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ON THE SINGULAR NUMBERS FOR SOME INTEGRAL OPERATORS

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Abstract

Two–sided estimates of Schatten–von Neumann norms for weighted Volterra integral operators are established. Analogous problems for some potential-type operators defined on $\mathbb{R}^n$ are solved.

Let H be a separable Hilbert space and let $\sigma_\infty(H)$ be the class of all compact operators $T : H \to H$, which forms an ideal in the normed algebra $\mathbb{B}$ of all bounded linear operators on H. To construct a Schatten-von Neumann ideal $\sigma_p(H)$ $(0 < p \le \infty)$ in $\sigma_\infty(H)$, the sequence of singular numbers $s_j(T) \equiv \lambda_j(|T|)$ is used, where the eigenvalues $\lambda_j(|T|)$ ($|T| \equiv (T^*T)^{1/2}$) are non-negative and are repeated according to their multiplicity and arranged in decreasing order. A Schatten-von Neumann quasinorm (norm if $1 \le p \le \infty$) is defined as follows:

$$\|T\|_{\sigma_p(H)} \equiv \Big(\sum_j s_j^p(T)\Big)^{1/p}, \quad 0 < p < \infty,$$

with the usual modification if $p = \infty$. Thus we have $\|T\|_{\sigma_\infty(H)} = \|T\|$ and $\|T\|_{\sigma_2(H)}$ is the Hilbert- Schmidt norm given by the formula

$$\|T\|_{\sigma_2(H)} = \Big(\int\int |T_1(x,y)|^2 dxdy\Big)^{1/2} \tag{1}$$

for an integral operator

$$Tf(x) = \int T_1(x,y) f(y) dy.$$

We refer, for example, to [2], [6], [7] for more information concerning Schatten-von Neumann ideals.

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In this paper necessary and sufficient conditions for the weighted Volterra integral operator

$$K_v f(x) = v(x) \int_0^x f(y)k(x,y)dy, \quad x \in (0,a),$$

to belong to Schatten-von Neumann ideals are established, where v is a measurable function on $(0,a)$ $(0 < a \le \infty)$.

Two-sided estimates of Schatten-von Neumann p-norms for the weighted Riemann–Liouville operator

$$R_{\alpha,v} f(t) = v(x) \int_0^x f(t)(x-t)^{\alpha-1} dt,$$

when $\alpha > 1/2$ and $p > 1/\alpha$, were established in [13] (for $\alpha = 1$ and $p > 1$ see [14]). Analogous results for the weighted Hardy operator

$$H_{v,u} f(x) = v(x) \int_0^x u(y) f(y) dy$$

were obtained in [3]. Similar problems for the Riemann-Liouville operator with two weights

$$R_{\alpha,v,u} f(x) = v(x) \int_0^x u(t) f(t)(x-t)^{\alpha-1} dt,$$

when $\alpha \in \mathbb{N}$ and $p \ge 1$, were solved in [4]. Further, upper and lower bounds for Schatten–von Neumann p-norms ($p \ge 2$) of certain Volterra integral operators, involving $R_{\alpha,v,u}$ only for $\alpha \ge 1$, were proved in [4] and [18].

Our main goal is to generalize the results of [13] and [14] for integral transforms with kernels and to give two-sided estimates of the above-mentioned norms for these operators in terms of their kernels.

We denote by $L^p_w(\Omega)$, $\Omega \subseteq \mathbb{R}^n$, a weighted Lebesgue space with respect to the weight w defined on Ω.

Throughout the paper the expression $A \approx B$ is interpreted as $c_1 A \le B \le c_2 A$ with some positive constants c_1 and c_2.

Let us recall some definitions from [10] (see also [8]).

We say that a kernel $k : \{(x,y) : 0 < y < x < a\} \to \mathbb{R}_+$ belongs to V $(k \in V)$ if there exists a positive constant d_1 such that for all x, y, z with $0 < y < z < x < a$ the inequality

$$k(x,y) \le d_1 k(x,z)$$

holds. Further, $k \in V_\lambda$ $(1 < \lambda < \infty)$ if there exists a positive constant d_2 such that for all $x,\ x \in (0,a)$, the inequality

$$\int_{x/2}^{x} k^{\lambda'}(x,y)dy \le d_2 x k^{\lambda'}\big(x, x/2\big), \quad \lambda' = \frac{\lambda}{\lambda - 1}.$$

is fulfilled.

For example, if $k_1(x) = x^{\alpha-1}$, where $\frac{1}{\lambda} < \alpha \le 1$, then $k(x,y) = k_1(x-y)$ belongs to $V \cap V_\lambda$ (for other examples of kernel k see [10], [8]).

First we investigate the mapping properties of K_v in Lebesgue spaces.

The following statements in equivalent form were proved in [10] (see also [8], [11]).

Theorem A. *Let $1 < p \le q < \infty$, $a = \infty$ and let $k \in V \cap V_p$. Then*

(a) *K_v is bounded from $L^p(0,\infty)$ into $L^q(0,\infty)$ if and only if*

$$D_\infty \equiv \sup_{j \in \mathbb{Z}} D_\infty(j) \equiv \sup_{j\in\mathbb{Z}} \Big(\int_{2^j}^{2^{j+1}} k^q(x, x/2) x^{q/p'} |v(x)|^q dx \Big)^{\frac{1}{q}} < \infty.$$

Moreover, $\|K_v\| \approx D_\infty$.

(b) *K_v acts compactly from $L^p(0,a)$ into $L^q(0,a)$ if and only if $D_\infty < \infty$ and $\lim_{j\to+\infty} D_\infty(j) = \lim_{j\to-\infty} D_\infty(j) = 0$.*

Theorem B. *Let $1 < p \le q < \infty$, $a < \infty$ and let $k \in V \cap V_p$. Then*

(a) *K_v is bounded from $L^p(0,a)$ to $L^q(0,a)$ if and only if*

$$D_a \equiv \sup_{j \ge 0} D_a(j) \equiv \sup_{j\ge 0} \Big(\int_{2^{-(j+1)}a}^{2^{-j}a} |v(x)|^q k^q(x, x/2) x^{q/p'} dx \Big)^{\frac{1}{q}} < \infty.$$

Moreover, $\|K_v\| \approx D_a$.

(b) *K_v acts compactly from $L^p(0,a)$ into $L^q(0,a)$ if and only if $D_a < \infty$ and $\lim_{j\to+\infty} D_a(j) = 0$;*

Analogous problems for the Riemann-Liouville operator for $\alpha > 1/p$ were solved in [9] (For boundedness two-weight criteria of general integral operators with positive kernels see [5], Chapter 3).

Let $0 < a \le \infty$, $k : \{(x,y) : 0 < y < x < a\} \to \mathbb{R}^1_+$ be a kernel and let $k_0(x) \equiv x k^2(x, x/2)$.

We denote by $l^p(L^2_{k_0}(0,a))$ the set of all measurable functions $g : (0,a) \to \mathbb{R}^1$ for which

$$\|g\|_{l^p(L^2_{k_0}(0,\infty))} = \Big(\sum_{n\in\mathbb{Z}} \Big(\int_{2^n}^{2^{n+1}} |g(x)|^2 k_0(x)dx \Big)^{p/2} \Big)^{1/p} < \infty$$

if $a = \infty$ and

$$\|g\|_{l^p(L^2_{k_0}(0,a))} = \Big(\sum_{n=0}^{+\infty} \Big(\int_{2^{-(n+1)}a}^{2^{-n}a} |g(x)|^2 k_0(x)dx \Big)^{p/2} \Big)^{1/p} < \infty$$

if $a < \infty$, with the usual modification for $p = \infty$.

We shall need the following interpolation result (see, e.g., [19], p. 147 for the interpolation properties of the Schatten classes, and p. 127 for the corresponding properties of the sequence spaces. See also [1], Theorem 5.1.2):

Proposition A. *Let $0 < a \le \infty$, $1 \le p_0, p_1 \le \infty$, $0 \le \theta \le 1$, $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$. If T is a bounded operator from $lp_i(L^2_{k_0}(0,a))$ into $\sigma_{p_i}(L^2(0,a))$, where $i = 0,1$, then it is also bounded from $l^p(L^2_{k_0}(0,a))$ into $\sigma_p(L^2(0,a))$. Moreover,*

$$\|T\|_{l^p(L^2_{k_0})\to\sigma_p(L^2)} \le \|T\|^{1-\theta}_{l^{p_0}(L^2_{k_0})\to\sigma_{p_0}(L^2)} \|T\|^{\theta}_{l^{p_1}(L^2_{k_0})\to\sigma_{p_1}(L^2)}.$$

The next statement is obvious when $p = \infty$; and when $1 \le p < \infty$ it follows from Lemma 2.11.12 of [15].

Proposition B. *Let $1 \le p \le \infty$ and let $\{f_k\}$, $\{g_k\}$ be orthonormal systems in a Hilbert space H. If $T \in \sigma_p(H)$, then*

$$\|T\|_{\sigma_p(H)} \ge \Big(\sum_n |\langle Tf_n, g_n\rangle|^p \Big)^{1/p}.$$

Now we prove the main results.

In the sequel we shall assume that $v \in L^2_{k_0}(2^n, 2^{n+1})$ for all $n \in \mathbb{Z}$.

Theorem 1. *Let $a = \infty$, $2 \le p < \infty$ and let $k \in V \cap V_2$. Then K_v belongs to $\sigma_p(L^2(0,\infty))$ if and only if $v \in l^p(L^2_{k_0}(0,\infty))$. Moreover, there exist positive constants b_1 and b_2 such that*

$$b_1\|v\|_{l^p(L^2_{k_0}(0,\infty))} \le \|K_v\|_{\sigma_p(L^2(0,\infty))} \le b_2\|v\|_{l^p(L^2_{k_0}(0,\infty))}.$$

Proof. *Sufficiency.* Note that the fact $k \in V \cap V_2$ implies

$$I(x) \equiv \int_0^x k^2(x,y)dy \le ck_0(x) \tag{2}$$

for some positive constant c independent of x. Indeed, by the condition $k \in V \cap V_2$ we have

$$I(x) = \int_0^{x/2} k^2(x,y)dy + \int_{x/2}^x k^2(x,y)dy \le c_1k_0(x) + c_2k_0(x) = c_3k_0(x).$$

Consequently, using the Hilbert-Schmidt formula (1) and taking into account (2), we find that

$$\|K_v\|_{\sigma_2(L^2(0,\infty))} = \left(\int_0^\infty \int_0^x k^2(x,y)v^2(x)dxdy\right)^{1/2}$$

$$= \left(\int_0^\infty v^2(x)\left(\int_0^x k^2(x,y)dy\right)dx\right)^{1/2} \le c_4\left(\int_0^\infty v^2(x)k_0(x)dx\right)^{1/2}$$

$$= c_4\left(\sum_{n\in\mathbb{Z}} \int_{2^n}^{2^{n+1}} v^2(x)k_0(x)dx\right)^{1/2} = c_4\|v\|_{l^2(L^2_{k_0}(0,\infty))}.$$

On the other hand, in view of Theorem A we see that there exist positive constants c_5 and c_6 such that

$$c_5\|v\|_{l^\infty(L^2_{k_0}(0,\infty))} \le \|K_v\|_{\sigma_\infty(L^2(0,\infty))} \le c_6\|v\|_{l^\infty(L^2_{k_0}(0,\infty))}.$$

Further, Proposition A yields

$$\|K_v\|_{\sigma_p(L^2(0,\infty))} \le c_7\|v\|_{l^p((L^2_{k_0}(0,\infty))},$$

where $2 \le p < \infty$.

Necessity. Let $K_v \in \sigma_p(L^2(0,\infty))$ and let

$$f_n(x) = \chi_{[2^n,2^{n+1})}(x)2^{-n/2},$$

$$g_n(x) = v(x)x^{1/2}\chi_{[3\cdot 2^{n-1},2^{n+1})}(x)k(x,x/2)\alpha_n^{-1/2},$$

where

$$\alpha_n = \int_{3\cdot 2^{n-1}}^{2^{n+1}} v^2(y) k_0(y) dy.$$

Then it is easy to verify that $\{f_n\}$ and $\{g_n\}$ are orthonormal systems. Further, by virtue of Proposition B (for $p \geq 1$) we have

$$\infty > \|K_v\|_{\sigma_p(L^2(0,\infty))} \geq \left(\sum_{n\in\mathbb{Z}} |\langle K_v f_n, g_n\rangle|^p \right)^{1/p}$$

$$= \left(\sum_{n\in\mathbb{Z}} \left(\int_{3\cdot 2^{n-1}}^{2^{n+1}} \left(\int_{2^n}^{x} 2^{-n/2} k(x,y) dy \right) v^2(x) x^{1/2} k(x, x/2) \alpha_n^{-1/2} dx \right)^p \right)^{1/p}$$

$$\geq c_8 \left(\sum_{n\in\mathbb{Z}} \left(\alpha_n^{-1/2} \int_{3\cdot 2^{n-1}}^{2^{n+1}} 2^{-n/2} k(x, x/2) v^2(x) (x - 2^n) x^{1/2} dx \right)^p \right)^{1/p}$$

$$\geq c_9 \left(\sum_{n\in\mathbb{Z}} \left(\alpha_n^{-1/2} \int_{3\cdot 2^{n-1}}^{2^{n+1}} k_0(x) v^2(x) dx \right)^p \right)^{1/p} = c_9 \left(\sum_{n\in\mathbb{Z}} \alpha_n^{p/2} \right)^{1/p}.$$

Now let

$$f_n'(x) = \chi_{[3\cdot 2^{n-2}, 3\cdot 2^{n-1})}(x) (3\cdot 2^{n-2})^{-1/2}$$

and

$$g_n'(x) = v(x) x^{1/2} \chi_{[2^n, 3\cdot 2^{n-1})}(x) k(x, x/2) \beta_n^{-1/2},$$

where

$$\beta_n = \int_{2^n}^{3\cdot 2^{n-1}} v^2(y) k_0(y) dy.$$

Then it is easy to verify that $\{f_m'\}$ and $\{g_m'\}$ are orthonormal systems. Further,

$$\infty > \|K_v\|_{\sigma_p(L^2(0,\infty))} \geq \left(\sum_{n\in\mathbb{Z}} |\langle K_v f_n', g_n'\rangle|^p \right)^{1/p}$$

$$= \left(\sum_{n\in\mathbb{Z}} \left(\int_{2^n}^{3\cdot 2^{n-1}} \left(\int_{3\cdot 2^{n-2}}^{x} (3\cdot 2^{n-2})^{-1/2} k(x,y) dy \right) \right.\right.$$

$$\left.\left. \times v^2(x) x^{1/2} k(x, x/2) \beta_n^{-1/2} dx \right)^p \right)^{1/p}$$

$$\geq c_{10}\Big(\sum_{n\in\mathbb{Z}}\Big(\beta_n^{-1/2}\int_{2^n}^{3\cdot 2^{n-1}} 2^{-(n-2)/2}k^2(x,x/2)v^2(x)$$

$$\times(x-3\cdot 2^{n-2})x^{1/2}dx\Big)^p\Big)^{1/p}$$

$$\geq c_{11}\Big(\sum_{n\in\mathbb{Z}}\Big(\beta_n^{-1/2}\int_{2^n}^{3\cdot 2^{n-1}} k_0(x)v^2(x)dx\Big)^p\Big)^{1/p} = c_{11}\Big(\sum_{n\in\mathbb{Z}}\beta_n^{p/2}\Big)^{1/p},$$

where $p \geq 1$. Consequently

$$\Big(\sum_{n\in\mathbb{Z}}\Big(\int_{2^n}^{2^{n+1}} v^2(x)k_0(x)dx\Big)^{p/2}\Big)^{1/p} \leq \Big(\sum_{n\in\mathbb{Z}}(\beta_n+\alpha_n)^{p/2}\Big)^{1/p}$$

$$\leq c_{12}\|K_v\|_{\sigma_p(L^2(0,\infty))} + c_{12}\|K_v\|_{\sigma_p(L^2(0,\infty))}$$

$$\leq c_{13}\|K_v\|_{\sigma_p(L^2(0,\infty))} < \infty.$$

■

Let us now consider the case $a < \infty$. We have the following statement:

Theorem 2. *Let* $0 < a < \infty$, $2 \leq p < \infty$ *and let* $k \in V \cap V_2$. *Then* K_v *belongs to* $\sigma_p(L^2(0,a))$ *if and only if* $v \in l^p(L^2_{k_0}(0,a))$. *Moreover, there exists positive constants* b_1 *and* b_2 *such that*

$$b_1\|v\|_{l^p(L^2_{k_0}(0,a))} \leq \|K_v\|_{\sigma_p(L^2(0,a))} \leq b_2\|v\|_{l^p(L^2_{k_0}(0,a))}.$$

Proof. *Sufficiency.* The Hilbert– Schmidt formula and the condition $k \in V \cap V_2$ yield

$$\|K_v\|_{\sigma_p(L^2(0,a))} = \Big(\int_0^a v^2(x)\Big(\int_0^x k^2(x,y)dy\Big)dx\Big)^{1/2}$$

$$\leq c_1\Big(\int_0^a v^2(x)k_0(x)dx\Big)^{1/2}$$

$$= c_1\Big(\sum_{n=0}^{\infty}\int_{2^{-(n+1)}a}^{2^{-n}a} v^2(x)k_0(x)dx\Big)^{1/2} = c_1\|v\|_{l^2(L^2_{k_0}(0,a))}.$$

In view of Theorem B (part (a)) we arrive at

$$\|K_v\|_{\sigma_\infty(L^2(0,a))} \approx \|v\|_{l^\infty(L^2_{k_0}(0,a))}.$$

Using Proposition A we derive

$$\|K_v\|_{\sigma_p(L^2(0,a))} \le c_2 \|v\|_{l^p(L^2_{k_0}(0,a))}$$

when $p \ge 2$.

To prove necessity we take the orthonormal systems of functions defined on $(0,a)$:

$$f_n(x) = \chi_{[2^{-(n+1)}a, 2^{-n}a)}(x)(2^{-(n+1)}a)^{-1/2}$$

and

$$g_n(x) = v(x)x^{1/2}\chi_{[3\cdot 2^{-(n+2)}a, 2^{-n}a)}(x)k(x,x/2)\alpha_n^{-1/2},$$

where

$$\alpha_n = \int_{3\cdot 2^{-(n+2)}a}^{2^{-n}a} v^2(y)k_0(y)dy$$

and $n = 0,1,2,\cdots$. Consequently Proposition B yields

$$\infty > \|K_v\|_{\sigma_p(L^2(0,a))} \ge \left(\sum_{n=0}^{+\infty} |\langle K_v f_n, g_n\rangle|^p\right)^{1/p}$$

$$= \left(\sum_{n=0}^{\infty}\left(\int_{3\cdot 2^{-(n+2)}a}^{2^{-n}a} x^{1/2}v^2(x)k(x,x/2)\right.\right.$$

$$\left.\left.\times\left(\int_{2^{-(n+1)}a}^{x}(2^{-(n+1)}a)^{-1/2}k(x,y)dy\right)\alpha_n^{-1/2}dx\right)^p\right)^{1/p}$$

$$\ge c_3\left(\sum_{n=0}^{\infty}\alpha_n^{p/2}\right)^{1/p}.$$

If we take the following orthonormal systems:

$$f'_n(x) = \chi_{[3\cdot 2^{-(n+3)}a, 3\cdot 2^{-(n+2)}a)}(x)(3\cdot 2^{-(n+3)}a)^{-1/2},$$

$$g'_n(x) = v(x)x^{1/2}\chi_{[2^{-(n+1)}a, 3\cdot 2^{-(n+2)}a)}(x)k(x,x/2)\beta_n^{-1/2},$$

where

$$\beta_n = \int_{2^{-(n+1)}a}^{3\cdot 2^{-(n+2)}a} v^2(y)k_0(y)dy,$$

then we arrive at the estimate

$$\|K_v\|_{\sigma_p(L^2(0,a))} \geq c_4 \left(\sum_{n=0}^{\infty} \beta_n^{p/2} \right)^{1/p}.$$

Finally we have the lower estimate for $\|K_v\|_{\sigma_p(L^2(0,a))}$. ■

Remark 1. It follows from the proof of Theorems 1 and 2 that the lower estimate of $\|K_v\|_{\sigma_p(L^2(0,a))}$ holds for $1 \leq p \leq \infty$.

Now we formulate and prove the next statement.

Proposition 1. *Let $1 \leq p < \infty$. Then*

$$\|v\|_{l^p(L^2_{k_0}(0,\infty))} \approx J(v,p),$$

where

$$J(v,p) = \left(\int_0^{\infty} \left(\int_{x/2}^{2x} v^2(y)k^2(y,y/2)dy \right)^{p/2} x^{p/2-1}dx \right)^{1/p}.$$

Proof. We have

$$\|v\|_{l^p(L^2_{k_0}(0,\infty))} = \left(\sum_{n\in\mathbb{Z}} \left(\int_{2^n}^{2^{n+1}} v^2(x)k_0(x)dx \right)^{p/2} \right)^{1/p}$$

$$\leq \left(\sum_{n\in\mathbb{Z}} \left(\int_{2^n}^{2^{n+1}} v^2(x)k^2(x,x/2)dx \right)^{p/2} 2^{(n+1)p/2} \right)^{1/p}$$

$$= c_1 \left(\sum_{n\in\mathbb{Z}} \left(\int_{2^n}^{2^{n+1}} v^2(x)k^2(x,x/2)dx \right)^{p/2} 2^{np/2} \right)^{1/p}$$

$$\leq c_2 \left(\sum_{n\in\mathbb{Z}} \int_{2^n}^{2^{n+1}} y^{p/2-1} \left(\int_{2^n}^{2^{n+1}} v^2(x)k^2(x,x/2)dx \right)^{p/2} dy \right)^{1/p}$$

$$\leq c_2 \left(\sum_{n\in\mathbb{Z}} \int_{2^n}^{2^{n+1}} y^{p/2-1} \left(\int_{y/2}^{2y} v^2(x)k^2(x,x/2)dx \right)^{p/2} dy \right)^{1/p} = c_2 J(v,p).$$

To prove the reverse inequality we observe that

$$J(v,p)=\Bigg(\sum_{n\in\mathbb{Z}}\int_{2^n}^{2^{n+1}} y^{p/2-1}\Bigg(\int_{y/2}^{2y} v^2(x)k^2(x,x/2)dx\Bigg)^{p/2}dy\Bigg)^{1/p}$$

$$\leq\Bigg(\sum_{n\in\mathbb{Z}}\Bigg(\int_{2^n}^{2^{n+1}} y^{p/2-1}dy\Bigg)\Bigg(\int_{2^{n-1}}^{2^{n+2}} v^2(x)k^2(x,x/2)dx\Bigg)^{p/2}\Bigg)^{1/p}$$

$$\leq c_3\Bigg(\sum_{n\in\mathbb{Z}}2^{np/2}\Bigg(\int_{2^{n-1}}^{2^n} v^2(x)k^2(x,x/2)dx\Bigg)^{p/2}\Bigg)^{1/p}$$

$$+c_3\Bigg(\sum_{n\in\mathbb{Z}}2^{np/2}\Bigg(\int_{2^n}^{2^{n+1}} v^2(x)k^2(x,x/2)dx\Bigg)^{p/2}\Bigg)^{1/p}$$

$$+c_3\Bigg(\sum_{n\in\mathbb{Z}}2^{np/2}\Bigg(\int_{2^{n+1}}^{2^{n+2}} v^2(x)k^2(x,x/2)dx\Bigg)^{p/2}\Bigg)^{1/p}\leq c_4\|v\|_{l^p(L^2_{k_0}(0,\infty))}.$$

■

From Theorem 1 and Proposition 1 we easily derive the following statement:

Theorem 3. *Let $2\leq p<\infty$ and let $k\in V\cap V_\lambda$. Then*

$$\|K_v\|_{\sigma_p(L^2(0,\infty))}\approx J(v,p).$$

A result analogous to Theorem 1 was obtained in [13] for the Riemann-Liouville operator $R_{\alpha,v}$, assuming that $\alpha>1/2$ and $p>1/\alpha$ (see [14] for $\alpha=1$ and $p>1$).

Let us now consider the multidimensional case. In particular, we shall deal with the operator

$$B^\alpha_{+,v}f(x)=v(x)\int\limits_{|y|<|x|}\frac{\Big(|x|^2-|y|^2\Big)^\alpha}{|x-y|^n}f(y)dy,\quad \alpha>0,$$

where v is a Lebesgue-measurable function on $\mathbb{R}^n$ with $v\in L^2(\{2^n<|y|<2^{n+1}\})$ for all $n\in\mathbb{Z}$ (for the definition and some properties of $B_{+,v}$, where $v\equiv 1$, see, e.g., [16], Chapter 7, and [17], Section 29).

Let w be a measurable a.e. positive function on $\mathbb{R}^n$. We denote by $l^p(L^2_w(\mathbb{R}^n))$ a set of all measurable functions $\varphi : \mathbb{R}^n \to \mathbb{R}^1$ for which

$$\|\varphi\|_{l^p(L^2_w(\mathbb{R}))} = \left(\sum_{k\in\mathbb{Z}} \left(\int\limits_{2^k<|x|<2^{k+1}} \varphi^2(x)w(x)dx \right)^{p/2} \right)^{1/p} < \infty.$$

The next result is from [19] (pp. 127, 147).

Proposition C. *Let* $1 \le p_0,\ p_1 \le \infty,\ 0 \le \theta \le 1,\ \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$. *If* T *is a bounded operator from* $l^{p_i}(L^2_w(\mathbb{R}^n))$ *into* $\sigma_{p_i}(L^2_w(\mathbb{R}^n))$, *where* $i = 0, 1$, *then it is also bounded from* $l^p(L^2_w(\mathbb{R}^n))$ *into* $\sigma_p(L^2(\mathbb{R}^n))$.

In the sequel we shall use the notation $l^p(L^2_{|x|^\beta}(\mathbb{R}^n)) \equiv l^p(L^2_\beta(\mathbb{R}^n))$.

First we formulate some statements concerning the mapping properties of $B^\alpha_{+,v}$.

Theorem C ([12]). *Let* $1 < p \le q < \infty,\ \alpha > \frac{n}{p}$. *Then* $B^\alpha_{+,v}$ *acts boundedly from* $L^p(\mathbb{R}^n)$ *into* $L^q(\mathbb{R}^n)$ *if and only if*

$$F \equiv \sup_{j\in\mathbb{Z}} F(j) \equiv \sup_{j\in\mathbb{Z}} \left(\int\limits_{2^j<|x|<2^{j+1}} |v(x)|^q |x|^{q(2\alpha-n/p)} dx \right)^{1/q} < \infty.$$

Moreover, $\left\|B^\alpha_{+,v}\right\| \approx F$.

The following result can be obtained in the same as Theorem 5 from [12], therefore we omit the proof (see also [11]).

Theorem D. *Let* $1 < p \le q < \infty$ *and let* $\alpha > \frac{n}{p}$. *Then* $B^\alpha_{+,v}$ *acts compactly from* $L^p(\mathbb{R}^n)$ *into* $L^q(\mathbb{R}^n)$ *if and only if* $F < \infty$ *and* $\lim\limits_{j\to-\infty} F(j) = \lim\limits_{j\to+\infty} F(j) = 0$.

Now we state and prove the following Theorem:

Theorem 4. *Let* $2 \le p < \infty$ *and let* $\alpha > n/2$. *Then* $B^\alpha_{+,v} \in \sigma_p(L^2(\mathbb{R}^n))$ *if and only if* $v \in l^p(L^2_{4\alpha-n}(\mathbb{R}^n))$. *Moreover, there exist positive constants* b_1 *and* b_2 *such that*

$$b_1\|v\|_{l^p(L^2_{4\alpha-n}(\mathbb{R}^n))} \le \|B^\alpha_{+,v}\|_{\sigma_p(L^2(\mathbb{R}^n))} \le b_2\|v\|_{l^p(L^2_{4\alpha-n}(\mathbb{R}^n))}.$$

Proof. For sufficiency, we use the Hilbert-Schmidt formula (1) and the condition $\alpha > \frac{n}{2}$. Thus,

$$\|B^{\alpha}_{+,v}\|_{\sigma_2(L^2(\mathbb{R}^n))} = \Bigg(\int\limits_{\mathbb{R}^n} v^2(x)\Bigg(\int\limits_{|y|<|x|} \frac{\Big(|x|^2-|y|^2\Big)^{2\alpha}}{|x-y|^{2n}}dy\Bigg)dx\Bigg)^{\frac{1}{2}}$$

$$\leq c_1\Bigg(\int\limits_{\mathbb{R}^n} |x|^{2\alpha}v^2(x)\Bigg(\int\limits_{|y|<|x|} |x-y|^{(\alpha-n)2}dy\Bigg)dx\Bigg)^{\frac{1}{2}}$$

$$\leq c_2\Bigg(\int\limits_{\mathbb{R}^n} |x|^{4\alpha-n}v^2(x)dx\Bigg)^{\frac{1}{2}} = c_2\Big(\sum_{k=-\infty}^{+\infty} a_k^2\Big)^{\frac{1}{2}},$$

where

$$a_k = \Bigg(\int\limits_{2^k<|y|<2^{k+1}} |x|^{4\alpha-n}\, v^2(x)dx\Bigg)^{1/2}.$$

Moreover, using Theorem C we arrive at the following two-sided inequality:

$$c_3\|v\|_{l^\infty(L^2_{4\alpha-n}(\mathbb{R}^n))} \leq \left\|B^{\alpha}_{+,v}\right\|_{\sigma_\infty(L^2(\mathbb{R}^n))} \leq c_4\,\|v\|_{l^\infty(L^2_{4\alpha-n}(\mathbb{R}^n))}.$$

By Proposition C we conclude that

$$\left\|B^{\alpha}_{+,v}\right\|_{\sigma_p(L^2(\mathbb{R}^n))} \leq c_5\,\|v\|_{l^p(L^2_{4\alpha-n}(\mathbb{R}^n))}\,,\quad 2\leq p<\infty.$$

Now we prove necessity. For this we take the orthonormal systems $\{f_k\}$ and $\{g_k\}$, where

$$f_k(x) = \chi_{\left\{2^{k-2}<|y|<2^{k-1}\right\}}(x)2^{-(k-2)n/2}\cdot\lambda_n^{-\frac{1}{2}},$$

$$g_k(x) = \chi_{\left\{2^{k}\leq|y|<2^{k+1}\right\}}(x)\,|x|^{2\alpha-\frac{n}{2}}\,v(x)\alpha_k^{-\frac{1}{2}},$$

$\lambda_n = (2^n-1)\pi^{n/2}/\Gamma(n/2+1)$ and

$$\alpha_k = \int\limits_{2^k\leq|x|<2^{k+1}} v^2(x)\,|x|^{4\alpha-n}\,dx.$$

Then in view of Proposition B we have

$$\infty > \|B^{\alpha}_{+,v}\|_{\sigma_p(L^2(\mathbb{R}^n))} \geq c_6 \Bigg(\sum_{k\in\mathbb{Z}} \Bigg(\alpha_k^{-1/2} \int\limits_{2^k<|x|<2^{k+1}} v^2(x)|x|^{2\alpha-\frac{n}{2}}$$

$$\times \Bigg(\int\limits_{2^{k-2}<|y|<2^{k-1}} \frac{(|x|^2-|y|^2)^{\alpha}}{|x-y|^n} 2^{-(k-2)n/2} dy \Bigg) dx \Bigg)^p \Bigg)^{\frac{1}{p}}$$

$$\geq c_7 \left(\sum_{k\in\mathbb{Z}} \alpha_k^{p/2} \right)^{1/p} = c_7 \|v\|_{l^p(L^2_{4\alpha-n}(\mathbb{R}^n))}$$

which completes the proof. ■

The following result is also true:

Theorem 5. *Let $2 \leq p < \infty$ and let $\alpha > n/2$. Then $B^{\alpha}_{+,v} \in \sigma_p(L^2(\mathbb{R}^n))$ if and only if*

$$I(v,p,\alpha) \equiv \Bigg(\int\limits_{\mathbb{R}^n} \Bigg(\int\limits_{\frac{|x|}{2}<|y|<2|x|} v^2(y)|y|^{4\alpha-2n} dy \Bigg)^{p/2} |x|^{np/2-n} dx \Bigg)^{\frac{1}{p}} < \infty.$$

Moreover,

$$c_1 I(v,p,\alpha) \leq \left\| B^{\alpha}_{+,v} \right\|_{\sigma_p(L^2(\mathbb{R}^n))} \leq c_2 I(v,p,\alpha)$$

for some positive constants c_1 and c_2.

Proof. Taking into account Theorem 4, the statement will be proved if we show that

$$\|v\|_{l^p(L^2_{4\alpha-n}(\mathbb{R}^n))} \approx I(v,p,\alpha).$$

Indeed, we have

$$\|v\|_{l^p(L^2_{4\alpha-n}(\mathbb{R}^n))} \leq \Bigg(\sum_{k\in\mathbb{Z}} \Bigg(\int\limits_{2^k<|x|<2^{k+1}} v^2(x)|x|^{4\alpha-2n} dx \Bigg)^{p/2} 2^{(k+1)np/2} \Bigg)^{\frac{1}{p}}$$

$$= b_1 \Bigg(\sum_{k\in\mathbb{Z}} \int\limits_{2^k<|y|<2^{k+1}} |y|^{np/2-n} \Bigg(\int\limits_{\frac{|y|}{2}<|x|<2|y|} v^2(x)|x|^{4\alpha-2n} dx \Bigg)^{p/2} dy \Bigg)^{1/p}$$

$$= b_1 I(v, p, \alpha).$$

The reverse inequality follows similarly. ■

Remark 2. Some results of this paper were announced in [11].

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