REVISTA MATEMÁTICA COMPLUTENSE ISSN 1139-1138
Vol. 15 Núm. 2 (2002), 571-583

EXISTENCE OF WEAK-RENORMALIZED SOLUTION FOR A NONLINEAR SYSTEM

B. CLIMENT

Abstract

We prove an existence result for a coupled system of the reaction-diffusion kind. The fact that no growth condition is assumed on some nonlinear terms motivates the search of a weak-renormalized solution.

1 Introduction. Description of the problem

This paper investigates the existence of a solution for the nonlinear system

$$\begin{cases} -\Delta u - \nabla \cdot (\beta(v) X'(u)) = f & \text{in } \Omega, \\ -\Delta v - \nabla \cdot (\beta'(v) X(u)) = g & \text{in } \Omega, \\ u = 0, \quad v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where Ω denotes a bounded open subset of $\mathbb{R}^N$, X is a C^1 bounded $\mathbb{R}^N$-valued function on $\mathbb{R}$, i.e.

$$X \in (C^1(\mathbb{R}))^N \cap (C^0_b(\mathbb{R}))^N, \tag{2}$$

β is a function whose second derivatives are bounded, i.e.

$$\beta \in W^{2,\infty}(\mathbb{R}) \tag{3}$$

and

$$f, g \in H^{-1}(\Omega). \tag{4}$$

Here, the main difficulty to find a solution is that no growth restrictions are assumed on X'. Since f and g belong to $H^{-1}(\Omega)$, it is natural to look for solutions u and v belonging to $H^1_0(\Omega)$. Thus, it is not clear how

2000 Mathematics Subject Classification: 35D05, 35J65.
Servicio de Publicaciones. Universidad Complutense. Madrid, 2002

to give a sense to $\nabla\cdot(\beta(v)X'(u))$. This inconvenient can be overcome by introducing a weak-renormalized formulation of this problem, essentially obtained through pointwise multiplication of the first equation of (1) by $h(u)$, where h belongs to $C_0^1(\mathbb{R})$, that is, $h \in C^1(\mathbb{R})$ and its support is compact.

Remark. We can view this system as a simplified model of a nonlinear elasticity problem characterized by a constitutive law of the form

$$\sigma = \sigma_l + Y(u),$$

where

$$(\sigma_l)_{ij} = \sum a_{ijkl}\varepsilon_{kl}(u), \quad \varepsilon_{kl}(u) = \frac{1}{2}(\frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k}), \quad Y_{ij} \in C^0(\mathbb{R}^2).$$

Indeed, the conservation of momentum reads

$$\nabla\cdot\sigma = F$$

(F is given), which is in some sense a generalization of (1). In this paper, we study the case in which

$$Y(u) = \begin{pmatrix} \beta(u_2)X_1'(u_1) & \beta'(u_2)X_1(u_1) \\ \beta(u_2)X_2'(u_1) & \beta'(u_2)X_2(u_1). \end{pmatrix}$$

2 The main result

Theorem 2.1. *Under the assumptions* (2), (3), (4), *there exists* $\{u, v\}$, *with* u, $v \in H_0^1(\Omega)$, *such that the second equation in* (1) *is satisfied in the usual weak or distributional sense and the first equation holds in the following sense:*

$$\left\{\begin{array}{l} -\nabla\cdot(h(u)\nabla u) + \nabla u\cdot\nabla h(u) - \nabla\cdot(\beta(v)h(u)X'(u)) \\ \qquad +\beta(v)X'(u)\cdot\nabla h(u) = fh(u) \text{ in } \mathcal{D}'(\Omega) \qquad \forall h \in C_0^1(\mathbb{R}). \end{array}\right. \tag{5}$$

A couple $\{u, v\}$ as above will be called a weak-renormalized solution to (1).

Remark. In (5), every term belongs to $\mathcal{D}'(\Omega)$. Indeed, $h(u)$ belongs to $H_0^1(\Omega)$, the first term is in $H^{-1}(\Omega)$. The second one is in $L^1(\Omega)$. For instance, since h has a compact support, we can put

$$h(u)X'(u) = h(u)X'(T_M(u)) \quad \text{and} \quad h'(u)X'(u) = h'(u)X'(T_M(u))$$

for some $M > 0$, where T_M is the usual truncation at level M. Thus, we see that the third term in the left belongs to $W^{-1,\infty}(\Omega)$ and the fourth term belongs to $L^2(\Omega)$.

Remark. Renormalized solutions to PDE's were introduced by R. Di-Perna and P.L. Lions in [4] in the framework of the Boltzmann equation. They have been used in connection with various nonlinear elliptic equations by P. Benilan et al. [2], L. Boccardo et al. [3] and P.L. Lions and F. Murat [6] (see also [7]). In the analysis of existence results for systems, weak-renormalized solutions were first considered by R. Lewandowski [5] (see also [1]).

In this paper, in order to solve (1), we will extend the techniques used in [3] in the context of a single equation.

Remark. With regard to uniqueness, it is an open problem. If we follow the classical argument of considering two solutions u^i, v^i for $i = 1, 2$ of (1), and we compute the difference of (5) written for u^1, v^1 and for u^2, v^2, we find expressions with terms of the form $X'(\cdot)u$ that we are not able to estimate. There is another argument, due to P. L. Lions and F. Murat [7], which leads to the uniqueness of renormalized solutions, but it cannot be applied here.

3 The proof of theorem 2.1

First step. The introduction of a family of approximations.
For each $\varepsilon > 0$, let us put $X^\varepsilon(s) = X(T_{1/\varepsilon}(s))$ for all $s \in \mathbb{R}$. We will introduce the following approximation to (1):

$$\begin{cases} -\Delta u^\varepsilon - \nabla \cdot (\beta(v^\varepsilon)(X^\varepsilon)'(u^\varepsilon)) = f & \text{in } \Omega, \\ -\Delta v^\varepsilon - \nabla \cdot (\beta'(v^\varepsilon)X(u^\varepsilon)) = g & \text{in } \Omega, \\ \quad u^\varepsilon, v^\varepsilon \in H_0^1(\Omega), \end{cases} \tag{6}$$

In order to solve (6), we will apply Schauder's theorem. Thus, for any given ε and $\{u,v\} \in L^2(\Omega) \times L^2(\Omega)$, we set $R^\varepsilon(\{u,v\}) = \{u^\varepsilon, v^\varepsilon\}$, with $\{u^\varepsilon, v^\varepsilon\}$ being the unique solution to the linear system

$$\begin{cases} -\Delta u^\varepsilon = f + \nabla \cdot (\beta(v)(X^\varepsilon)'(u)) & \text{in } \Omega, \\ -\Delta v^\varepsilon = g + \nabla \cdot (\beta'(v)X(u)) & \text{in } \Omega, \\ \quad u^\varepsilon, v^\varepsilon \in H_0^1(\Omega), \end{cases} \tag{7}$$

Obviously, $R^\varepsilon = R_3 \circ R_2 \circ R_1^\varepsilon$, where

- $R_1^\varepsilon : L^2(\Omega) \times L^2(\Omega) \mapsto H^{-1}(\Omega) \times H^{-1}(\Omega)$ is the nonlinear continuous mapping given by

$$\begin{cases} R_1^\varepsilon(\{u,v\}) = \{f + \nabla \cdot (\beta(v)(X^\varepsilon)'(u)), g + \nabla \cdot (\beta'(v)X(u))\} \\ \quad \forall \{u,v\} \in L^2(\Omega) \times L^2(\Omega), \end{cases}$$

- $R_2 : H^{-1}(\Omega) \times H^{-1}(\Omega) \mapsto H_0^1(\Omega) \times H_0^1(\Omega)$ associates to each $\{f,g\} \in H^{-1}(\Omega) \times H^{-1}(\Omega)$ the unique solution $\{w,z\}$ of the following linear system

$$\begin{cases} -\Delta w = f & \text{in } \Omega, \\ -\Delta z = g & \text{in } \Omega, \\ \quad w, z \in H_0^1(\Omega), \end{cases}$$

- R_3 is the compact embedding of $H_0^1(\Omega) \times H_0^1(\Omega)$ into $L^2(\Omega) \times L^2(\Omega)$.

Since R_1^ε maps the whole space $L^2(\Omega) \times L^2(\Omega)$ inside a ball, Schauder's theorem can be applied and (6) possesses at least one solution $\{u^\varepsilon, v^\varepsilon\}$.

Second step. A priori estimates and weak convergence.
Choosing u^ε and v^ε as test functions in the first and second equation in (6) respectively, one finds:

$$\int_\Omega \nabla u^\varepsilon \nabla u^\varepsilon + \int_\Omega \beta(v^\varepsilon)(X^\varepsilon)'(u^\varepsilon) \cdot \nabla u^\varepsilon = \langle f, u^\varepsilon \rangle_{H^{-1}, H_0^1}. \tag{8}$$

$$\int_\Omega \nabla v^\varepsilon \nabla v^\varepsilon + \int_\Omega \beta'(v^\varepsilon) X(u^\varepsilon) \cdot \nabla v^\varepsilon = \langle g, v^\varepsilon \rangle_{H^{-1}, H_0^1}. \tag{9}$$

For ε sufficiently small, $X = X \circ T_{1/\varepsilon} = X^\varepsilon$, whence we can replace $X(u^\varepsilon)$ by $X^\varepsilon(u^\varepsilon)$ in (9).

Let us introduce the function $H = (H_1, H_2, ..., H_n)$, with

$$H_i(t,s) = \int_0^s \beta(0)(X_i^\varepsilon)'(\theta)d\theta + \int_0^t \beta'(\theta)X_i^\varepsilon(s)\,d\theta.$$

Then,

$$\int_\Omega \beta(v^\varepsilon)(X_\varepsilon)'(u^\varepsilon)\cdot\nabla u^\varepsilon + \int_\Omega \beta'(v^\varepsilon)X_\varepsilon(u^\varepsilon)\cdot\nabla v^\varepsilon = \int_\Omega \nabla\cdot H(u^\varepsilon, v^\varepsilon) = 0$$

thanks to Stokes' theorem. Summing (8) and (9), we obtain

$$\int_\Omega |\nabla u^\varepsilon|^2 + \int_\Omega |\nabla v^\varepsilon|^2 = \langle f, u^\varepsilon\rangle_{H^{-1},H_0^1} + \langle g, v^\varepsilon\rangle_{H^{-1},H_0^1}$$

and

$$\|u^\varepsilon\|^2_{H_0^1} + \|v^\varepsilon\|^2_{H_0^1} \le \|f\|^2_{H^{-1}} + \|g\|^2_{H^{-1}}\,.$$

Consequently, at least for a subsequence, still indexed by ε, we can conclude that

$$\begin{aligned} &u^\varepsilon \to u,\ v^\varepsilon \to v \quad \text{weakly in } H_0^1(\Omega),\\ &u^\varepsilon \to u,\ v^\varepsilon \to v \quad \text{strongly in } L^p(\Omega) \quad \forall p\in[1,2^\star) \text{ and a.e.} \end{aligned} \tag{10}$$

Here, we have denoted by $2^\star$ the exponent furnished by the Sobolev embedding theorem, that is

$$\begin{cases} 2^\star = \frac{2N}{N-2} \quad \text{if } N\ge 3,\\ 2^\star < +\infty \text{ arbitrarily large if } N = 2. \end{cases}$$

Third step. The strong convergence of v^ε in H_0^1.
It is easy to see that v is a weak solution to the problem

$$\begin{cases} -\Delta v - \nabla\cdot(\beta'(v)X(u)) = g \quad \text{in } \Omega,\\ \quad v = 0 \quad \text{on } \partial\Omega \end{cases} \tag{11}$$

Indeed, since β' and X are continuous and bounded, it is clear that $\beta'(v^\varepsilon) \to \beta'(v)$ strongly in L^p for all $p\in[1,2^\star)$ and $X(u^\varepsilon)\to X(u)$

strongly in L^r for all $r \in [1, +\infty)$. This enables us to pass to the limit in the second equation in (6).

From (11), we also see that

$$\int_\Omega |\nabla v|^2 = -\int_\Omega \beta'(v) X(u) \cdot \nabla v + \int_\Omega gv. \tag{12}$$

Let us use v^ε as a test function in the second equation in (6). We find:

$$\int_\Omega |\nabla v^\varepsilon|^2 = -\int_\Omega \beta'(v^\varepsilon) X(u^\varepsilon) \cdot \nabla v^\varepsilon + \int_\Omega g v^\varepsilon. \tag{13}$$

Arguing as above, we can pass to the limit in the right hand side in (13). Accordingly, we have:

$$\int_\Omega |\nabla v^\varepsilon|^2 \to -\int_\Omega \beta'(v) X(u) \cdot \nabla v + \int_\Omega gv.$$

This, combined with (12), gives the convergence in norm in H_0^1 for v^ε and, consequently,

$$v^\varepsilon \to v \quad \text{strongly in } H_0^1. \tag{14}$$

Fourth step. The strong convergence of u^ε in H_0^1.
We will first prove that

$$\lim_{K \to +\infty} \left(\limsup_{\varepsilon \to 0} \int_{\{|u^\varepsilon| > K\}} |\nabla u^\varepsilon|^2 \right) = 0 \tag{15}$$

Thus, let us consider the test functions $u^\varepsilon - T_K(u^\varepsilon)$ in the first equation in (6). Notice that

$$\nabla(u^\varepsilon - T_K(u^\varepsilon)) = \begin{cases} \nabla u^\varepsilon & \text{if } |u^\varepsilon| \geq K, \\ 0 & \text{if } |u^\varepsilon| < K. \end{cases}$$

Hence,

$$\begin{aligned} \int_{\{|u^\varepsilon| \geq K\}} |\nabla u^\varepsilon|^2 + \int_\Omega \beta(v^\varepsilon)(1 - T_K'(u^\varepsilon))(X^\varepsilon)'(u^\varepsilon) \cdot \nabla u^\varepsilon \\ = \langle f, u^\varepsilon - T_K(u^\varepsilon) \rangle. \end{aligned} \tag{16}$$

We can put $(1 - T_K'(u^\varepsilon))(X^\varepsilon)'(u^\varepsilon) \cdot \nabla u^\varepsilon = \nabla \cdot Y_K^\varepsilon(u^\varepsilon)$, where

$$(Y_K^\varepsilon)_i(t) = \int_0^t (1 - T_K'(\theta))(X^\varepsilon)'(\theta)\, d\theta.$$

Thus, the second term in the left hand side of (16) can be written in the form

$$\int_\Omega (\nabla \cdot Y_K^\varepsilon(u^\varepsilon))\beta(v^\varepsilon) = -\int_\Omega Y_K^\varepsilon(u^\varepsilon) \cdot \nabla\beta(v^\varepsilon).$$

Moreover,

$$Y_K^\varepsilon(s) = \begin{cases} X^\varepsilon(s) - X^\varepsilon(K) & \text{if } s > K, \\ 0 & \text{if } |u^\varepsilon| \le K, \\ X^\varepsilon(s) - X^\varepsilon(-K) & \text{if } s < -K. \end{cases}$$

Since $X \in C_b^0(\mathbb{R})^N$, for $\varepsilon > 0$ sufficiently small, Y_K^ε is independent of ε and $Y_K^\varepsilon(u^\varepsilon)$ is bounded by a constant independent of ε. We also have

$$\limsup_{\varepsilon\to 0} |Y_K^\varepsilon(u^\varepsilon)| \le |X(u) - X(K)|\mathbb{1}_{\{u>K\}} + |X(u) - X(-K)|\mathbb{1}_{\{u<-K\}}$$

for all $K > 0$. Therefore,

$$\begin{cases} \limsup\limits_{\varepsilon\to 0} \displaystyle\int_{\{|u^\varepsilon|>K\}} |\nabla u^\varepsilon|^2 \le \int_\Omega |X(u) - X(K)| \cdot |\nabla\beta(v)|\mathbb{1}_{\{u>K\}} \\ \quad + \displaystyle\int_\Omega |X(u) - X(-K)| \cdot |\nabla\beta(v)|\mathbb{1}_{\{u<-K\}} + \langle f, u - T_K(u)\rangle, \end{cases} \tag{17}$$

whence

$$\begin{cases} \lim\limits_{K\to+\infty} \left(\limsup\limits_{\varepsilon\to 0} \displaystyle\int_{\{|u^\varepsilon|>K\}} |\nabla u^\varepsilon|^2 \right) \\ \quad \le \lim\limits_{K\to+\infty} \left[\displaystyle\int_\Omega |X(u) - X(K)| \cdot |\nabla\beta(v)|\mathbb{1}_{\{u>K\}} \right. \\ \quad \left. + \displaystyle\int_\Omega |X(u) - X(-K)| \cdot |\nabla\beta(v)|\mathbb{1}_{\{u<-K\}} \right] \\ \quad + \lim\limits_{K\to+\infty} \langle f, u - T_K(u)\rangle = 0. \end{cases} \tag{18}$$

This proves (15). Let us introduce the sets $F^\varepsilon_{i,j}$,

$$F^\varepsilon_{i,j} = \{x \in \Omega : |u^\varepsilon - T_j(u)| \le i\}.$$

We are now going to prove that

$$\lim_{j\to+\infty}\left(\limsup_{\varepsilon\to 0}\int_{F^\varepsilon_{i,j}} |\nabla(u^\varepsilon - T_j(u))|^2\right) = 0 \quad \forall i \ge 1. \tag{19}$$

Thus, let us use $T_i(u^\varepsilon - T_j(u))$ as test function in the first equation of (6). We obtain

$$\int_\Omega \nabla u^\varepsilon \cdot \nabla T_i(u^\varepsilon - T_j(u)) + \int_\Omega \beta(v^\varepsilon)(X^\varepsilon)'(u^\varepsilon) \cdot \nabla T_i(u^\varepsilon - T_j(u)) \\ = \langle f, T_i(u^\varepsilon - T_j(u))\rangle. \tag{20}$$

Let us notice that

$$\nabla T_i(u^\varepsilon - T_j(u)) = 0 \text{ in } \Omega \setminus F^\varepsilon_{i,j}.$$

We can then write (20) in the form

$$\int_{F^\varepsilon_{i,j}} \nabla u^\varepsilon \cdot \nabla T_i(u^\varepsilon - T_j(u)) + \int_{F^\varepsilon_{i,j}} \beta(v^\varepsilon)(X^\varepsilon)'(u^\varepsilon) \cdot \nabla T_i(u^\varepsilon - T_j(u)) \\ = \langle f, T_i(u^\varepsilon - T_j(u))\rangle. \tag{21}$$

Since

$$|u^\varepsilon| \le |u^\varepsilon - T_j(u)| + |T_j(u)| \le i + j \quad \text{if } x \in F^\varepsilon_{i,j}\,,$$

we can write $T_{1/\varepsilon}(u^\varepsilon) = T_{i+j}(u^\varepsilon)$ for all $x \in F^\varepsilon_{i,j}$ whenever ε is sufficiently small. This gives:

$$(X^\varepsilon)'(u^\varepsilon) = X'(T_{i+j}(u^\varepsilon))T'_{i+j}(u^\varepsilon) = X'(T_{i+j}(u^\varepsilon)) \quad \text{in } F^\varepsilon_{i,j}.$$

Thus, for small $\varepsilon > 0$, the second term in the left in (21) is

$$\int_{F^\varepsilon_{i,j}} \beta(v^\varepsilon)X'(T_{i+j}(u^\varepsilon)) \cdot \nabla T_i(u^\varepsilon - T_j(u))$$

and converges to

$$\int_\Omega \beta(v)X'(T_{i+j}(u)) \cdot \nabla T_i(u - T_j(u)) \tag{22}$$

as $\varepsilon \to 0$, since

$$T_i(u^\varepsilon - T_j(u)) \to T_i(u - T_j(u)) \text{ weakly in } H_0^1$$

and $\beta(v^\varepsilon)X'(T_{i+j}(u^\varepsilon))$ is bounded in $(L^\infty(\Omega))^N$ and converges a.e. to $\beta(v)X'(T_{i+j}(u))$.

Let us introduce $H^{i,j} = (H_1^{i,j}, H_2^{i,j}, ..., H_N^{i,j})$, with

$$H^{i,j}(s) = \int_0^s T_i'(\theta - T_j(\theta))(1 - T_j'(\theta))X'(T_{i+j}(\theta))\, d\theta.$$

Then (22) can be rewritten in the form

$$\int_\Omega (\nabla \cdot H_K^{i,j}(u))\beta(v) = -\int_\Omega H^{i,j}(u) \cdot \nabla\beta(v)$$

Moreover, it is not difficult to check that

$$H^{i,j}(u) = \begin{cases} X(i+j) - X(j) & \text{if } j < |u| < i+j, \\ 0 & \text{otherwise.} \end{cases}$$

For any i, we have $H^{i,j}(u) \to 0$ a. e. as $j \to +\infty$. Since X is bounded, $H^{i,j}(u)$ is also bounded. Thus, we obtain from Lebesgue's theorem that

$$\int_\Omega H^{i,j}(u) \cdot \nabla\beta(v) \to 0 \quad \text{as } j \to \infty.$$

for all $i \geq 1$. Recalling (20) we see we have proved the following:

$$\lim_{j\to+\infty}\left(\lim_{\varepsilon\to 0}\int_{F_{i,j}^\varepsilon} \nabla u^\varepsilon \cdot \nabla T_i(u^\varepsilon - T_j(u))\right) = \lim_{j\to+\infty}\langle f, T_i(u - T_j(u))\rangle. \tag{23}$$

On the other hand,

$$\lim_{j\to+\infty}\left(\lim_{\varepsilon\to 0}\int_{F_{i,j}^\varepsilon} \nabla T_j(u) \cdot \nabla T_i(u^\varepsilon - T_j(u))\right)$$
$$= \lim_{j\to+\infty}\int_\Omega \nabla T_j(u) \cdot \nabla T_i(u - T_j(u)).$$

Consequently,

$$\begin{aligned}&\lim_{j\to+\infty}\left(\lim_{\varepsilon\to 0}\int_{F^\varepsilon_{i,j}}|\nabla(u^\varepsilon-T_j(u))|^2\right)\\&=\lim_{j\to+\infty}\left(\langle f,T_i(u-T_j(u))\rangle-\int_\Omega\nabla T_j(u)\cdot\nabla T_i(u-T_j(u))\right).\end{aligned}\tag{24}$$

Notice that, the terms on the right hand side of (24) can be bounded as follows:

$$\begin{aligned}\langle f,T_i(u-T_j(u))\rangle-&\int_\Omega\nabla T_j(u)\cdot\nabla T_i(u-T_j(u))\\&\le(\|f\|_{H^{-1}}+\|u\|)\,\|u-T_j(u)\|\end{aligned}$$

and this converges to 0 as $j\to+\infty$. Therefore, (19) is satisfied.

We can now prove that u^ε converges strongly in H^1_0. Indeed, obseve that, if $x\in\Omega\setminus F^\varepsilon_{i,j}$, then

$$|u^\varepsilon|\ge|u^\varepsilon-T_j(u)|-|T_j(u)|\ge i-j,$$

so that $\Omega\setminus F^\varepsilon_{i,j}\subset E^\varepsilon_{i-j}$, with

$$E^\varepsilon_{i-j}=\{x\in\Omega:|u^\varepsilon(x)|\ge i-j\}.$$

Therefore,

$$\begin{aligned}\tfrac12\int_\Omega|\nabla(u^\varepsilon-u)|^2&\le\frac12\int_{F^\varepsilon_{i,j}}|\nabla(u^\varepsilon-u)|^2+\frac12\int_{E^\varepsilon_{i-j}}|\nabla(u^\varepsilon-u)|^2\\&\le\int_{F^\varepsilon_{i,j}}|\nabla(u^\varepsilon-T_j(u))|^2+\int_{F^\varepsilon_{i,j}}|\nabla(T_j(u)-u)|^2\\&+\int_{E^\varepsilon_{i-j}}|\nabla u^\varepsilon|^2+\int_{E^\varepsilon_{i-j}}|\nabla u|^2\le 2(A^\varepsilon_{ij}+B^\varepsilon_{ij}+C^\varepsilon_{ij}+D^\varepsilon_{ij}).\end{aligned}\tag{25}$$

We have seen in (19) that

$$\lim_{j\to+\infty}\limsup_{\varepsilon\to 0}A^\varepsilon_{ij}=0\quad\forall i\ge 1\tag{26}$$

The second term B^ε_{ij} satisfies

$$\limsup_{\varepsilon\to0}B^\varepsilon_{ij}\le\int_\Omega|\nabla(T_j(u)-u)|^2,$$

whence we also have

$$\lim_{j\to+\infty}\limsup_{\varepsilon\to 0} B^{\varepsilon}_{ij} = 0 \quad \forall i \geq 1 \tag{27}$$

From (15) we know that

$$\lim_{j\to+\infty}\limsup_{\varepsilon\to 0} C^{\varepsilon}_{ij} = 0 \quad \text{as } i,j \to +\infty,\ i-j \to +\infty. \tag{28}$$

Finally, this is also true for D^{ε}_{ij}, since $u \in H^1_0$:

$$\lim_{j\to+\infty}\limsup_{\varepsilon\to 0} D^{\varepsilon}_{ij} = 0 \quad \text{as } i,j \to +\infty,\ i-j \to +\infty. \tag{29}$$

From (25) and (26)–(29), we deduce at once that $u^{\varepsilon} \to u$ strongly in H^1_0 as $\varepsilon \to 0$.

Fifth step. End of the proof of theorem 1.1.
Let us chose $h \in C^1_c(\mathbb{R})$ and $\varphi, \psi \in \mathcal{D}$. Multiplying the first equation in (6) by $h(u^{\varepsilon})\varphi$ and the second one by ψ and integrating by parts, we obtain:

$$\left\{\begin{array}{l} \displaystyle\int_{\Omega}(\nabla u^{\varepsilon} + \beta(v^{\varepsilon})(X^{\varepsilon})'(u^{\varepsilon}))\cdot\nabla(h(u^{\varepsilon})\varphi) = \langle f, h(u^{\varepsilon})\varphi\rangle \\ \displaystyle\int_{\Omega}(\nabla v^{\varepsilon} + \beta'(v^{\varepsilon})X^{\varepsilon}(u^{\varepsilon}))\cdot\nabla\psi = \langle g, \psi\rangle. \end{array}\right. \tag{30}$$

Since h and h' have compact support on $\mathbb{R}$, for ε sufficiently small we have

$$(X^{\varepsilon})'(t)h(t) = X'(t)h(t), \qquad (X^{\varepsilon})'(t)h'(t) = X'(t)h'(t).$$

Both functions belong to $(C^0(\mathbb{R}) \cap L^{\infty}(\mathbb{R}))^N$. Thus, we can write (30) as follows

$$\left\{\begin{array}{l} \displaystyle\int_{\Omega} h(u^{\varepsilon})\nabla u^{\varepsilon}\cdot\nabla\varphi + \int_{\Omega} h'(u^{\varepsilon})|\nabla u^{\varepsilon}|^2\varphi + \int_{\Omega}\beta(v^{\varepsilon})h(u^{\varepsilon})X'(u^{\varepsilon})\cdot\nabla\varphi \\ \qquad \displaystyle + \int_{\Omega}\beta(v^{\varepsilon})h'(u^{\varepsilon})(X'(u^{\varepsilon})\cdot\nabla u^{\varepsilon})\varphi = \langle f, h(u^{\varepsilon})\varphi\rangle \\ \displaystyle\int_{\Omega}\nabla v^{\varepsilon}\nabla\psi + \int_{\Omega}\beta'(v^{\varepsilon})X(u^{\varepsilon}))\cdot\nabla\psi = \langle g, \psi\rangle. \end{array}\right. \tag{31}$$

Now, using the strong convergence of u^ε to u in $H_0^1(\Omega)$, it is easy to pass to the limit in each term of (31); this yields

$$\begin{cases} \displaystyle\int_\Omega h(u)\nabla u\cdot\nabla\varphi + \int_\Omega h'(u)|\nabla u|^2\varphi + \int_\Omega \beta(v)h(u)X'(u)\cdot\nabla\varphi \\ \qquad\displaystyle + \int_\Omega \beta(v)h'(u)(X'(u)\cdot\nabla u)\varphi = \langle f, h(u)\varphi\rangle \\ \displaystyle\int_\Omega \nabla v\cdot\nabla\psi + \int_\Omega \beta'(v)X(u)\cdot\nabla\psi = \langle g,\psi\rangle. \end{cases}$$

This completes the proof.

Acknowledgment. The author is indebted to E. Fernández-Cara, for several fruitful discussions.

References

[1] J. Baranger, A. Mikelić, *Solutions stationnaires pour un écoulement quasi-newtonien avec echauffement visqueux,* C. R. Acad. Sci. Paris, t. 319, Série I, p. 637–642, 1994.

[2] P. Benilan, L. Boccardo, T. Gallouët, R. Gariepy, J.L. Vázquez, *On the p-Laplacian on L^1,* to appear.

[3] L. Boccardo, J.I. Díaz, D. Giachetti, F. Murat, *Existence of a solution for a weaker form of a nonlinear elliptic equation,* In "Recent Advances in Nonlinear Elliptic and Parabolic Problems", P. Benilan et al. eds., Pitman Research Notes in Math., 208, Longman, Harlow 1989.

[4] R. DiPerna, P.L. Lions, *On the Cauchy problem for Boltzmann equations: global existence and weak stability,* Annals of Math. (2) 130 (1989), No. 2, p. 321–366.

[5] R. Lewandowski, *Les équations de Stokes et de Navier-Stokes couplées avec l'équation de l'énergie cinétique turbulente,* C. R. Acad. Sci. Paris, t. 381, Série I, p. 1097–1102, 1994.

[6] P.L. Lions, F. Murat, *Solutions renormalisées d'équations elliptiques,* to appear.

[7] F. Murat, *Soluciones renormalizadas de EDP elípticas no lineales,* Lectures at the University of Sevilla (may 1992), Research Report R93023, LAN, University Paris VI, 1993.

Department of Differential Equations and Numerical Analysis
University of Sevilla
Tarfia s/n, E-41012 Sevilla
Spain
E-mail: blanca@numer.us.es.

Recibido: 13 de Marzo de 2000
Revisado: 13 de Diciembre de 2001