)/h(t_1) \leq \left(\ln\frac{e}{t_2}\right)^{\alpha} \Big/ \left(\ln\frac{e}{t_1}\right)^{\alpha}. \tag{3.3}$$

Indeed, this inequality is trivial if $h(t_2) = h(t_1)$. If ever $h(t_2) > h(t_1)$ then, by continuity of all considered functions, there exists a value $t_0 \in (t_1, t_2]$ such that

$$h(t_2) = \left(\ln\frac{e}{t_0}\right)^{\alpha} \widetilde{g}(t_0) \leq \left(\ln\frac{e}{t_2}\right)^{\alpha} \widetilde{g}(t_1).$$

On the other hand, $h(t_1) \geq \left(\ln\frac{e}{t_1}\right)^{\alpha} \widetilde{g}(t_1)$, which immediately entails the inequality (3.3).

Using this inequality, we are able to estimate the derivative $h'(t)$ that exists almost everywhere due to monotonicity of $h(t)$. For any point t, where $h'(t)$ exists, we can write that

$$\begin{aligned} h'(t) &= \lim_{\Delta t \to +0} \frac{h(t+\Delta t) - h(t)}{\Delta t} \\ &\leq h(t)\left(\ln\frac{e}{t}\right)^{-\alpha} \lim_{\Delta t \to +0} \frac{1}{\Delta t}\left[\left(\ln\frac{e}{t+\Delta t}\right)^{\alpha} - \left(\ln\frac{e}{t}\right)^{\alpha}\right] \\ &= h(t)\left(\ln\frac{e}{t}\right)^{-\alpha}\left[\left(\ln\frac{e}{t}\right)^{\alpha}\right]' = -\frac{\alpha h(t)}{t\left(\ln\frac{e}{t}\right)}. \end{aligned}$$

On the last step we define the function $f(t) = \left(\ln\frac{e}{t}\right)^{-\alpha} h(t)$, for which we have

$$\begin{aligned} f'(t) &= -\alpha\left(\ln\tfrac{e}{t}\right)^{-\alpha-1}\left(-\tfrac{1}{t}\right) h(t) + \left(\ln\tfrac{e}{t}\right)^{-\alpha} h'(t) \\ &\leq \tfrac{h(t)}{t}\left(\ln\tfrac{e}{t}\right)^{-\alpha-1}(\alpha - \alpha) = 0, \end{aligned}$$

i.e., f is decreasing and $f^* = f$. Thus $\|f\|_{L_{\infty,\alpha,E}} = \|\left(\ln\frac{e}{t}\right)^{\alpha} f(t)\|_{\widetilde{E}} = \|h\|_{\widetilde{E}} < \infty$ and $Tf \in B_{q,\alpha,E}$ for the operator T, defined by (3.1). On the other hand,

$$\begin{aligned} Tf(t) &= t^{-1/q}\int_{t^m}^{1}\left(\ln\frac{e}{s}\right)^{-\alpha} h(s)\frac{ds}{s} \\ &\geq \tfrac{\min(1,m)}{1-\alpha}\, t^{-1/q} h(t)\left[\left(\ln\tfrac{e}{t^m}\right)^{1-\alpha} - 1\right]. \end{aligned}$$

Considering only $t \leq e^{-1/m}$, we obtain that

$$\begin{aligned}&\left(\ln\tfrac{e}{t^m}\right)^{1-\alpha} - 1 = \left(\ln\tfrac{e}{t^m}\right)^{1-\alpha}\left[1-\left(\ln\tfrac{e}{t^m}\right)^{\alpha-1}\right]\\ &\geq \left(1-2^{\alpha-1}\right)\left(\ln\tfrac{e}{t^m}\right)^{1-\alpha} \approx \left(\ln\tfrac{e}{t}\right)^{1-\alpha},\end{aligned}$$

i.e., for all such t,

$$Tf(t) \gtrsim t^{-1/q}\left(\ln\frac{e}{t}\right)^{1-\alpha} h(t) \geq t^{-1/q}\, \widetilde{g}(t)\ln\frac{e}{t} \geq g^{**}(t) \geq g^{*}(t).$$

This means that the function $g^*(t)\chi_{(0,e^{-1/m}]}(t)$ belongs to any rearrangement invariant space, into which the operator (3.1) is acting. But for all remaining $t > e^{-1/m}$, the function $g^*(t)\chi_{(e^{-1/m},1)}(t)$ is bounded and thus belongs to *any* rearrangement invariant space on the interval $(0, 1)$. Since the function $g(t)$ was taken arbitrarily, this proves the theorem.

∎

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Recibido: 26 de Junio de 2000
Revisado: 19 de Octubre de 2000