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COMPLEX SPRAYS AND COMPLEX CURVES

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Abstract

After defining what is meant by a complex spray X on a complex manifold M, we introduce the notion of a *spray complex curve* associated to X. Several equivalent formulations are derived and we give necessary and sufficient conditions for M to admit spray complex curves for X through each point and in each direction. Refinements of this result are then derived for some special cases, for example when X is the horizontal radial vector field associated to a complex Finsler metric.

1 Introduction

Let F be a complex Finsler metric on a complex manifold M. A prominent rôle in the investigation of the complex geometry of F is played by complex curves which are in some sense compatible with F. For example, the complex geodesics of Vesentini [13] and the stationary discs of Lempert [8, 9, 10] in the study of the Kobayashi and Carathéodory metrics of a domain M in $\mathbb{C}^n$. Following Lempert's results on the Kobayashi metric for a strongly convex domain, a greater interest has been focused on the differential-geometric aspects of such curves. Here we have the totally geodesic curves of Royden [12], Faran [4] and later Pang [11], and the *geodesic complex curves* of Abate and Patrizio [2]. The curves used by the last three listed authors have been shown to be special cases of the notion of a *horizontal complex curve (or h.c.c.)* for a complex Finsler metric in [15]. Although h.c.c.'s appear in a disguised form in Faran's paper [4], he neither interprets them geometrically nor develops their properties. The main existence theorem for the various complex curves encountered in [11, 2, 15] features an integrability criterion which involves the Lie bracket of a certain vector field X with its conjugate

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(Faran takes a dual approach involving differential forms). In the case of [11], X is the geodesic spray of F and is a vector field on the cotangent bundle: in [2] and [15], X is the horizontal radial vector field of F and is a vector field on the holomorphic tangent bundle $\mathcal{O}M$. Two features common to [11, 2, 15] are

(a) the vector field X corresponds to a real spray on M which satisfies a homogeneity condition with respect to complex multiplication in the fibres of the bundle on which X is defined and

(b) if the projection to M of an integral path of X meets one of the relevant complex curves tangentially then the projected path actually stays within that curve near the point of contact. (These projected paths are the *geodesics* in [11, 2] and the *horizontal paths* in [15].)

In this paper we take these two features as our starting point to first define the notions of a *complex spray* and of a *spray complex curve* (or *s.c.c.* for short). In deriving several equivalent characterisations of s.c.c.'s we prove some of their elementary properties. Next we consider the question of existence of spray complex curves. If for each $(z, v) \in \mathcal{O}M$ there exists a spray complex curve for X through z and tangent to v then we call X *integrable*. Our main result (theorem 3.7) gives necessary and sufficient conditions for a complex spray to be integrable. We also show that under these conditions each $z \in M$ has a neighbourhood U_z for which $U_z \backslash \{z\}$ is foliated by those spray complex curves that pass through z. In the final section we specialise to *metrical spray complex curves* (in which case the domain of the complex curve possesses a compatible Hermitian metric) and then to *horizontal complex curves* (in which case the spray X arises from a complex Finsler metric). Our main goal in each case is the derivation of an appropriate existence theorem. The resulting theorem 4.18 for horizontal complex curves was announced in [15] (although it was not proven there in full generality).

2 Complex Sprays

In what follows, M denotes an n-dimensional complex manifold. Its holomorphic tangent bundle $\mathcal{O}M$ is a $2n$-dimensional complex manifold and the natural projection $\pi\colon \mathcal{O}M \to M$ is a holomorphic submersion. The bundle $\mathcal{O}M$ is naturally identified with the sub-bundle $T^{1,0}M$ of the complexified tangent bundle $TM \equiv T^{\mathbb{R}}M \otimes \mathbb{C}$, where $T^{\mathbb{R}}M$ denotes the real tangent bundle of M and $T^{1,0}M$ denotes the i-eigenspace of the complex structure J in TM ($\mathrm{i} = \sqrt{-1}$). If $f\colon M \to N$ is a smooth map, then its tangent mapping $f_*\colon TM \to TN$ is complex linear on fibres and f_* restricts to give the usual tangent mapping from $T^{\mathbb{R}}M$ to $T^{\mathbb{R}}N$. Moreover, f is holomorphic if and only if $f_*(T^{1,0}M) \subset T^{1,0}N$. A reference for the basic theory (with a different notation) is [6]. We use $\mathcal{O}M_o$ to denote the complement of the zero section in $\mathcal{O}M$. The holomorphic vertical bundle $V\mathcal{O}M$ of M is the rank-n, holomorphic sub-bundle of $\mathcal{O}(\mathcal{O}M)$ given by

$$V\mathcal{O}M = \ker(\pi_*)$$

where $\pi_*\colon \mathcal{O}(\mathcal{O}M) \to \mathcal{O}M$ is the derivative of π. The Einstein summation convention (under which repeated indices are summed over their common range) is to be understood unless stated otherwise.

The following definition is a natural complex version of the usual notion of a spray.

Definition 2.1. *A complex spray on a complex manifold M is a vector field X of type $(1,0)$ on the holomorphic tangent bundle $\mathcal{O}M$ satisfying*

(i) $\pi_*(X_{(z,v)}) = v$ *for all* $(z,v) \in \mathcal{O}M$*, where* $\pi\colon \mathcal{O}M \to M$ *is the natural projection, and*

(ii) $(\mu_a)_*(X_{(z,v)}) = \frac{1}{a} X_{(z,av)}$ $\forall a \in \mathbb{C} \setminus \{0\}$*, where* $\mu_a\colon \mathcal{O}M \to \mathcal{O}M$ *is scalar multiplication by a in each fibre. That is* $\mu_a(z,v) = (z,av)$*.*

If (z^i) are local complex coordinates on M and (z^i, v^i) are the corresponding coordinates induced on $\mathcal{O}M$, then

$$\left\{ \frac{\partial}{\partial v^1}, \dots, \frac{\partial}{\partial v^n} \right\}$$

is a local basis for $V\mathcal{O}M$ and it is easy to verify that the coordinate expression for a complex spray X is given by

$$X_{(z,v)} = v^j \left.\frac{\partial}{\partial z^j}\right|_{(z,v)} - b^k(z,v) \left.\frac{\partial}{\partial v^k}\right|_{(z,v)}$$

where each function b^k is homogeneous of degree $(2,0)$ with respect to v i.e. $b^k(z,av) = a^2 b^k(z,v)$ for all $a \in \mathbb{C}$. Each function b^k is as smooth as is X.

Example 2.2. Let N be a complex homogeneous nonlinear connection on M as defined in [15]. That is, N is a rank-n sub-bundle of $\mathcal{O}(\mathcal{O}M)$ such that

(a) $\mathcal{O}_{(z,v)}(\mathcal{O}M) = N_{(z,v)} \oplus V_{(z,v)}\mathcal{O}M$ for each $(z,v) \in \mathcal{O}M$.

(b) $N_{(z,0)} = \mathcal{O}_z M$ for each $z \in M$ (where we identify M with the image of the zero section of $\mathcal{O}M$).

(c) $(\mu_a)_* (N_{(z,v)}) = N_{(z,av)}$ for each $(z,v) \in \mathcal{O}M$.

With respect to local coordinates $(z^1, \dots, z^n, v^1, \dots, v^n)$ as above on $\mathcal{O}M$, $N_{(z,v)}$ has a basis of the form

$$\left\{ \left.\frac{\partial}{\partial z^j}\right|_{(z,v)} - N_j^k(z,v) \left.\frac{\partial}{\partial v^k}\right|_{(z,v)} \right\}_{j=1}^{n}$$

The condition *(c)* ensures that the functions N_j^i are homogeneous of degree $(1,0)$ in the v variable. That is,

$$N_j^i(z,av) = aN_j^i(z,v) \quad \forall a \in \mathbb{C}.$$

The *horizontal radial vector field* X associated to N is the vector field on $\mathcal{O}M$ which is given locally by

$$X_{(z,v)} = v^j \left(\left.\frac{\partial}{\partial z^j}\right|_{(z,v)} - N_j^k(z,v) \left.\frac{\partial}{\partial v^k}\right|_{(z,v)} \right)$$

The vector field X is a complex spray on M.

In particular, a type (1,0) complex affine connection ∇ on M gives rise to a homogeneous nonlinear connection N whose coefficients $\{N_j^i\}$ are given by

$$N_j^i(z,v) = \Gamma_{kj}^i(z) v^k$$

where $\Gamma_{jk}^i(z)$ are the connection coefficients of ∇ with respect to (z^i) and are given by

$$\nabla_{\frac{\partial}{\partial z^k}} \left(\frac{\partial}{\partial z^j} \right) = \Gamma_{jk}^i(z) \frac{\partial}{\partial z^i}$$

The corresponding complex spray on M then has the local form

$$X_{(z,v)} = v^j \left(\frac{\partial}{\partial z^j}\bigg|_{(z,v)} - \Gamma^k_{ij}(z) v^i \, \frac{\partial}{\partial v^k}\bigg|_{(z,v)} \right).$$

Example 2.3. Let F be a strongly pseudoconvex complex Finsler metric on M (see [2]). Then $G = F^2$ is a positive-valued function on $\mathcal{O}M$ which is homogeneous of degree $(1, 1)$ in the v variable and the Hermitian matrix

$$[G_{i\bar{j}}(z, v)] = \left[\frac{\partial^2 G}{\partial v^i \partial \overline{v}^j} \right]$$

is defined and positive definite (for each nonzero (z, v)). We then obtain (see [2] for details) a homogeneous nonlinear connection N on M whose coefficients N^i_j are given by

$$N^i_j(z, v) = G^{i\bar{k}} G_{\bar{k};j}$$

where $[G^{i\bar{k}}]$ is the matrix inverse of $[G_{i\bar{k}}(z, v)]$ and

$$G_{\bar{k};j} = \frac{\partial^2 G}{\partial \bar{v}^k \partial z^j}$$

The corresponding complex spray X is called the *horizontal radial vector field associated to F*.

Example 2.4. For a complex spray X on a 1-dimensional complex manifold M we have

$$X_{(w,u)} = u \, \frac{\partial}{\partial w}\bigg|_{(w,u)} - b(w, u) \, \frac{\partial}{\partial u}\bigg|_{(w,u)}$$

where w is a local complex coordinate on M, (w, u) are the induced coordinates on $\mathcal{O}M$ and where $b(w, u) = u^2 b(w, 1)$. Therefore, setting $\Gamma(w) = b(w, 1)$ we have

$$X_{(w,u)} = u \, \frac{\partial}{\partial w}\bigg|_{(w,u)} - u^2 \Gamma(w) \, \frac{\partial}{\partial u}\bigg|_{(w,u)}.$$

Moreover, it is possible to define a type (1,0) complex affine connection on M by piecing together the local formulae

$$\nabla_{\frac{\partial}{\partial w}}\left(\frac{\partial}{\partial w}\right) = \Gamma(w)\frac{\partial}{\partial w}.$$

Thus each complex spray on a 1-dimensional complex manifold arises from a type-$(1,0)$ complex affine connection.

Definition 2.5. *Let X be a complex spray on M and β an integral path of X in $\mathcal{O}M$ which starts at (z,v). Then the path $\gamma = \pi \circ \beta$ is called a* horizontal path (for X) in M*. The path γ starts at z and has initial velocity v.*

Note 2.6. In terms of local coordinates (z^j), one may verify that a path γ is horizontal for a spray $X = v^i \frac{\partial}{\partial z^i} - b^i(z,v)\frac{\partial}{\partial v^i}$ if and only if the functions $\gamma^i(t) = z^i(\gamma(t))$ are solutions to the system of equations

$$\ddot{\gamma}^j(t) + b^j(\gamma^i(t), \dot{\gamma}^i(t)) = 0 \quad \text{for } j = 1, \dots, n.$$

Note 2.7. Local existence and uniqueness of integral paths for a $\mathcal{C}^1$ vector field ensures that if X is a $\mathcal{C}^1$ complex spray on M then for each $v \in \mathcal{O}_z M$ there exists a unique horizontal path for X starting at z with velocity v.

Note 2.8. The horizontal paths for the horizontal radial vector field of a complex Finsler metric F are called the horizontal paths of F. In general they will differ from the geodesics of F. However, if F is weakly Kähler, a generalisation of the usual Kähler condition for Hermitian metrics (see [2]), then each horizontal path for F will be a geodesic. In particular, if F^2 is a Hermitian metric, then its horizontal paths will coincide with its geodesics if and only if F^2 is Kähler.

Terminology. For a real spray on a real manifold the paths corresponding to our horizontal paths are often referred to as geodesics. We prefer to reserve the term *geodesic* to denote those paths in M which are extremal for the energy integral of a given complex Finsler metric. Our use is consistent with the fact that the horizontal paths of a complex Finsler metric do not always coincide with its geodesics.

3 Spray Complex Curves

In what follows, we will denote the *vertical radial vector field of M* by Y. With respect to induced local coordinates (z^i, v^i) the vector field Y is given by

$$Y_{(z,v)} = v^i \left.\frac{\partial}{\partial v^i}\right|_{(z,v)}$$

For a *Riemann surface* ($\equiv$ a connected, one-dimensional complex manifold) we will use χ to denote complex sprays and η to denote the vertical radial vector field.

We now introduce the notion of a spray complex curve associated to a complex spray.

Definition 3.1. *Let X be a complex spray on a complex manifold M and let $\phi\colon W \to M$ be a holomorphic mapping of a Riemann surface W into M. Then ϕ is called a* spray complex curve for X *(abbreviated* s.c.c.*) if*

(i) *ϕ is an immersion and*

(ii) $X_{\phi_*(w,u)} \in \phi_{**}(\mathcal{O}_{(w,u)}(\mathcal{O}W))$ *for all* $(w,u) \in \mathcal{O}W$.

When X is understood, we simply say that ϕ is a spray complex curve.

Remark 3.2. If $\phi\colon W \to M$ is an s.c.c. for X and if $f\colon U \to W$ is a holomorphic immersion, then $\phi \circ f$ is also an s.c.c. for X. In particular, this holds if $f\colon U \to W$ is the universal covering space of W. Thus, if desired, we may replace any spray complex curve for X by another whose image is the same subset of M but whose domain is a simply-connected Riemann surface.

The following proposition gives some equivalent formulations of the notion of a spray complex curve for X.

Proposition 3.3. *Let X be a complex spray on M and $\phi\colon W \to M$ a holomorphic immersion (W a Riemann surface). For each $w \in W$ let U_w be an open neighbourhood of w on which ϕ is a holomorphic embedding. Then the following conditions are equivalent*

(i) $X_{\phi_*(w,u)} \in \phi_{**}(\mathcal{O}_{(w,u)}(\mathcal{O}W)) \quad \forall (w,u) \in \mathcal{O}W$

(ii) *For each $w \in W$ there exists $u \in (\mathcal{O}_w W) \setminus \{0\}$ such that* $X_{\phi_*(w,u)} \in \phi_{**}(\mathcal{O}_{(w,u)}(\mathcal{O}W))$

(iii) *For each $w \in W$ there exists $u \in (\mathcal{O}_w W) \setminus \{0\}$ such that*

$$\phi_{**}(\mathcal{O}_{(w,u)}(\mathcal{O}W)) = \phi_{**}(V_{(w,u)}(\mathcal{O}W)) \oplus \mathbb{C}X_{\phi_*(w,u)}.$$

(iv) *For each nonzero $(w,u) \in \mathcal{O}W$,*

$$\phi_{**}(\mathcal{O}_{(w,u)}(\mathcal{O}W)) = \phi_{**}(V_{(w,u)}(\mathcal{O}W)) \oplus \mathbb{C}X_{\phi_*(w,u)}.$$

(v) *There exists a complex spray χ on W such that $\phi_{**}(\chi_{(w,u)}) = X_{\phi_*(w,u)}$ for all $(w,u) \in \mathcal{O}W$.*

(vi) *Each horizontal path $\gamma\colon (-\epsilon,\epsilon) \to M$ through $\gamma(0) = \phi(w)$ which is tangent to $\phi(U_w)$ at $t = 0$ remains in $\phi(U_w)$ for $|t|$ sufficiently small.*

(vii) *For each $w \in W$ there exists a horizontal path $\gamma\colon (-\epsilon,\epsilon) \to M$ such that γ is nonconstant, $\gamma(0) = \phi(w)$, and $\gamma(t) \in \phi(U_w)$ for $|t|$ sufficiently small.*

Proof. Clearly *(i)* implies *(ii)*.
(ii) ⇒ (iii). Assume that *(ii)* holds. Then for each w there exists $u \in (\mathcal{O}_w W) \setminus \{0\}$ such that

$$X_{\phi_*(w,u)} \in \phi_{**}(\mathcal{O}_{(w,u)}(\mathcal{O}W)).$$

Therefore

$$\phi_{**}(V_{(w,u)}\mathcal{O}W) \oplus \mathbb{C}X_{\phi_*(w,u)} \subset \phi_{**}(\mathcal{O}_{(w,u)}(\mathcal{O}W)).$$

But the left hand side is 2-dimensional over $\mathbb{C}$ (because $\phi_{**}(V\mathcal{O}W)$ is nonzero and is contained in the vertical sub-bundle of $\mathcal{O}M$ which is everywhere linearly independent of X). Since the right side has dimension at most 2, this inclusion must actually be an equality. Hence *(iii)* holds.

(iii) ⇒ (iv). Assume *(iii)* is true and choose a nonzero element (w,u) of $\mathcal{O}W$ such that

$$\mathbb{C}X_{\phi_*(w,u)} \oplus \phi_{**}(V_{(w,u)}\mathcal{O}W) = \phi_{**}(\mathcal{O}_{(w,u)}(\mathcal{O}W)).$$

Denote by μ_a the map of $\mathcal{O}M$ to itself given by multiplication in each fibre by $a \in \mathbb{C}\setminus\{0\}$. Then, since $\frac{1}{a}X_{\phi_*(w,au)} = (\mu_a)_*(X_{\phi_*(w,u)})$, it follows that at $\phi_*(w,au)$ we have

$$\begin{aligned}
\mathbb{C}X \oplus \phi_{**}(V_{(w,au)}\mathcal{O}W) &= (\mu_a)_*\mathbb{C}X_{\phi_*(w,u)} \oplus \phi_{**}(\mu_a)_*(V_{(w,u)}\mathcal{O}W) \\
&= (\mu_a)_*\mathbb{C}X_{\phi_*(w,u)} \oplus (\mu_a)_*\phi_{**}(V_{(w,u)}\mathcal{O}W) \\
&= (\mu_a)_*(\phi_{**}(\mathcal{O}_{(w,u)}(\mathcal{O}W))) \quad \text{by hypothesis} \\
&= \phi_{**}(\mu_a)_*(\mathcal{O}_{(w,u)}(\mathcal{O}W)) \\
&= \phi_{**}(\mathcal{O}_{(w,au)}(\mathcal{O}W))
\end{aligned}$$

where the second and fourth equalities both use the fact that $\phi_* \circ \mu_a = \mu_a \circ \phi_*$ (by holomorphy of ϕ). Hence

$$\mathbb{C}X_{\phi_*(w,u)} \oplus \phi_{**}(V_{(w,u)}\mathcal{O}W) = \phi_{**}(\mathcal{O}_{(w,u)}(\mathcal{O}W))$$

for all nonzero (w,u), and we have shown that *(iii)* implies *(iv)*.

(iv) $\Rightarrow$ *(i)*. Obvious.

Thus properties *(i)* through *(iv)* are equivalent.

(iv) $\Rightarrow$ *(v)*. Assume that

$$\phi_{**}(\mathcal{O}_{(w,u)}\mathcal{O}W) = \phi_{**}(V_{(w,u)}\mathcal{O}W) \oplus \mathbb{C}X_{\phi_*(w,u)}$$

for all $(w,u) \in \mathcal{O}W_o$. Therefore there exists a unique vector $\chi_{(w,u)}$ in $\mathcal{O}_{(w,u)}\mathcal{O}W$ such that $\phi_{**}(\chi_{(w,u)}) = X_{\phi_*(w,u)}$ for each nonzero (w,u). Set $\chi_{(w,0)} = 0$. We need to verify that χ is a $\mathcal{C}^1$ complex spray. First,

$$\begin{aligned}
\phi_*\pi_*(\chi_{(w,u)}) &= \pi_*\phi_{**}(\chi_{(w,u)}) \\
&= \pi_*(X_{\phi_*(w,u)}) \qquad \text{by definition of } \chi \\
&= \phi_*(w,u) \qquad \text{since } X \text{ is a complex spray.}
\end{aligned}$$

But by assumption, $\phi_*|_{\mathcal{O}_wW}$ is injective. Therefore $\pi_*(\chi_{(w,u)}) = u \in \mathcal{O}_wW$.

That $(\mu_a)_*(\chi_{(w,u)}) = \frac{1}{a}\chi_{(w,au)}$ for $a \in \mathbb{C}\setminus\{0\}$ and $(w,u) \in \mathcal{O}W_o$ follows by applying ϕ_{**} separately to each side of this desired equation and using the homogeneity of X to show that the two resulting expressions are equal. Then the desired equality follows by applying the definition of χ. Finally, since X is $\mathcal{C}^1$ and both ϕ_* and ϕ_{**} are $\mathcal{C}^\infty$ it follows from

$$\chi_{(w,u)} = (\phi_{**})^{-1}(X_{\phi_*(w,u)})$$

that χ is also $\mathcal{C}^1$ for nonzero (w, u). Since χ is a complex spray on a 1-dimensional complex manifold, this is equivalent to χ being $\mathcal{C}^1$ everywhere. Thus *(iv)* implies *(v)*.

(v) $\Rightarrow$ *(vi)*. Assume that *(v)* holds and fix $u \in \mathcal{O}_w W \setminus \{0\}$. Let σ be the horizontal path for χ through w with initial velocity u and let γ be the horizontal path for X in M with $\gamma(0) = \phi(w)$ and $\gamma'(0) = \phi_*(u)$. Since σ' is an integral path for χ and $\phi_{**}(\chi) = X$ it follows that $\phi_* \circ (\sigma')$ is an integral path for X. Therefore $\pi \circ \phi_* \circ (\sigma') = \phi \circ \sigma$ is a horizontal path for X through $\phi(w)$ with initial velocity $\phi_*(\sigma'(0)) = \phi_*(u)$. The uniqueness of horizontal paths for X implies that $\gamma(t) = \phi(\sigma(t))$ for $|t|$ sufficiently small and hence $\gamma(t)$ lies in $\phi(U_w)$ for $|t|$ sufficiently small. Since $(w, u) \in \mathcal{O}W$ is arbitrary, we see that property *(vi)* holds.

(vi) $\Rightarrow$ *(vii)*. Obvious.

(vii) $\Rightarrow$ *(ii)*. Assume that *(vii)* holds. Let $\gamma\colon (-\epsilon, \epsilon) \to M$ be a nonconstant horizontal path through $\phi(w)$ and contained in $\phi(U_w)$. Then $\gamma = \phi \circ \sigma$ for some path σ through w in W. Therefore $\gamma' = \phi_* \circ \sigma' \in \mathcal{O}_\gamma M$ and $\gamma'' = (\gamma')'\colon (-\epsilon, \epsilon) \to \mathcal{O}(\mathcal{O}M)$ is given by

$$\gamma'' = \phi_{**} \circ (\sigma'').$$

Since $\gamma'' = X_{\gamma'}$ (γ being a horizontal path for X) we obtain

$$X_{\gamma'} \in \phi_{**}(\mathcal{O}_{\sigma'}(\mathcal{O}W)).$$

Evaluating at $t = 0$ and observing that $\sigma'(0) \neq 0$ (because $\gamma'(0) \neq 0$) we find that condition *(ii)* holds at the point $\sigma'(0) \in \mathcal{O}_w W \setminus \{0\}$.

It is now clear that all seven properties are equivalent. q.e.d.

Remark 3.4. If $\phi\colon W \to M$ is a holomorphic immersion (W a Riemann surface) and $w \in W$ has a neigbourhood U_w such that $\phi(U_w)$ is contained in the union of the horizontal segments through $\phi(w)$ which are tangent to $\phi(U_w)$ at w, then $\phi|_{U_w}\colon U_w \to M$ is an s.c.c. for X. This can be seen by noting that our hypothesis implies that condition *(vii)* of proposition 3.3 is valid at each point of U_w.

We now turn our attention to existence conditions for spray complex curves.

Definition 3.5. *A complex spray X on M is* integrable *if for each $(z, v) \in \mathcal{O}M_o$ there exists a spray complex curve $\phi\colon W \to M$ (depending on (z, v)) for X such that $(z, v) \in \phi_*(\mathcal{O}W)$.*

Informally, X is integrable if for each $(z, v) \in \mathcal{O}M$ one can find an s.c.c. for X which is tangent to v at z.

If $\phi\colon W \to M$ is an s.c.c. for X, we saw in proposition 3.3 that there exists a complex spray χ on W which is ϕ_*-related to X. Hence the Lie bracket $[\chi, \overline{\chi}]$ is ϕ_*-related to $[X, \overline{X}]$. But an easy calculation shows that any $\mathcal{C}^1$ complex spray χ on W satisfies an equation of the form

$$[\chi, \overline{\chi}] = \alpha\eta - \overline{\alpha\eta}$$

for some continuous complex-valued function α (where η is the vertical radial vector field of W, as usual). Applying ϕ_{**} to both sides of this equation and using the identity $\phi_{**}\eta = Y_{\phi_*}$ which is true for any holomorphic map ϕ, we obtain

$$[X, \overline{X}]_{\phi_*(w,u)} = \alpha(w, u)Y_{\phi_*(w,u)} - \overline{\alpha}(w, u)\overline{Y}_{\phi_*(w,u)}.$$

Thus we deduce that a necessary condition for X to be integrable is that

$$[X, \overline{X}] = fY - \overline{fY}$$

for some continuous complex-valued function f on $\mathcal{O}M_o$. The converse to this is also true. Our proof relies on the following variant of Frobenius's theorem which characterises when a sub-bundle of the tangent bundle TA (of a complex manifold A) arises from a foliation whose leaves are complex submanifolds of A.

Frobenius's Theorem. *Let E be a $\mathcal{C}^k$, rank r complex sub-bundle of the tangent bundle TA of a complex manifold A. Assume that*

(i) $\overline{E} = E$

(ii) $[\Gamma^k(E), \Gamma^k(E)] \subset \Gamma^{k-1}(E)$*, where* $\Gamma^p(E)$ *denotes the set of* $\mathcal{C}^p$ *sections of E*

(iii) $J(\mathrm{Re}(E)) = \mathrm{Re}(E)$ *where* $\mathrm{Re}(E) = \{v \in E : \overline{v} = v\} = E \cap T^{\mathbb{R}}A$ *and* $J\colon TA \to TA$ *is the complex structure tensor of A.*

Then E is the tangent bundle of a $\mathcal{C}^k$ foliation of A by $2r$-dimensional manifolds. Moreover, each leaf of this foliation is actually a complex r-dimensional submanifold of A.

Proof. The first assertion follows because conditions *(i)* and *(ii)* ensure that the usual version of Frobenius's theorem for real manifolds applies. If L is a leaf of the resulting foliation then

$$J(T_a^{\mathbb{R}}L) = J(\mathrm{Re}(E_a)) = \mathrm{Re}(E_a) = T_a^{\mathbb{R}}L \quad \forall a \in L$$

It is standard (see [5] for example) that this implies that L is a complex submanifold of A. q.e.d.

The next proposition collects some identities which are valid for any complex spray and which we use in proving the above-mentioned converse.

Proposition 3.6. *Let X be a $\mathcal{C}^1$ complex spray on M. Then*

(a) $[X,Y] = -X$

(b) $[X,\overline{Y}] = 0$

(c) $[X,\overline{X}]$ *is a vertical vector field. That is,* $[X,\bar{X}] \in V\mathcal{O}M \oplus \overline{V\mathcal{O}M}$

Proof. With respect to local complex coordinates (z^i) on M we may write

$$X_{(z,v)} = v^i\frac{\partial}{\partial z^i} - b^i\frac{\partial}{\partial v^i}$$

where $b^i(z,v)$ is $(2,0)$-homogeneous in v. Thus

$$\begin{aligned}
[X,Y] &= [v^j\frac{\partial}{\partial z^j} - b^j\frac{\partial}{\partial v^j}, v^k\frac{\partial}{\partial v^k}] \\
&= [v^j\frac{\partial}{\partial z^j}, v^k\frac{\partial}{\partial v^k}] - [b^j\frac{\partial}{\partial v^j}, v^k\frac{\partial}{\partial v^k}] \\
&= -v^j\frac{\partial}{\partial z^j} - (b^j\frac{\partial}{\partial v^j} - v^k\frac{\partial b^j}{\partial v^k}\frac{\partial}{\partial v^j}) \\
&= -v^j\frac{\partial}{\partial z^j} - (b^j\frac{\partial}{\partial v^j} - 2b^j\frac{\partial}{\partial v^j}) \text{ by homogeneity of } b^j \\
&= -(v^j\frac{\partial}{\partial z^j} - b^j\frac{\partial}{\partial v^j}) \\
&= -X.
\end{aligned}$$

For *(b)*, we have

$$\begin{aligned}[X,\bar{Y}] &= [v^j\frac{\partial}{\partial z^j},\bar{v}^k\frac{\partial}{\partial \bar{v}^k}] - [b^j\frac{\partial}{\partial v^j},\bar{v}^k\frac{\partial}{\partial \bar{v}^k}]\\ &= 0-\left(b^j\frac{\partial \bar{v}^k}{\partial v^j}\frac{\partial}{\partial \bar{v}^k} - \bar{v^k}\frac{\partial b^j}{\partial \bar{v}^k}\frac{\partial}{\partial v^j}\right)\\ &= 0\end{aligned}$$

because $\bar{v^k}\dfrac{\partial b^j}{\partial \bar{v}^k}=0$ by the $(2,0)$-homogeneity of $b^j(z,v)$ in the variable v.

(c). From $X = v^j\frac{\partial}{\partial z^j} - b^j\frac{\partial}{\partial v^j}$ we get

$$\begin{aligned}[X,\bar{X}] &= [v^j\frac{\partial}{\partial z^j} - b^j\frac{\partial}{\partial v^j},\bar{v}^k\frac{\partial}{\partial \bar{z}^k} - \bar{b}^k\frac{\partial}{\partial \bar{v}^k}]\\ &= 0-[b^j\frac{\partial}{\partial v^j},\bar{v}^k\frac{\partial}{\partial \bar{z}^k}] - [v^j\frac{\partial}{\partial z^j},\bar{b}^k\frac{\partial}{\partial \bar{v}^k}] + [b^j\frac{\partial}{\partial v^j},\bar{b}^k\frac{\partial}{\partial \bar{v}^k}]\\ &= \bar{v}^k\frac{\partial b^j}{\partial \bar{z}^k}\frac{\partial}{\partial v^j} - v^j\frac{\partial \bar{b}^k}{\partial z^j}\frac{\partial}{\partial \bar{v}^k} + b^j\frac{\partial \bar{b}^k}{\partial v^j}\frac{\partial}{\partial \bar{v}^k} - \bar{b}^k\frac{\partial b^j}{\partial \bar{v}^k}\frac{\partial}{\partial v^j}\\ &= \bar{X}(b^j)\frac{\partial}{\partial v^j} - X(\bar{b^j})\frac{\partial}{\partial \bar{v}^j}\end{aligned}$$

which is clearly vertical. q.e.d.

For a complex manifold M, denote by $\mathbb{P}M$ the holomorphic fibre bundle over M whose fibre over $z\in M$ is the space of complex lines in $\mathcal{O}_zM$. If $(z,v)\in\mathcal{O}M$ is nonzero we will denote the complex line $\mathbb{C}v\subset\mathcal{O}_zM$ by $(z,[v])\in\mathbb{P}M$. The map $q\colon\mathcal{O}M_o\to\mathbb{P}M$ defined by

$$q(z,v)=(z,[v])$$

is a holomorphic submersion and the kernel of $q_*\colon T^{1,0}(\mathcal{O}M_o)\to T^{1,0}(\mathbb{P}M)$ is spanned by Y. Let $p\colon\mathbb{P}M\to M$ be the holomorphic projection given by $p(z,[v])=z$. Then

$$\pi = p\circ q\colon\mathcal{O}M_o\to M$$

where we recall that $\pi\colon\mathcal{O}M_o\to M$ is the natural projection.

Theorem 3.7. *Let X be a $\mathcal{C}^1$ complex spray on the complex manifold M. Assume that*

$$[X,\overline{X}] = fY-\overline{fY}$$

for some complex-valued continuous function f on $\mathcal{O}M_o$. Then X is integrable. In addition, the spray complex curve $\phi\colon W \to M$ through a fixed point $(z_0, v_0) \in \mathcal{O}M_o$ can be chosen to satisfy

(i) *the induced map*

$$\Phi\colon W \to \mathbb{P}M$$

defined by

$$w \mapsto q(\phi(w), \phi_*(\mathcal{O}_w W))$$

is injective

(ii) *ϕ is maximal in the sense that if $\psi\colon U \to M$ is another spray complex curve for X with $(z_0, v_0) \in \psi_*(\mathcal{O}U)$ then*

$$\Psi(U) \subset \Phi(W)$$

where $\Psi(u) = q(\psi(u), \psi_*(\mathcal{O}_u U))$ for each $u \in U$. In particular, $\psi(U)$ is contained in $\phi(W)$.

Moreover, if $\gamma\colon (a,b) \to M$ is any horizontal path for X, then $\gamma((a,b))$ is contained in the image of a spray complex curve for X.

Proof. The idea of the proof is to show that X determines a $\mathcal{C}^1$ foliation of $\mathbb{P}M$ by Riemann surfaces and to show that the p-image of each such surface is a spray complex curve.

Let $\mathcal{E}$ be the $\mathcal{C}^1$ sub-bundle of $T(\mathcal{O}M_o)$ spanned by X, $\overline{X}$, Y and $\overline{Y}$ and let $\mathcal{D}$ be the sub-bundle of $T(\mathbb{P}M)$ given by

$$\mathcal{D}_{(z,[v])} = \operatorname{span}\{q_*(X_{(z,v)}), q_*(\overline{X}_{(z,v)})\} = q_*(\mathcal{E}_{(z,v)}).$$

The homogeneity of X ensures that $\mathcal{D}_{(z,[v])}$ is well-defined. Moreover, if s is a smooth local section of $q\colon \mathcal{O}M_o \to \mathbb{P}$ then one may verify that

$$\{q_*(X_{s(z,[v])}), q_*(\overline{X}_{s(z,[v])})\}$$

is a local $\mathcal{C}^1$ frame for $\mathcal{D}$ and hence $\mathcal{D}$ is $\mathcal{C}^1$. The assumption that $[X, \overline{X}] = fY - \overline{fY}$ implies that $\mathcal{E}$ satisfies the hypotheses of theorem 3 and hence arises as the tangent bundle of a foliation $\mathcal{F}$ of $\mathcal{O}M_o$ by 2-dimensional complex submanifolds.

We show that $\mathcal{D}$ arises from a $\mathcal{C}^1$ foliation $\mathcal{F}'$ of $\mathbb{P}M$ by Riemann surfaces by exhibiting such a surface through an arbitrary point $(z_0, [v_0])$ whose tangent space coincides with $\mathcal{D}$ everywhere. Let A denote the leaf of $\mathcal{F}$ through (z_0, v_0). For any $(z, v) \in A$, the tangent space $T^{1,0}_{(z,v)}A$ is spanned by $X_{(z,v)}$ and $Y_{(z,v)}$. Now $q_*(Y) \equiv 0$. Since $p_*(q_*(X_{(z,v)})) = \pi_*(X_{(z,v)}) = v \neq 0$, we must have $q_*(X_{(z,v)}) \neq 0$. Thus the restriction of q to A has complex rank 1 everywhere. It follows that (z_0, v_0) has a connected neighbourhood U in A whose image $q(U)$ is an embedded 1-dimensional complex submanifold of $\mathbb{P}M$. As $T^{1,0}_{(z,v)}q(U)$ is spanned by $q_*(X_{(z,v)})$ for each $(z, v) \in U$, the Riemann surface $q(U)$ is everwhere tangent to $\mathcal{D}$, as desired.

Now fix $(z_0, [v_0]) \in \mathbb{P}M$ and let B be the maximal leaf of $\mathcal{F}'$ through $(z_0, [v_0])$. So B is an injectively immersed Riemann surface in $\mathbb{P}M$. From $p_*(q_*(X_{(z,v)})) = v \neq 0$ and the fact that $T^{1,0}_{(z,[v])}B$ is spanned by $q_*(X_{(z,v)})$, it follows that the restriction

$$p|_B \colon B \to M$$

is a holomorphic immersion and that the holomorphic tangent space of $p(B)$ at z is spanned by v for each $(z, [v]) \in B$. Consider now the maximal integral path $\beta\colon (a, b) \to \mathcal{O}M_o$ of X through (z, v) (where $(z, [v]) \in B$). As $(q \circ \beta)'(t) = q_*(X_{\beta(t)})$ for all t, the path $q \circ \beta$ is everywhere tangent to $\mathcal{D}$. Hence $q \circ \beta$ lies in a single leaf of $\mathcal{F}'$ (by connectedness) and this leaf must be B. Thus $\pi \circ \beta = p \circ q \circ \beta$ lies in $p(B)$. It follows that $\pi \circ \beta$ is a nonconstant horizontal path for X which starts at z with velocity v and stays within $p(B)$. Part *(vii)* of proposition 3.3 now implies that

$$p|_B \colon B \to M$$

is a spray complex curve for X through z_0 in the direction v_0. As each horizontal path for X arises by applying π to an integral path of X, the above argument also shows that each horizontal path for X will be contained in the image of some spray complex curve for X.

It remains to show the maximality of $p(B)$. Let $\psi\colon W \to M$ be a spray complex curve for X which is tangent to v_0 at z_0 and let χ be the induced complex spray on W. The associated map $\Psi\colon W \to \mathbb{P}M$ passes through $(z_0, [v_0])$. If we show that Ψ satisfies

$$\Psi_*(T_w W) = \mathcal{D}_{\Psi(w)} \quad \forall w \in W.$$

then its image must be contained in a single leaf of $\mathcal{F}$ (by connectedness) and, since it passes through $(z_0, [v_0])$, this leaf must be B.

Fix $(w, u) \in \mathcal{O}W$ and let $\sigma\colon (-\epsilon, \epsilon) \to W$ be a horizontal path for χ with $\sigma'(0) = (w, u)$. Since $\sigma'(t)$ spans $\mathcal{O}_{\sigma(t)}W$ it follows that

$$\Psi(\sigma(t)) = q(\psi_*(\sigma'(t)))$$

But $\psi_* \circ \sigma'$ is an integral path for X (since $\psi_{**}(\chi) = X$). Therefore

$$\Psi_*(\sigma'(t)) = q_*(X_{\psi_*(\sigma'(t))}) \quad \forall t$$

At $t = 0$ this yields $\Psi_*(w, u) = q_*(X_{\psi_*(w,u)})$ from which $\Psi_*(T_w W) = \mathcal{D}_{\psi_*(w,u)}$ is immediate. q.e.d.

Remark 3.8. If X is a complex spray which is $\mathcal{C}^2$ on $\mathcal{O}M_o$, then each point $z \in M$ has a neighbourhood U_z with the property that any two points of U_z may be joined by a unique horizontal path for X which is contained in U_z. (Details of this are contained in [14]). We say that U_z is *simple and convex* for X. The existence of simple convex neighbourhoods for a complex spray generalises the corresponding existence of such neighbourhods for an affine connection (see [7]).

Corollary 3.9. *Assume that X is a $\mathcal{C}^1$ complex spray which is $\mathcal{C}^2$ on $\mathcal{O}M_o$ and which satisfies the condition*

$$[X, \overline{X}] = fY - \overline{fY} \qquad \text{on } \mathcal{O}M_o$$

for some complex-valued function f. Then each point $z \in M$ has a neighbourhood U_z such that for any two points z_1 and z_2 in U_z there exists a spray complex curve for X whose image contains both z_1 and z_2 and is itself contained in U_z .

Proof. Choose U_z to be a simple convex neighbourhood of z. Then z_1 and z_2 may be joined by a horizontal path for X whose image is contained in U_z. This path is then contained in the image of a spray complex curve for X. Since the horizontal segment from z_1 to z_2 is compact, we can truncate the domain of the spray complex curve to ensure that the resulting image is contained in U_z. q.e.d.

The local flow α of a $\mathcal{C}^1$ complex spray X is a $\mathcal{C}^1$ map

$$\alpha\colon A \to \mathcal{O}M$$

(where A is an open neighbourhood of $\mathcal{O}M \times \{0\}$ in $\mathcal{O}M \times \mathbb{R}$) such that

$$X_{\alpha(z,v,s)} = \left.\frac{\partial\alpha(z,v,t)}{\partial t}\right|_{t=s} \qquad \forall s.$$

The exponential map of X is the $\mathcal{C}^1$ map

$$\exp\colon \{(z,v) \in \mathcal{O}M : (z,v,1) \in A\} \to M$$

given by

$$\exp(z,v) = \pi(\alpha(z,v,1)).$$

For any $z \in M$, the derivative at 0 of the restriction of exp to $\mathcal{O}_z M$ is the identity and hence exp maps some neighbourhood V_z of $0 \in \mathcal{O}_z M$ $\mathcal{C}^1$-diffeomorphically onto an open neighbourhood U_z of z in M. If V_z also satisfies

$$av \in V_z, \quad \forall v \in V_z, a \in \mathbb{C} \text{ with } |a| \leq 1$$

then we refer to both V_z and the corresponding U_z as *normal neighbourhoods.*

Teorem 3.10. *Let X be an integrable $\mathcal{C}^1$ complex spray on M and let U_z be a normal neighbourhood of $z \in M$. Then $U_z \setminus \{z\}$ is foliated by those spray complex curves for X which pass through z.*

Proof. We use the same notation as above. Let V_z be the normal neighbourhood of 0 in $\mathcal{O}_z M$ which corresponds to U_z. The holomorphic foliation

$$\{\mathbb{C}v \cap (V_z \setminus \{0\}) : v \in V_z \setminus \{0\}\}$$

of $V_z \setminus \{0\}$ yields the $\mathcal{C}^1$ foliation

$$\{\exp(\mathbb{C}v \cap (V_z \setminus \{0\})) : v \in V_z \setminus \{0\}\}$$

of $U_z \setminus \{z\}$ by 2-dimensional real submanifolds. We show that each such leaf actually coincides with the image of some spray complex curve for X.

Fix $v \in \partial V_z$ and consider the continuous map $g\colon V_z \cap (\mathbb{C}v) \to \mathbb{P}M$ given by

$$g(te^{i\theta}v) = q\left(\frac{\partial}{\partial t}(\exp(te^{i\theta}v))\right) \quad \text{for } t \in (-1,1) \text{ and } \theta \in [0,2\pi]$$

(This map is well-defined at 0 because $q(\mathrm{e}^{\mathrm{i}\theta}v) = q(v)$ for all θ.) Let $\mathcal{F}$ denote the foliation of $\mathbb{P}M$ determined by X and let $B \subset \mathbb{P}M$ be the maximal leaf through $q(v)$. For each fixed θ the path

$$t \mapsto \exp(t\mathrm{e}^{\mathrm{i}\theta}v)$$

is horizontal for X (it is thus $\mathcal{C}^2$) and hence the path

$$t \mapsto q\left(\frac{\partial}{\partial t}(\exp(t\mathrm{e}^{\mathrm{i}\theta}v))\right)$$

is everywhere tangent to $\mathcal{F}$. As this path passes through $q(v)$ it must be contained in the leaf B. It follows that $g(V_z \cap (\mathbb{C}v)) \subset B$. Injectivity of g implies that the set $W = g(V_z \cap (\mathbb{C}v))$ is open in B (by invariance of domain). Then

$$p|_W : W \to M$$

is a spray complex curve for X with

$$\exp((\mathbb{C}v) \cap V_z) = p(W)$$

and

$$\exp((\mathbb{C}v) \cap (V_z \setminus \{0\})) = p(W \setminus \{q(v)\})$$

q.e.d.

4 (h, X)-Horizontal Complex Curves

Next we examine holomorphic mappings from a Riemann surface W into M which take geodesics for a fixed but arbitrary Hermitian metric h on W into horizontal paths for a given complex spray X on M. These are a restricted type of spray complex curve for X. If we assume that X arises from a complex Finsler metric, then we recover the horizontal complex curves of [15].

Definition 4.1. *Let h be a Hermitian metric on W and let X be a $\mathcal{C}^1$ complex spray on M. A nonconstant holomorphic mapping $\phi\colon W \to M$ is called a (h, X)-*spray complex curve *if $\phi \circ \sigma$ is a horizontal path for*

X whenever σ is a geodesic for h. When we wish to suppress mention of the metric h we will call ϕ a metrical spray complex curve *for X.*

Remark 4.2. If ϕ is a (h, X)-spray complex curve then its nonconstancy combines with its holomorphy to imply that ϕ is an immersion. For if $\phi_*(\mathcal{O}_w W) = \{0\}$ then ϕ would map each geodesic through w to the constant path at $\phi(w)$ and hence ϕ would be constant (by the identity theorem for holomorphic maps).

Remark 4.3. By condition *(vi)* of proposition 3.3, we see that a (h, X)-spray complex curve is a spray complex curve for X in the sense of definition 3.1. Roughly speaking, the added condition is that each horizontal path of X which is tangent to $\phi(W)$ pulls back to give a h-geodesic in W.

The following proposition gives some equivalent characterisations of (h, X)-spray complex curves.

Proposition 4.4. *Let X be a complex spray on a complex manifold M, h a Hermitian metric on a Riemann surface W and $\phi\colon W \to M$ a holomorphic immersion. Denote the horizontal radial vector field for h by χ. Then the following conditions are equivalent*

(i) *ϕ is a (h, X)-spray complex curve.*

(ii) *The mapping ϕ satisfies*

$$\phi_{**}(\chi_{(w,u)}) = X_{\phi_*(w,u)} \ \forall\ (w, u) \in \mathcal{O}W.$$

(iii) *For each $w \in W$ there exists $u \in \mathcal{O}_w W \setminus \{0\}$ such that*

$$\phi_{**}(\chi_{(w,u)}) = X_{\phi_*(w,u)}.$$

(iv) *$\phi_{**}(\nu_{(w,u)}) = \mathbb{C}X_{\phi_*(w,u)}\ \forall (w, u) \in \mathcal{O}W$, where ν denotes the nonlinear connection on W arising from h.*

Proof. The proof of the equivalence of the first three is omitted, the techniques used being very similar to some of those in the proof of proposition 3.3. The equivalence of *(ii)* and *(iv)* follows because $\nu_{(w,u)}$ is spanned by $\chi_{(w,u)}$. q.e.d.

Remark 4.5. If $\phi\colon W \to M$ is holomorphic and takes each h-geodesic through a given point $w \in W$ into a horizontal path for X then one can show that the restriction of ϕ to any normal neighbourhood of w in W is a (h, X)-spray complex curve.

Next we wish to formulate and prove an analogue of theorem 3.7 for metrical spray complex curves. We saw in proposition 3.3 that for each spray complex curve $\phi\colon W \to M$ the complex spray X induces a complex spray χ on W. Thus we would like to know when the complex spray χ arises from a Hermitian metric. The following lemma answers this in the case $W = \mathbb{C}$ or $\mathbb{D}_r$, the disc of radius $r > 0$ and centre 0 in $\mathbb{C}$.

Lemma 4.6. *A $\mathcal{C}^1$ complex spray χ on $\mathbb{D}_r$ (or $\mathbb{C}$) arises from a Hermitian metric if and only if*

$$[\chi, \bar{\chi}] = \alpha(\eta - \bar{\eta})$$

for some continuous real-valued function α on $\mathcal{O}\mathbb{D}_r \setminus \mathbb{D}_r$ (respectively $\mathcal{O}\mathbb{C} \setminus \mathbb{C}$). Moreover, this Hermitian metric (if it exists) is uniquely determined up to a factor of a positive constant.

Proof. Write $\chi = u(\frac{\partial}{\partial w} - ub(w)\frac{\partial}{\partial u})$. The proof of the first assertion is in two steps. First we show that χ arises from a metric if and only if $\frac{\partial b}{\partial \bar{w}}$ is real-valued everywhere. The second step (which we omit) is a straightforward calculation to show that $\frac{\partial b}{\partial \bar{w}}$ is real valued if and only if $[\chi, \bar{\chi}] = \alpha(\eta - \bar{\eta})$ for some real valued function α.

Assume first that χ arises from the Hermitian metric $h(w)dwd\bar{w}$. Then $b(w) = \frac{\partial \log h}{\partial w}$ and therefore

$$\frac{\partial b}{\partial \bar{w}} = \frac{1}{4}\Delta(\log h)$$

where Δ denotes the standard Laplacian in $\mathbb{C}$. The right hand side is real-valued since h is real-valued.

Conversely, assume that $\frac{\partial b}{\partial \bar{w}}$ is real valued. We wish to find a strictly positive function h on $\mathbb{D}_r$ (respectively $\mathbb{C}$) such that $b = \frac{\partial \log h}{\partial w}$. The differential equation

$$\frac{\partial \psi}{\partial w} = b$$

on $\mathbb{D}_r$ (resp. $\mathbb{C}$) always has a solution defined everywhere. One way to see this is to note that by conjugating we get a $\bar{\partial}$-equation. These can always be solved on $\mathbb{D}_r$ (resp. $\mathbb{C}$) and the conjugate of such a solution is then a solution to our equation. If we can choose our solution ψ of $\frac{\partial \psi}{\partial w} = b$ so that it is real-valued, then $h = \mathrm{e}^{\psi}$ will provide us with the desired metric. Let ψ be any solution of $\frac{\partial \psi}{\partial w} = b$. Our hypothesis implies that $4\frac{\partial b}{\partial \bar{w}} = \Delta(\psi)$ is real-valued. Therefore

$$\Delta(\psi - \bar{\psi}) = \Delta(\psi) - \overline{\Delta(\psi)} = 0$$

so that $\psi - \bar{\psi}$ is harmonic and only takes values along the imaginary axis. Thus we may write

$$\psi - \bar{\psi} = f - \bar{f}$$

for some holomorphic function f on $\mathbb{D}_r$ (resp. $\mathbb{C}$). Hence

$$\psi + \bar{f} = \bar{\psi} + f$$

so that $\psi + \bar{f}$ is real valued. Moreover, since $\frac{\partial \bar{f}}{\partial w} = 0$, the function $\psi + \bar{f}$ satisfies

$$\frac{\partial}{\partial w}\left(\psi + \bar{f}\right) = b.$$

The uniqueness part follows from lemma 4.8 below whose proof is omitted (see [15] for details). q.e.d.

Note 4.7. The existence part of the above proof breaks down (and the lemma itself is untrue) if we try to replace $\mathbb{D}_r$ or $\mathbb{C}$ by a nonsimply-connected domain W in $\mathbb{C}$. For example, the complex spray χ on $W = \mathbb{D}_r \setminus \{0\}$ or on $W = \mathbb{C} \setminus \{0\}$ given by

$$\chi_{(w,u)} = u \left.\frac{\partial}{\partial w}\right|_{(w,u)} - \frac{\mathrm{i}u^2}{2w} \left.\frac{\partial}{\partial u}\right|_{(w,u)}$$

satisfies the hypothesis of the lemma. If χ were to arise from a Hermitian metric $h(w)dwd\bar{w}$ on W then we would have

$$\frac{\partial \log(h)}{\partial w} = \frac{\mathrm{i}}{2w}.$$

With respect to polar coordinates (r, θ) on W this equation takes the form

$$\frac{\partial \log(h)}{\partial \theta} = 1$$

from which it would follow that $\log(h)$ is a continuous (since h is continuous) branch of the argument function θ defined on all of W. But this is impossible.

Lemma 4.8. *Let g and h be two $\mathcal{C}^1$ Hermitian metrics on $\mathbb{D}_r$ with χ_g and χ_h their respective horizontal radial vector fields. Then the following are equivalent*

(i) *$g = ch$ for some constant $c > 0$.*

(ii) *$\chi_g = \chi_h$*

(iii) *$\chi_g(h) = 0$.*

Proposition 4.9. *A $\mathcal{C}^1$ complex spray χ on a simply-connected Riemann surface W arises from a Hermitian metric if and only if*

$$[\chi, \bar{\chi}] = \alpha(\eta - \bar{\eta})$$

for some continuous real-valued function α on $\mathcal{O}W \setminus W$. Moreover, this Hermitian metric (if it exists) is uniquely determined up to a factor of a positive constant.

Proof. If χ arises from a Hermitian metric then $[\chi, \bar{\chi}] = \alpha(\eta - \bar{\eta})$ follows by a straightforward calculation.

For the converse we need only consider the case where W is the Riemann sphere because, up to biholomorphism, the only other simply-connected Riemann surfaces are $\mathbb{D}$ and $\mathbb{C}$ (to which lemma 4.6 applies). However, lemma 4.6 implies that if

$$[\chi, \bar{\chi}] = \alpha(\eta - \bar{\eta})$$

then we can find Hermitian metrics h_1 and h_2 on $W_1 = W \setminus \{\text{north-pole}\}$ and on $W_2 = W \setminus \{\text{south-pole}\}$ respectively which give rise to the sprays obtained by restricting χ to W_1 and W_2. But on the intersection $W_1 \cap W_2$, each of h_1 and h_2 has χ as its horizontal radial vector field. It follows by lemma 4.8 that $\frac{h_1}{h_2}$ is locally constant on $W_1 \cap W_2$ and hence is constant everywhere (since $W_1 \cap W_2$ is connected). Let $c > 0$ be this constant value. Then the metric h on W defined by

$$h = \begin{cases} h_1 & \text{on } W_1 \\ ch_2 & \text{on } W_2 \end{cases}$$

gives rise to χ.

Lemma 4.8 implies that any two metrics on W which give rise to χ have a locally constant quotient. Connectedness of W implies that the same constant works globally. q.e.d..

Proposition 4.10. *Let $\phi\colon W \to M$ be a (h, X)-spray complex curve. Then*

$$[X, \overline{X}]_{\phi_*(w,u)} = \alpha(w,u)(Y_{\phi_*(w,u)} - \overline{Y}_{\phi_*(w,u)}) \quad \forall (w,u) \in \mathcal{O}W$$

where α is some real-valued function on $\mathcal{O}W$.

Proof. Let χ be the spray on W corresponding to h. Then

$$[\chi, \bar{\chi}] = \alpha(\eta - \bar{\eta})$$

for some real-valued function α. Applying ϕ_{**} to both sides of this equation and using the facts that χ and X are ϕ_*-related vector fields and that η and Y are also ϕ_*-related, yields

$$[X, \overline{X}]_{\phi_*} = \alpha(Y_{\phi_*} - \overline{Y}_{\phi_*})$$

q.e.d.

Corollary 4.11. *If a $\mathcal{C}^1$ complex spray X admits metrical spray complex curves for all initial conditions then there exists a continuous real-valued function f on $\mathcal{O}M_o$ such that*

$$[X, \overline{X}] = f(Y - \overline{Y}).$$

The converse to this corollary is also true.

Theorem 4.12. *Let X be a $\mathcal{C}^1$ complex spray on the complex manifold M. If*

$$[X, \overline{X}] = f(Y - \overline{Y}) \text{ on } \mathcal{O}M_o \text{ for some real function } f$$

then

for each $(z_0, v_0) \in \mathcal{O}M_o$ *there exists a metrical spray complex curve* $\phi\colon W \to M$ *such that* $\phi_*(w_0, u_0) = (z_0, v_0)$ *for some* (w_0, u_0) *in* $\mathcal{O}W_o$*.*

Proof. Assume that $[X, \overline{X}] = f(Y - \overline{Y})$. By theorem 3.7, there exists a Riemann surface W and a spray complex curve $\phi\colon W \to M$ for X such that $\phi_*(w_0, u_0) = (z_0, v_0)$ for some $(w_0, u_0) \in \mathcal{O}W$. By remark 3.2, we may assume that W is simply-connected. By *(v)* of proposition 3.3, ϕ and X induce a complex spray χ on W. Now $[\chi, \bar{\chi}]$ is a linear combination of η and $\bar{\eta}$. Since

$$\phi_{**}([\chi, \bar{\chi}]) = [X, \overline{X}]_{\phi_*} = f(Y_{\phi_*} - \overline{Y_{\phi_*}})$$

and $\phi_{**}(\eta) = Y_{\phi_*}$, it follows that

$$[\chi, \bar{\chi}] = (f \circ \phi_*)(\eta - \bar{\eta})$$

because ϕ_{**} is injective on fibres of $V\mathcal{O}M \oplus \overline{V\mathcal{O}M}$. Therefore by proposition 4.9, χ arises from a Hermitian metric h (say) on W. Thus ϕ is a (h, X)-spray complex curve. q.e.d.

4.1 Horizontal Complex Curves

We now consider spray complex curves for the horizontal radial vector field of a strongly pseudoconvex complex Finsler metric F on M.

Definition 4.13. *If* X *is the horizontal radial vector field of a strongly pseudoconvex complex Finsler metric* F*, then a* (h, X)*-spray complex curve* ϕ *will also be called a* (h, F)-horizontal complex curve*. When we wish to suppress mention of* h *and* F *we will call* ϕ *a* horizontal complex curve*.*

The following proposition summarises the properties of horizontal complex curves. The proof of these properties and further details on horizontal complex curves may be found in [15].

Proposition 4.14. *Let* $\phi\colon W \to M$ *be a horizontal complex curve for* (h, F)*. Then there exists a constant* $c > 0$ *such that*

(i) $\phi^*(F^2) = ch$*, where* $\phi^*(F^2) = F^2 \circ \phi_*$ *is the pull-back of* F^2*.*

(ii) $[X,\overline{X}]_{\phi_*} = \frac{K}{2c}\phi^*(F^2)(Y_{\phi_*} - \overline{Y}_{\phi*})$ *where* $K(w)$ *is the Gaussian curvature of* h *at* $w \in W$*.*

(iii) *The holomorphic curvature of* F *at* $\phi_*(\mathcal{O}_w W)$ *is* $\frac{1}{c}K(w)$ *for each* $w \in W$*.*

(iv) *The image* $\phi(W)$ *has a canonically defined holomorphic, normal bundle in* $\mathcal{O}M$*.*

Remark 4.15. As h and ch have the same geodesics ($c > 0$ constant) we see that any (h, F)-horizontal complex curve is also a (ch, F)-horizontal complex curve for each constant $c > 0$. By part *(i)* of proposition 4.14 it follows that ϕ is a $(\phi^*(F^2), F)$-horizontal complex curve. Then part *(ii)* implies that

$$[X,\overline{X}] = \phi^*(F^2)\frac{K}{2}(Y - \overline{Y})$$

where K is the Gaussian curvature of $\phi^*(F^2)$. By part *(iii)* of the proposition, $K(w)$ equals the holomorphic curvature of F at $\phi_*(w, u)$ for all $(w, u) \in \mathcal{O}W_o$. Conversely, it was demonstrated in [15] that if $\phi\colon W \to M$ is holomorphic and the Gaussian curvature of $\phi^*(F^2)$ at w coincides with the holomorphic curvature of F at $\phi_*(w, u)$ for each $(w, u) \in \mathcal{O}W_o$, then ϕ is a h.c.c. for $(\phi^*(F^2), F)$.

Remark 4.16. The opening line of the first remark combines with part *(i)* of proposition 4.14 to show that if ϕ is a horizontal complex curve for (h, F), then ϕ is a horizontal complex curve for (h', F) if and only if the metric h' is a constant multiple of h.

Remark 4.17. If X is the horizontal radial vector field of F, then a spray complex curve $\phi\colon W \to M$ for X is necessarily a horizontal complex curve. This can be seen as follows. Let χ be the spray on W induced by X. Then

$$\begin{aligned}\chi(\phi^*(F^2)) &= \phi_{**}(\chi)(F^2)\\ &= X_{\phi_*}(F^2)\\ &= 0 \quad \text{since } X(F^2) \equiv 0.\end{aligned}$$

Lemma 4.8 now implies that χ is the complex spray on W determined by the Hermitian metric $\phi^*(F^2)$. It follows that ϕ is a horizontal complex curve for $(\phi^*(F^2), F)$.

The existence theorem 4.12 for metrical spray complex curves combined with remark 4.17 yields

Theorem 4.18. *Let F be a strongly pseudoconvex complex Finsler metric on the complex manifold M and let X be the associated horizontal radial vector field. Then*

$$[X, \overline{X}] = \frac{\kappa}{2} F^2 (Y - \overline{Y}) \text{ on } \mathcal{O}M_o \text{ for some real-valued function } \kappa$$

if and only if

> *for each $(z_0, v_0) \in \mathcal{O}M_o$ there exists a horizontal complex curve in M through z_0 in the direction v_0.*

The function κ then coincides with the holomorphic curvature of F.

Remark 4.19. The assertion that κ is the holomorphic curvature of F was demonstrated in [15]. The existence part of this result was also stated there but the sketched proof required the added assumption

$$X(\kappa) = 0$$

which permitted a more geometric approach.

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