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NILPOTENT CONTROL SYSTEMS

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Abstract

We study the class of matrix controlled systems associated to graded filiform nilpotent Lie algebras. This generalizes the non-linear system corresponding to the control of the trails pulled by car.

1 Introduction

When we consider the problem of a mobile robot on the plane, then the front wheels of the driving car are subjected to two controls (driving and turning speed). If the driving car pulls a chain of n trailers, then a model for the kinematic behavior of this system is given by :

$$(1) \quad \begin{cases} \dot{x}_1 = u_1 \\ \dot{x}_2 = u_2 \\ \dot{x}_3 = x_2 u_1 \\ \dot{x}_4 = x_3 u_1 \\ \vdots \\ \dot{x}_n = x_{n-1} u_1 \end{cases}$$

where u_1 and u_2 are the control functions. This system can be written in the "canonical form":

$$\dot{X}(t) = [u_1(t)A_1 + u_2(t)A_2)]X(t)$$

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where A_1 and A_2 are the matrices

$$A_1 = \begin{pmatrix} 0 & & & & \\ 0 & 0 & & & \\ 0 & 1 & 0 & & \\ \vdots & \ddots & \ddots & \ddots & \\ 0 & \cdots & 0 & 1 & 0 \end{pmatrix} ; \; A_2 = \begin{pmatrix} 0 & & & & \\ 1 & 0 & & & \\ 0 & 0 & 0 & & \\ \vdots & \ddots & \ddots & \ddots & \\ 0 & \cdots & 0 & 0 & 0 \end{pmatrix}$$

and $X(t)$ is defined by

$$X(t) = \begin{pmatrix} 1 & & & & & & \\ x_2(t) & 1 & & & & & \\ x_3(t) & x_1(t) & 1 & & & & \\ x_4(t) & \frac{1}{2}x_1^2(t) & x_1(t) & \ddots & & & \\ \vdots & \vdots & \vdots & \ddots & \ddots & & \\ \vdots & \vdots & \vdots & \vdots & x_1(t) & \ddots & \\ x_n(t) & \frac{1}{(n-2)!}x_1^{n-2}(t) & \cdots & \cdots & \frac{1}{2}x_1^2(t) & x_1(t) & 1 \end{pmatrix}$$

We can see that the matrices A_1 and A_2 generate a n-dimensional nilpotent linear Lie algebra which is isomorphic to the filiform Lie algebra $\mathcal{L}_n$ ([G.K]), whose brackets are given by:

$$[X_1, X_i] = X_{i+1}$$

$i = 2, ..., n-1$, the non-defined brackets being equal to zero or obtained by antisymmetry. The corresponding matrix representation of $\mathcal{L}_n$ is :

$$\begin{pmatrix} 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ a_2 & 0 & & & & \vdots \\ a_3 & a_1 & 0 & & & \vdots \\ a_4 & 0 & a_1 & \ddots & & \vdots \\ \vdots & \vdots & \ddots & & 0 & \vdots \\ a_n & 0 & \cdots & 0 & a_1 & 0 \end{pmatrix}.$$

This matrix is the image of an element $\sum a_i X_i$ for the given faithful representation.

Remark. The writing of the previous non linear system is possible because we can use a nilpotent minimal representation of the Lie algebra $\mathcal{L}_n$. Note that, for a general nilpotent Lie algebra, there does not exist a procedure to determine the minimal possible degree of a faithful representation.

The aim of this work is to generalize to a class of nilpotent Lie algebras, including $\mathcal{L}_n$, the corresponding control systems.

2 Filiform nilpotent Lie algebras

2.1 Filiform nilpotent Lie algebras

Let $\mathcal{G}$ be a n-dimensional (real) Lie algebra. Let $\mathcal{C}^i\mathcal{G}$ be the characteristic ideal defined by

$$\begin{cases} \mathcal{C}^0\mathcal{G}=\mathcal{G} \\ \mathcal{C}^1\mathcal{G}=[\mathcal{G},\mathcal{G}] \\ \vdots \\ \mathcal{C}^i\mathcal{G}=[C^{i-1}\mathcal{G},\mathcal{G}], \quad i \geq 1 \end{cases}$$

The Lie algebra $\mathfrak{g}$ is *nilpotent* if there is an integer k such that

$$\mathcal{C}^k\mathcal{G}=\{0\}$$

Definition 1. *The n-dimensional nilpotent Lie algebra $\mathcal{G}$ is called filiform if the smallest k such that $\mathcal{C}^k\mathcal{G}=\{0\}$ is equal to $n-1$.*

In this case the descending sequence is

$$\mathcal{G}\supset C^1\mathcal{G}\supset \cdots \supset \mathcal{C}^{n-2}\mathcal{G}\supset \{0\} = \mathcal{C}^{n-1}\mathcal{G}$$

and we have

$$\begin{cases} \dim \mathcal{C}^1\mathcal{G}=n-2, \\ \dim \mathcal{C}^i\mathcal{G}=n-i-1, \quad i=1,...,n-1. \end{cases}$$

Examples.

1) The Lie algebra $\mathcal{L}_n$ is filiform.

2) The following n-dimensional (n-even) Lie algebra $\mathcal{Q}_n$ defined by

$$\begin{cases} [X_1,X_2]=X_3 & , & [X_2,X_{n-1}]=2X_n \\ \vdots & , & [X_3,X_{n-2}]=-2X_n \\ [X_1,X_{n-2}]=X_{n-1} & , & \vdots \\ [X_1,X_{n-1}]=X_n & , & [X_p,X_{p+1}]=(-1)^p 2X_n, \quad p=\frac{n}{2}. \end{cases}$$

is filiform.

For this algebra, we have the following linear representation :

$$\begin{pmatrix} 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ a_2 & 0 & \ddots & & & & & \vdots \\ a_3 & a_1 & \ddots & \ddots & & 0 & & \vdots \\ \vdots & 0 & a_1 & \ddots & \ddots & & & \vdots \\ a_i & \vdots & \ddots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & \vdots & 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ a_{n-1} & 0 & \cdots & \cdots & 0 & a_1 & 0 & \vdots \\ a_n & -a_{n-1} & \cdots & (-1)^i a_i & \cdots & -a_3 & a_1+a_2 & 0 \end{pmatrix}$$

2.2 Graded filiform Lie algebras

Let $\mathcal{G}$ be a filiform Lie algebra. It is naturally filtered by the ideals $\mathcal{C}^i\mathcal{G}$ of the descending sequence. Then we can associate to the filiform Lie algebra $\mathcal{G}$ a graded Lie algebra, noted $gr\mathcal{G}$, which is also filiform. This algebra is defined by

$$gr\mathcal{G} = \oplus_{i=0,\dots,n-1} \frac{\mathcal{C}^i\mathcal{G}}{\mathcal{C}^{i+1}\mathcal{G}}$$

We denote $\frac{\mathcal{C}^i\mathcal{G}}{\mathcal{C}^{i+1}\mathcal{G}}$ by $\mathcal{G}_{i+1}$. Then we have

$$gr\mathcal{G} = \mathcal{G}_1 \oplus \mathcal{G}_2 \oplus \dots \oplus \mathcal{G}_n$$

with $\dim \mathcal{G}_1 = 2$, $\dim \mathcal{G}_i = 1$ for $2 \leq i \leq n$ and

$$[\mathcal{G}_i, \mathcal{G}_j] \subset \mathcal{G}_{i+j} \quad i+j \leq n$$

Lemma 1. *There is a homogeneous basis* $\{X_1, X_2, ..., X_n\}$ *of* $gr\mathcal{G}$ *such that*

$$\begin{array}{rcl} X_1, X_2 & \in & \mathcal{G}_1, \quad X_i \in \mathcal{G}_i \qquad i = 2, ..., n \\ [X_1, X_i] & = & X_{i+1} \qquad i = 2, ..., n \ , \\ [X_i, X_j] & = & 0 \qquad 2 \leq i < j \quad i + j \neq n, \\ [X_i, X_{n-i}] & = & (-1)^i \alpha X_n \end{array}$$

with $\alpha \in \mathbb{R}$ *and* $\alpha = 0$ *if* n *is even.*

A Lie algebra $\mathcal{G}$ is called graded if it is isomorphic to its associated graded Lie algebra :

$$\mathcal{G} = gr\mathcal{G}$$

The classification of graded filiform Lie algebras is described by the following theorem :

Theorem 1. (V) *If* n *is odd, then there are only, up to isomorphism, two* n*-dimensional graded filiform Lie algebras:* $\mathcal{L}_n$ *and* $\mathcal{Q}_n$*.*

If n *is even, then* $\mathcal{L}_n$ *is , up to isomorphism, the only* n *-dimensional graded filiform Lie algebra.*

The preceding matricial presentation of $\mathcal{L}_n$ and $\mathcal{Q}_n$ shows that these algebras admit a faithful representation of degree the dimension of the algebra.

3 Control system on graded nilpotent Lie groups

3.1 Linear representation of the Lie group Q_n

From Vergne's theorem, without loss of generality we can restrict ourselves to consider the classes of nonlinear systems involving the matrix Lie groups L_n and Q_n associated to the Lie algebras $\mathcal{L}_n$ and $\mathcal{Q}_n$. The case L_n, considered in the introduction (corresponding to a car with trailers) has been studied in [S.L]. The system has the canonical form (1).

Let us consider now the linear representation of the Lie algebra $\mathcal{Q}_n$ given in the previous section. Taking the exponential of this matrix, we find the linear representation of the connected and simply-connected Lie group Q_n associated to $\mathcal{Q}_n$

$$\begin{pmatrix}
1 & & & & & & & & & \\
x_2 & 1 & & & & & & & & \\
x_3 & x_1 & 1 & & & & & & & \\
x_4 & \frac{(x_1)^2}{2} & x_1 & 1 & & & & & & \\
\vdots & \vdots & & \ddots & \ddots & & & & & \\
x_i & \vdots & \frac{(x_1)^i}{i!} & \cdots & x_1 & \ddots & & & & \\
\vdots & \vdots & & & & \ddots & \ddots & & & \\
\vdots & \vdots & & & & & \ddots & 1 & & \\
x_{n-1} & \frac{(x_1)^{n-3}}{(n-3)!} & \cdots & \cdots & \cdots & \cdots & \cdots & x_1 & 1 & \\
x_n & y_{n-1} & \cdots & \cdots & \cdots & \cdots & \cdots & y_3 & x_1+x_2 & 1
\end{pmatrix}$$

where y_i are polynomial functions of $x_1, ..., x_i$.

3.2 Controlled system associated to $\mathcal{Q}_n$

Let us consider the following non linear system

$$(2) \quad : \begin{cases} \dot{x}_1 = u_1(t) \\ \dot{x}_2 = u_2(t) \\ \dot{x}_3 = x_2 u_1(t) \\ \dot{x}_4 = x_3 u_1(t) \\ \vdots \\ \dot{x}_{n-1} = x_{n-2} u_1(t) \\ \dot{x}_n = x_{n-1}(u_1(t) + u_2(t)) \end{cases}$$

Proposition 1. *The system (2) can be written as*

$$\dot{X}(t) = [u_1(t)B_1 + u_2(t)B_2)]X(t)$$

where B_1 and B_2 are the matrices corresponding to the generators of the Lie algebra $\mathcal{Q}_n$.

Proof. Let be

$$B_1 : \begin{pmatrix} 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & 0 & & & & \vdots \\ 0 & 1 & 0 & & & \vdots \\ \vdots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & 0 & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & \cdots & 0 & 1 & 0 \end{pmatrix}, B_2 : \begin{pmatrix} 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ 1 & 0 & & & & \vdots \\ 0 & 0 & 0 & & & \vdots \\ \vdots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & 0 & \ddots & 0 & 0 & \vdots \\ 0 & \cdots & \cdots & 0 & 1 & 0 \end{pmatrix}$$

These matrices generate the Lie algebra $\mathcal{Q}_n$. In fact, if we put

$$B_i = [B_1, B_{i-1}] = B_1 B_{i-1} - B_{i-1} B_1$$

for $i = 3, ..., n$ then we also have

$$[B_i, B_{n-i+1}] = (-1)^i 2 B_n$$

for $i = 2, ..., p = n/2$. This corresponds to the brackets of $\mathcal{Q}_n$. Then we can identify $\mathcal{Q}_n$ with the Lie algebra of the matrices B_i and the Lie group Q_n associated to $\mathcal{Q}_n$ is the linear group :

$$Q_n = \begin{pmatrix} 1 & & & & & & & & & \\ x_2 & 1 & & & & & & & & \\ x_3 & x_1 & 1 & & & & & & & \\ x_4 & \frac{(x_1)^2}{2} & x_1 & 1 & & & & & & \\ \vdots & \vdots & & \ddots & \ddots & & & & & \\ x_i & \vdots & \frac{(x_1)^i}{i!} & \cdots & x_1 & \ddots & & & & \\ \vdots & \vdots & & & & \ddots & \ddots & & & \\ \vdots & \vdots & & & & & \ddots & 1 & & \\ x_{n-1} & \frac{(x_1)^{n-3}}{(n-3)!} & \cdots & \cdots & \cdots & \cdots & \cdots & x_1 & 1 & \\ x_n & y_{n-1} & \cdots & \cdots & \cdots & \cdots & \cdots & y_3 & x_1 + x_2 & 1 \end{pmatrix}$$

Thus we have

$$(u_1(t) B_1 + u_2(t) B_2)(X(t)) =$$

$$
\begin{pmatrix}
0 & 0 & 0 & & & & \\
u_2(t) & 0 & 0 & & & & \\
x_2u_1(t) & u_1(t) & 0 & & & & \\
x_3u_1(t) & x_1u_1(t) & u_1(t) & & & & \\
\vdots & \vdots & \vdots & & & & \\
x_iu_1(t) & \frac{(x_1)^{i-2}}{(i-2)!}u_1(t) & \frac{(x_1)^{i-3}}{(i-3)!}u_1(t) & & & & \\
\vdots & \vdots & & \ddots & & & \\
\vdots & \vdots & & \ddots & 0 & & \\
x_{n-2}u_1(t) & \frac{(x_1)^{n-4}}{(n-4)!}u_1(t) & & & u_1(t) & 0 & \\
x_{n-1}U(t) & \frac{(x_1)^{n-3}}{(n-3)!}U(t) & \cdots & \cdots & x_1U(t) & U(t) & 0
\end{pmatrix}
$$

with $U(t) = u_1(t) + u_2(t)$. This gives the required system.

Theorem 2. *The system (2) is controlable.*

Recall that the system is controlable if, given two distincts points X_0 and X_f in $\mathcal{Q}_n$, there is a finite time T and a function control $u(t) = (u_1(t), u_2(t))$ such that the solution satisfies $X(0) = X_0$ and $X(T) = X_f$. From [S.L] , such a system is controlable if and only if the matrices B_1 and B_2 generate $\mathcal{Q}_n$. From the definition of these matrices, $B_1, B_2 \in \mathcal{Q}_n - [\mathcal{Q}_n, \mathcal{Q}_n]$ and generate the Lie algebra $\mathcal{Q}_n$.

4 The system (2) as a perturbation of (1)

Let $\varepsilon \in \mathbb{C}$ and consider the linear isomorphism

$$f_\varepsilon : \mathcal{Q}_n \to \mathcal{Q}_n$$

given by $f_\varepsilon(X_1) = X_1$, $f_\varepsilon(X_i) = \varepsilon X_i$ for $i = 2, ..., n$. If we put $Y_i = f_\varepsilon(X_i)$, the bracket of $\mathcal{Q}_n$ in the basis $\{Y_1, ..., Y_n\}$ is defined by

$$
\begin{cases}
[Y_1, Y_i] = Y_{i+1}, \quad i = 2, ..., n-1 \\
[Y_2, Y_{n-1}] = 2\varepsilon Y_n \\
\vdots \\
[Y_p, Y_{p+1}] = (-1)^p 2\varepsilon Y_n
\end{cases}
$$

Observe that if ε tends to 0, the brackets of $\mathcal{Q}_n$ tend to those of $\mathcal{L}_n$:

$$\left\{ \; [Y_1, Y_i] = Y_{i+1}, \quad i = 2, ..., n-1. \right.$$

the other bracket beeing nul. This proves that $\mathcal{Q}_n$is a deformation of $\mathcal{L}_n$, or that $\mathcal{L}_n$ is a contraction of $\mathcal{Q}_n$. In this way we can follow the representation of $\mathcal{Q}_n$ and see the system (2) as a perturbation of the system (1). Let us consider the representation of $\mathcal{Q}_n$ given by the matrices

$$B^\varepsilon = \begin{pmatrix} 0 & & & & & & & \\ a_2 & 0 & & & & & & \\ a_3 & a_1 & \ddots & & & 0 & & \\ \vdots & & a_1 & \ddots & & & & \\ a_i & & & \ddots & \ddots & & & \\ \vdots & & & & \ddots & \ddots & & \\ a_{n-1} & & & & & a_1 & 0 & \\ a_n & -\varepsilon a_{n-1} & \dots & (-1)^i \varepsilon a_i & \dots & -\varepsilon a_3 & a_1 + \varepsilon a_2 & 0 \end{pmatrix}$$

If B_i^ε is the matrix defined by $a_i = 1, a_j = 0$ for $j \neq i$, then we have

$$\begin{cases} [B_1^\varepsilon, B_i^\varepsilon] = B_{i+1}^\varepsilon, \quad i = 2, ..., n-1 \\ \left[B_2^\varepsilon, B_{n-1}^\varepsilon\right] = 2\varepsilon B_n^\varepsilon \\ \vdots \\ \left[B_p^\varepsilon, B_{p+1}^\varepsilon\right] = (-1)^p 2\varepsilon B_n^\varepsilon \end{cases}$$

that is, the brackets on the new basis. But

$$\lim_{\varepsilon \to 0} B_\varepsilon = \begin{pmatrix} 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ a_2 & 0 & & & & & & \vdots \\ a_3 & a_1 & \ddots & & & 0 & & \vdots \\ \vdots & & a_1 & \ddots & & & & \vdots \\ a_i & & & \ddots & \ddots & & & \vdots \\ \vdots & & & & \ddots & \ddots & & \vdots \\ a_{n-1} & & & & & a_1 & 0 & \vdots \\ a_n & 0 & \dots & 0 & \dots & 0 & a_1 & 0 \end{pmatrix}.$$

These matrices correspond to the linear representation of $\mathcal{L}_n$ given before.

The nonlinear matrix system

$$\dot{X}(t) = [u_1(t)B_1^{\varepsilon} + u_2(t)B_2^{\varepsilon})]X(t)$$

is written :

$$(3) \quad \begin{cases} \dot{x}_1 = u_1 \\ \dot{x}_2 = u_2 \\ \dot{x}_3 = x_2 u_1 \\ \dot{x}_4 = x_3 u_1 \\ \vdots \\ \dot{x}_{n-1} = x_{n-2} u_1 \\ \dot{x}_n = x_{n-1} u_1 + \varepsilon x_{n-1} u_2 \end{cases} .$$

This system is a perturbation of the nonlinear matrix system associated to $\mathcal{L}_n$. In fact, if $\varepsilon \to 0$, we find again the equations of (1). It is clear that the systems (2) and (3) are isomorphic, as they are defined by equivalent representations of $\mathcal{Q}_n$.

We can interprete these equations by saying that the last trailer has a perturbation given by the term $\varepsilon x_{n-1} u_2$. This is natural, because the role of the first trailer is not the same as that of the last one.

4.1 Determination of the solutions

Recall that we can give a global solution of a matrix system associated to a nilpotent Lie algebra by

$$X(t) = e^{g_1(t)A_1} e^{g_2(t)A_2} ... e^{g_n(t)A_n}$$

where the matrices A_i are the elements of the Lie algebra.

4.1.1 Solution of (1)

A direct computation of $X(t) = e^{g_1(t)A_1} e^{g_2(t)A_2} ... e^{g_n(t)A_n}$ gives :

$$\begin{cases} x_1 = g_1 \\ x_2 = g_2 \\ x_3 = g_1 g_2 + g_3 \\ x_4 = \frac{g_1^2}{2!} g_2 + g_1 g_3 + g_4 \\ \vdots \\ x_n = \frac{g_1^{n-2}}{(n-2)!} g_2 + ... + g_n \end{cases}$$

The functions g_i depends on the control functions u_1 and u_2. These relations are defined comparing the derivates of the previous solutions and the equations of (1). We obtain :

$$\begin{cases} \dot{g}_1 = u_1 \\ \dot{g}_2 = u_2 \\ \dot{g}_3 = -g_1 \, \dot{g}_2 \\ \vdots \\ \dot{g}_i = (-1)^i \frac{g_1^{i-2}}{(i-2)!} \, \dot{g}_2 \\ \vdots \\ \dot{g}_n = \frac{g_1^{n-2}}{(n-2)!} \, \dot{g}_2 \end{cases} .$$

By quadrature, we obtain the expressions of the g_i.

4.1.2 Solutions of the system (2)

The same calculations for the system (2) give:

$$\begin{cases} x_1 = g_1 \\ x_2 = g_2 \\ x_3 = g_1 g_2 + g_3 \\ x_4 = \frac{g_1^2}{2!} g_2 + g_1 g_3 + g_4 \\ \vdots \\ x_n = \frac{g_1^{n-2}}{(n-2)!} g_2 + ... + g_n \end{cases}$$

The relations between the functions g_i and the control functions are given by :

$$\left\{ \begin{array}{l} \dot{g}_1 = u_1 \\ \dot{g}_2 = u_2 \\ \dot{g}_3 = g_1 \, \dot{g}_2 \\ \quad \vdots \\ \dot{g}_i = (-1)^i \frac{g_1^{i-2}}{(i-2)!} \, \dot{g}_2 \\ \quad \vdots \\ \dot{g}_{n-1} = \frac{-g_1^{n-3}}{(n-3)!} \, \dot{g}_2 \\ \dot{g}_n = \frac{g_1^{n-2}}{(n-2)!} \, \dot{g}_2 + \dot{g}_2 \left(\frac{g_1^{n-3}}{(n-3)!} g_2 + \frac{g_1^{n-4}}{(n-4)!} g_3 + \dots + g_{n-1} \right) \end{array} \right.$$

4.1.3 Solutions of the perturbed system (3)

The link between (1) and (2) is given by solving (3). We obtain :

$$\left\{ \begin{array}{l} x_1 = g_1 \\ x_2 = g_2 \\ x_3 = g_1 g_2 + g_3 \\ x_4 = \frac{g_1^2}{2!} g_2 + g_1 g_3 + g_4 \\ \vdots \\ x_n = \frac{g_1^{n-2}}{(n-2)!} g_2 + \dots + g_n \end{array} \right.$$

and find again the same expression as in (2). On the other hand, the perturbation can be read from the relations between the g_i and the control functions u_i :

$$\begin{cases} \dot{g}_1 = u_1 \\ \dot{g}_2 = u_2 \\ \dot{g}_3 = -g_1 \, \dot{g}_2 \\ \vdots \\ \dot{g}_i = (-1)^i \frac{g_1^{i-2}}{(i-2)!} \, \dot{g}_2 \\ \vdots \\ \dot{g}_{n-1} = \frac{-g_1^{n-3}}{(n-3)!} \, \dot{g}_2 \\ \dot{g}_n = \frac{g_1^{n-2}}{(n-2)!} \, \dot{g}_2 + \varepsilon \, \dot{g}_2 \left(\frac{g_1^{n-3}}{(n-3)!} g_2 + \frac{g_1^{n-4}}{(n-4)!} g_3 + \dots + g_{n-1} \right) \end{cases}$$

When $\varepsilon \rightarrow 0$, we find the expressions of the g_i of the system (1).

References

[A] Ancochea Bermúdez J.M., On the rigidity of solvable Lie algebras. In *Deformation Theory of Algebras and Structures and Applications.* Hazewinkel Ed. Kluwer Acad. Publis. 403-445, 1988 .

[A.C] Ancochea Bermúdez J.M.,; Campoamor Strusberg O., On certain families of naturally graded Lie algebras, to appear in J. Pure Appl. Algebra.

[G.K] Goze M; Khakimdjanov Y., *Nilpotent Lie Algebras.* Kluwer Academic Publishers, MIA 361, 336p, 1996.

[R] Remm E., Systèmes controlables sur les groupes de Lie. mémoire DEA. Université de Haute Alsace. Mulhouse, 1997.

[S.L] Silva Leite F., Control Systems on matrix Lie groups, Actas/Proceedings EAMA-97. 295-306, 1997

[V] Vergne M., Cohomologie des algèbres de Lie nilpotentes. *Bull. Soc. Math. France*, **98**, 81-116, 1970.

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