The geometry of abstract groups and their splittings

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ABSTRACT

A survey of splitting theorems for abstract groups and their applications. Topics covered include preliminaries, early results, Bass-Serre theory, the structure of G-trees, Serre's applications to SL_2 and length functions. Stallings' theorem, results about accessibility and bounds for splittability. Duality groups and pairs; results of Eckmann and collaborators on PD^2 groups. Relative ends, the JSJ theorems and the splitting results of Kropholler and Roller on PD^n groups.

Notions of quasi-isometry, of hyperbolic group, and of its boundary. We recall that convergence groups on the circle are Fuchsian, and survey results relating properties of the action of a hyperbolic group on its boundary to the structure of the group. Types of isometric action of a group on a Λ-tree, and the Λ-tree of a valued field, with mention of the applications made by Culler, Shalen and Morgan. Rips' theorem, and some of its applications.

Splittings over 2-ended groups and work of Sela and Bowditch, more general splitting theorems, characterisations of groups by their coarse geometry. Finally we survey the extent to which it is possible to push through the Thurston programme for PD^3 complexes and pairs: despite many advances, there remain more conjectures than theorems.

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Introduction

I will survey an area of research in group theory which is guided by geometric intuition: indeed, even the groups themselves are to be thought of as having geometric structure in some sense.

After a preliminary chapter to give some background and fix notations, I follow a roughly historical approach. Free products with amalgamation were first introduced by Hanna Neumann in 1948, and although the theory had important early applications, more intensive activity began with a major result of [Stallings 1968] and the roughly simultaneous development of Bass-Serre theory which rewrote the algebraic notion of group splitting as the geometric notion of a group acting on a tree. These led to much further work and a number of important applications, most of which were covered in the monograph [Dicks and Dunwoody 1989]. This corresponds to the first part of this survey, up to Chapter 6.

However, that monograph gives an austerely algebraic account of the theory, and appeared at about the same time as three significant developments, all inspired by geometry: the work of Gromov, and in particular the notion of (word) hyperbolic group; the breakthrough by Rips in the study of group actions on $\mathbb{R}$-trees; and the solution of the problem of convergence groups on the circle. These led to a period of renewed activity, results of greater depth, and solution of old problems. In Chapters 7-9 I survey these developments. In the final Chapter 10 I focus on a problem of particular interest to myself: the classification of Poincaré duality spaces and pairs of (formal) dimension 3: I survey the known results and formulate a number of conjectures.

In the first part of the paper I attempt to give an idea of the methods of proof of the results cited. In the second part this is no longer feasible, and I have to be content with a few indications. No originality is claimed for the results, except for Theorem 10.8, but I have sought to systematise notation throughout, and the concept of vertex pair of a splitting has enabled me to simplify the statement of many theorems.

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Our notation for references is to give the author(s) and year of publication; in the case of unpublished work first promulgated in 1999, say, we write P99. Distinct papers by the same author in the same year are distinguished by a letter (e.g. 1999a).

Contents

1. Preliminaries

We attempt in this section to summarise the preliminary knowledge required to make the rest of the article intelligible, and to fix some notation.

1.1. Generators and relations

Since most accounts of infinite groups begin by discussing generators and relations, and words; we do the same, but with the caveat that unless generators and relations have some structural origin, they are usually not a good means to understanding the group. The standard reference for arguments from generators and relators is [Lyndon and Schupp 1977].

If X is a subset of a group G we say that X *generates* G if each element $g \in G$ may be expressed as a product $y_1 y_2 \dots y_r$ such that, for each i, either $y_i \in X$ or $y_i^{-1} \in X$. Such a product is called a *word*. If G admits a finite set of generators, it is *finitely generated*: we will abbreviate this to f.g. We may always require that in this expression there is no i with $y_i y_{i+1} = 1$: we then say we have a *reduced word*. If the expression for any $g \in G$ as a reduced word is always unique, we say that G is freely generated by X, and is a *free group*. We can construct the free group $F(X)$ generated by X directly: the elements are just the reduced words; product is defined by concatenation and subsequent cancellation. The *rank* of the free group $F(X)$ is the cardinality of the set X; in general, the rank of a group is the minimum number of elements in a generating set. A free group of rank 1 is infinite cyclic (isomorphic to $\mathbb{Z}$).

If X generates G, there is a natural epimorphism $\alpha : F(X) \to G$. A subset R of $F(X)$ is said to be a set of defining *relators* for G if every element of $\operatorname{Ker} \alpha$ may be expressed as a product of terms $f^{-1} r f$ with $f \in F(X)$ and either $r \in R$ or $r^{-1} \in R$. Conversely, given any subset R of $F(X)$, the set of elements of $F(X)$ expressible as such a product is a normal subgroup $\langle R \rangle_{F(X)}$ of $F(X)$, and the quotient group

$F(X)/\langle R\rangle_{F(X)}$ is uniquely determined up to isomorphism. A set X of generators of G together with a set R of defining relators is referred to as a *presentation* of G. If G has a presentation with both X and R finite sets, we say it is *finitely presented*, or f.p. for short. A presentation is usually written as $\langle X|R\rangle$: for example, a cyclic group of order 2 is presented as $\langle x|x^2\rangle$.

If the set of generators already has some structure it may be natural to modify this description. Suppose, in particular, that we have two subgroups A, B of G and G is generated by $A\cup B$. Then we consider words which are formal products of elements of A and elements of B. In this situation, a reduced word is one in which no term in the product is equal to 1, and elements of A alternate with those of B (as always, the empty word is allowed). If each element of G is uniquely expressible as a reduced word in this sense, G is said to be the free product of A and B, and we write $G \cong A * B$. Again, we can use reduced words to synthesise a group which is a free product. For example, the free product of free groups of ranks r and s is a free group of rank $r+s$.

Clearly if A_1 can be generated by r_1 elements and A_2 by r_2 elements, the free product $A_1 * A_2$ can be generated by $r_1 + r_2$ elements. A classical result in this area is a converse.

Theorem 1.1 *[Gruško 1940] The rank of the free product $A_1 * A_2$ is the sum of those for A_1 and A_2.*

Suppose that G is a group and S a subset. The *Cayley graph* of G with respect to S is the graph $\Gamma(G;S)$ with vertex set G and edge set $G\times S$, where the edge (g,s) joins the vertices g and gs. Then $\Gamma(G;S)$ has no loop if $1 \notin S$; if $S\cap S^{-1} = \emptyset$, two vertices are joined by at most one edge. The graph is connected if and only if S generates G, and is locally finite if and only if S is finite. The group G acts freely on $\Gamma(G;S)$ on the left: $h.g = hg$ and $h.(g,s) = (hg,s)$. The quotient by the action has just one vertex, and has one edge for each element of S; it is compact if and only if S is finite.

1.2. Homology of groups

We will need some basic results about homology and cohomology of groups. There are several texts available: [Brown 1982] is closest in spirit to this survey, but contains much more detail than we require here.

Usually one works over the ground ring $\mathbb{Z}G$, but to allow more flexibility let k be any commutative ring, which may be $\mathbb{Z}$ but which we often wish to take as $\mathbb{F}_2$. For a constructive definition one contemplates a *resolution*, i.e. an exact sequence

$$\ldots \to P_n \xrightarrow{d_n} P_{n-1} \ldots \to P_1 \xrightarrow{d_1} P_0 \to k \to 0,$$

with the P_n projective kG-modules. In particular there is the standard complex, given by taking P_n the free kG-module with basis G^n and d_n given by a well known formula. However if a smaller complex can be found this may be more convenient.

In particular we say (if $k = \mathbb{Z}$) that G satisfies FP_n if there is a resolution with P_k finitely generated for all $k \leq n$.

We can always take $P_0 = kG$. The kernel of the natural map $\epsilon : kG \to k$ (defined by $\epsilon(G) = 1$) is called the *augmentation ideal* of the group ring kG; we will denote it by $\mathfrak{G}$. If X generates the group G, the elements $\{g - 1 \,|\, g \in X\}$ generate the ideal $\mathfrak{G} \lhd kG$. Thus we can take P_1 to be the free module with basis X; in particular, if G is f.g. then it satisfies FP_1. The converse is also true.

If G is presented as $\langle X|R\rangle$, we can take P_1 and P_2 as free kG-modules with bases X, R respectively. Thus if G is f.p. then it satisfies FP_2. A counterexample of [Bestvina and Brady 1997] shows that the converse does not hold. One says that a group satisfying FP_2 is *almost finitely presented*, or a.f.p. for short.

For any kG-module M the homology group $H_n(G; M)$ is the n^{th} homology group of the chain complex $P_* \otimes_{kG} M$. Elementary homological algebra shows that this is independent of the choice of resolution, that a module homomorphism $\alpha : M \to M'$ induces homomorphisms $\alpha_* : H_n(G; M) \to H_n(G; M')$, and that a short exact sequence $0 \to M' \to M \to M'' \to 0$ of kG-modules induces a long exact sequence

$\ldots H_n(G; M') \to H_n(G; M) \to H_n(G; M'') \to H_{n-1}(G; M') \to H_{n-1}(G; M) \ldots$

of homology groups. Similarly the cohomology group $H^n(G; M)$ is the n^{th} cohomology group of the cochain complex $\mathrm{Hom}_{kG}(P_*, M)$; this is independent of the choice of resolution; a module homomorphism $\alpha : M \to M'$ induces homomorphisms $\alpha^* : H^n(G; M) \to H^n(G; M')$, and a short exact sequence $0 \to M' \to M \to M'' \to 0$ of kG-modules induces a long exact sequence

$\ldots H^n(G; M') \to H^n(G; M) \to H^n(G; M'') \to H^{n+1}(G; M') \to H^{n+1}(G; M) \ldots$

of cohomology groups.

If H is a subgroup of G, we can regard any kG-module M as a kH-module, and if the first is projective, so is the second. When we wish to emphasise the distinction, we write $\mathrm{Res}\,^H_G M$ for M regarded as kH-module: here Res stands for *restriction*. The restriction $\mathrm{Res}\,^H_G P_*$ is a resolution for H, and the natural projection of $P_* \otimes_{kH} \mathrm{Res}\,^H_G M$ to $P_* \otimes_{kG} M$ induces maps $H_q(H; \mathrm{Res}\,^H_G M) \to H_q(G; M)$. Similarly, there is a natural inclusion of $\mathrm{Hom}_{kG}(P_*, M)$ in $\mathrm{Hom}_{kH}(P_*, \mathrm{Res}\,^H_G M)$ inducing maps $H^q(G; M) \to H^q(H; \mathrm{Res}\,^H_G M)$. These are called restriction maps; we will write $\mathrm{res}\,^H_G : H^q(G; M) \to H^q(H; \mathrm{Res}\,^H_G M)$ or, more briefly, $\mathrm{res} : H^q(G; M) \to H^q(H; M)$.

Dually, if N is a kH-module, the *induced* kG module $\mathrm{Ind}\,^G_H N$ is given by $kG \otimes_{kH} N$ and the *coinduced* module $\mathrm{Coind}\,^G_H N := \mathrm{Hom}_{kH}(kG, N)$. We can write $\mathrm{Ind}\,^G_H N = \bigoplus_{g \in G/H} gN$, as abelian group, where the sum is over left cosets, and g runs over a set of representatives (a left transversal of H in G). Then

$$H_q(G; \mathrm{Ind}\,^G_H N) = H_q(H; N) \quad \text{for all } q.$$

For a resolution P_* for kG gives one for kH, and $P_* \otimes_{kG} \mathrm{Ind}\,^G_H N = P_* \otimes_{kH} N$. Similarly,

$$H^q(G; \mathrm{Coind}\,^G_H N) = H^q(H; N) \quad \text{for all } q,$$

since $\mathrm{Hom}_{kG}(P_*, \mathrm{Hom}_{kH}(kG, N)) = \mathrm{Hom}_{kH}(P_*, N)$. These results are known as *Shapiro's lemma*.

In particular, taking H to be the trivial group and $N = k$, we find that $H^q(G; \mathrm{Hom}_k(kG, k))$ and $H_q(G; kG \otimes_k k)$ vanish for all $q \geq 1$. These modules will be important for us. The module $kG \otimes_k k = kG$ is a direct sum of copies of k indexed by G; the module $\mathrm{Hom}_k(kG, k)$ is a direct product of copies of k indexed by G: we will denote it by $\overline{kG}$, indicating that infinite linear combinations of elements of G are permitted.

1.3. Topological notions

This article is written somewhat from the viewpoint of a topologist, and many of the applications of group theory will be to topology. Unless otherwise stated, our topological spaces will be supposed to be simplicial complexes, or at least CW-complexes. We recall that, for any group G, there exist connected CW-complexes X, with fundamental group G, and with vanishing higher homotopy groups: $\pi_r(X) = 0$ for $r \geq 2$. The space X is determined by these conditions up to homotopy equivalence; it is called an *Eilenberg-Maclane complex*, of type $K(G, 1)$, or simply 'a $K(G, 1)$'. The homology and cohomology of the abstract group G coincide with those of the space $K(G, 1)$.

If the coefficient field k is understood, we define the *Betti numbers* of a space X by $\beta_i(X) := \dim_k H_i(X; k) = \dim_k H^i(X; k)$. If these vanish for large i (e.g. if dim X is finite), the *Euler characteristic* is $\chi(X) = \sum_i (-1)^i \beta_i(X)$.

We seek parallels between definitions and constructions in topology and in group theory. For example, the group theoretic notion of free product corresponds neatly to the topological notion of 'wedge' (attaching two spaces by identifying a single point from each). Two topological notions which particularly concern us are those of dimension and of manifold. For the analogue to manifold, see §5.1.

If a space has dimension n, it follows that its homology and cohomology, with any coefficients, vanish in dimensions higher than n. In [Wall 1965] I obtained a converse: if $H_i(\tilde{X}; \mathbb{Z}) = 0$ for all $i > n$ and $H^{n+1}(X; M) = 0$ for any coefficient bundle M then X is homotopy equivalent to a CW-complex of dimension $\max(n, 3)$.

We can impose the hypothesis of the preceding paragraph on $K(G, 1)$. This gives the notion of *cohomological dimension* (c.d.), which can be reformulated in many ways. In particular, the following are equivalent:

(i) c.d. $G \leq n$,

(ii) for any $\mathbb{Z}G$-module M, $H^q(G; M) = 0$ if $q > n$,

(iii) there is a resolution P_* for G with $P_k = 0$ for $k > n$.

We also consider other coefficient rings, and define c.d.$_k G \leq n$ if, for any kG-module M and any $q > n$, $H^q(G; M) = 0$. References for fuller treatments of cohomological dimension are [Brown 1982] and [Bieri 1976]. One further useful result: if we already know that c.d.$G < \infty$, then

$$\text{c.d.}G = \max\{n \,|\, \text{for some free } \mathbb{Z}G - \text{module } F,\ H^n(G; F) \neq 0\}.$$

It follows from Shapiro's lemma that if H is a subgroup of G and $\text{c.d.}_k G \leq n$ then $\text{c.d.}_k H \leq n$.

We denote the order of a group G by $|G|$ and the index in G of the subgroup H by $|G:H|$. A group is *torsion free* if it contains no element (other than the identity) of finite order. If H is finite and $\text{c.d.}H < \infty$, then H is trivial (the easiest way to see this is to use the fact that a cyclic group has non-zero cohomology in all even dimensions). Thus any group G with $\text{c.d.}G < \infty$ is torsion free. More generally, if H is finite and $\text{c.d.}_k H < \infty$, then $|H|$ is invertible in the ring k.

If $|G:H| < \infty$ and $\text{c.d.}G < \infty$ then $\text{c.d.}H = \text{c.d.}G$. Conversely, a theorem of Serre states that if $|G:H| < \infty$, $\text{c.d.}H < \infty$, and G is torsion free, then $\text{c.d.}H = \text{c.d.}G$. More generally, if k is a coefficient ring, say G has no k-torsion if the order of every element of G of finite order is invertible in k. Then if $|G:H| < \infty$, $\text{c.d.}_k H < \infty$, and G is k-torsion free, then [Bieri 1976, Theorem 5.7] $\text{c.d.}_k H = \text{c.d.}_k G$.

There is a relative version of dimension. Let (Y, X) be a CW pair with Y connected and $\pi_1(Y) = G$; define $\text{c.d.}(Y, X) \leq n$ if

(i) $H_i(Y, X; \mathbb{Z}G) = 0$ for all $i > n$, and

(ii) $H^{n+1}(Y, X; M) = 0$ for any $\mathbb{Z}G$-module M.

It again follows from the methods of [Wall 1965] that if $\text{c.d.}(Y, X) \leq n$ then Y is homotopy equivalent to a CW-complex obtained from X by adding cells of dimension at most $\max(n, 3)$. If $n = 1$ we will look in §4.1 for a result where $\max(n, 3)$ is replaced by 1.

1.4. Groups: concepts and examples

We have already defined free products and free groups; in particular we have the infinite cyclic group $\mathbb{Z}$, which is free on one generator. A group G with a normal subgroup N with quotient group $G/N \cong H$ is said to be an extension of N by H. If N belongs to a class $\mathcal{K}$ of groups and H to a class $\mathcal{H}$, we say 'G is $\mathcal{K}$-by-$\mathcal{H}$'. We define a *virtually* $\mathcal{K}$ *group* G to be a group possessing a subgroup of finite index which belongs to $\mathcal{K}$. If $\mathcal{K}$ is closed under taking subgroups of finite index, this is the same as a $\mathcal{K}$-by-finite group. One of our concerns below is the structure of virtually free groups.

Iterating this construction, we define a *poly-$\mathcal{K}$-group* to be one with a finite chain of subgroups $1 \subset G_1 \subset G_2 \ldots \subset G_n = G$, each normal in the next, such that the quotient groups $G_i/G_{i-1} \in \mathcal{K}$. If G is a polycyclic group, it has a subgroup G' of finite index having a chain of subgroups with all the quotients infinite cyclic. The length of this chain is the *Hirsch length* of G', or of G.

A partially ordered set satisfies the ascending chain condition (a.c.c. for short) if it contains no infinite strictly increasing sequence. A group G is said to be *noetherian* if the set of subgroups of G, ordered by inclusion, satisfies the a.c.c. It is immediate that a subgroup of a noetherian group is noetherian, and that a noetherian group is f.g.

If $\{G_i\}$ is a strictly increasing sequence of subgroups of G, with augmentation ideals $\mathfrak{G}_i$, then $\mathfrak{G}_i\mathbb{Z}G$ is a strictly increasing sequence of right ideals of the group ring $\mathbb{Z}G$. Thus if the group ring $\mathbb{Z}G$ is noetherian, so is the group G: the converse is not clear. It was shown by [Hall 1954] that a solvable group G with noetherian group ring is polycyclic-by-finite. No other example of a noetherian group ring seems to appear in the literature. It would be interesting to know if others do exist.

If G is a group and H a subgroup, the *centraliser* of H in G is

$$Z_G(H) := \{g \in G \,|\, (\forall h \in H)\, gh = hg\},$$

and the *normaliser* is

$$N_G(H) := \{g \in G \,|\, (\forall h \in H)\, g^{-1}hg \in H\},$$

or equivalently, $\{g \in G \,|\, g^{-1}Hg = H\}$. We write $H \lhd G$ to denote that H is a normal subgroup of G. The set of left cosets Hg of H in G is denoted $H\backslash G$; a section to the projection $G \to H\backslash G$ called a (left) *transversal* to H in G.

Two subgroups H, H' of a group G are said to be *commensurable* if $H \cap H'$ has finite index in both. Two abstract groups H, H' will be called commensurable if they have subgroups K, K' of finite index which are isomorphic to each other. The *commensuriser* $\mathrm{Comm}_G(H)$ is the set of $g \in G$ such that $g^{-1}Hg$ is commensurable to H.

The group of automorphisms of G is denoted $\mathrm{Aut}\,(G)$, and the (normal) subgroup of inner automorphisms by $\mathrm{Inn}\,(G)$; we write $\mathrm{Out}\,(G)$ for the quotient group $\mathrm{Aut}\,(G)/\mathrm{Inn}\,(G)$.

A closed oriented surface S_g is determined up to homeomorphism by its genus g; the fundamental group admits the presentation

$$G_g := \langle a_1, b_1, \ldots, a_g, b_g \,|\, a_1b_1a_1^{-1}b_1^{-1} \ldots a_gb_ga_g^{-1}b_g^{-1}\rangle.$$

We call these groups *surface groups*, and distinguish the case $g = 1$ where the group — a *torus group* — is isomorphic to $\mathbb{Z} \times \mathbb{Z}$, so is abelian, the universal covering of S is the euclidean plane $\mathbb{R}^2$ and G_1 acts on it by translations; from the cases $g > 1$ where G_g acts freely by isometries on the hyperbolic plane $\mathbb{H}^2$ with quotient S_g. The group of (orientation-preserving) isometries of $\mathbb{H}^2$ is $PSL_2(\mathbb{R})$, and a discrete subgroup of it is called a *Fuchsian group*. We are interested in characterisations of these and related groups; they also play a particular role in splitting theorems.

We call G a *2-orbifold group* if there is a faithful action of G on $\mathbb{R}^2$ which is *proper*, i.e. with compact stabilisers, and *uniform*, i.e. with compact quotient. If the action preserves orientation, G has a presentation

$$\langle a_1, b_1, \ldots, a_g, b_g, c_1, \ldots, c_r \,|\, a_1b_1a_1^{-1}b_1^{-1} \ldots a_gb_ga_g^{-1}c_1^{m_1} \ldots c_r^{m_r}\rangle.$$

The group with this presentation is assigned the orbifold Euler characteristic $\chi(G) = 2 - 2g + \sum_1^r (m_i^{-1} - 1)$.

If $\chi(G) < 0$, G acts uniformly by isometries on $\mathbb{H}^2$ and is a Fuchsian group. If $\chi(G) = 0$, G acts uniformly by isometries on $\mathbb{R}^2$. We will not be interested in the cases $\chi(G) > 0$, when G is finite: here G acts by isometries on the sphere S^2 (unless $g = 0, r = 1$ or $g = 0, r = 2, m_1 \neq m_2$).

The cases $g = 0, r = 3$ are referred to as triangle groups: here the presentation is $\Delta_{p,q,r} := \langle x, y, z \mid x^p, y^q, z^r, xyz \rangle$, and $\chi(\Delta_{p,q,r}) = p^{-1} + q^{-1} + r^{-1} - 1$. The Fuchsian triangle groups (where $\chi < 0$) are counterexamples in various contexts. A good introduction to 2-orbifolds and their properties is [Scott 1983].

Another important class of examples are the Baumslag-Solitar groups defined by $BS_{m,n} := \langle x, y \mid y^{-1}x^m y x^{-n} \rangle$. We have a matrix representation

$$x \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad y \mapsto \begin{pmatrix} m & 0 \\ 0 & n \end{pmatrix}.$$

In the cases when $m = 1$ or $n = 1$, the representation is faithful and the group is solvable.

2. Actions of groups on trees

2.1. Building a group from subgroups

We are interested in decomposing a group in some way into subgroups. A prototype for this is the above notion of free product: the group $A * B$ is decomposed as the free product of A and B. A first generalisation of this is, given groups A and B with a common subgroup C, to modify the free product $A * B$ by requiring the two copies of C to coincide. We will denote the result by $A *_C B$. This construction can be made precise (see below); the original definition is due to [Hanna Neumann 1948].

Given an arbitrary diagram of groups G_i and homomorphisms $\phi_e : G_{o(e)} \to G_{t(e)}$, we consider systems consisting of a group G and homomorphisms $\alpha_i : G_i \to G$ such that, for all e, $\alpha_{o(e)} = \alpha_{t(e)} \circ \phi_e$. There is a universal pushout object in the category of such systems $\{G; \alpha_i\}$, which can be defined by generators and relators. The generators are those of the several G_i; we have the relators of each G_i together with, for each edge e and generator for $G_{o(e)}$, a relator expressing the desired commutativity. We want this group to be built out of subgroups corresponding to the G_i, i.e. the α_i to be injective. For this it is not enough that the ϕ_e are injective.

Suppose given two groups G_1, G_2 and two injective homomorphisms $\phi_1, \phi_2 : G_1 \to G_2$. If we insist that $\alpha_2 \circ \phi_1 = \alpha_2 \circ \phi_2$, α_1 cannot be injective unless $\phi_1 = \phi_2$. Instead, we introduce a further generator t (referred to as the *stable letter*) and impose the equations $\alpha_2(\phi_2(x)) = t^{-1}\alpha_2(\phi_1(x))t$. The resulting group G is known as an *HNN group*, after the paper [Higman, Neumann and Neumann 1949], where it was introduced. Our notation for G, when $G_2 = A$ and $G_1 = C$, will be $A*_C$ (for precision, we should also specify the injective homomorphisms ϕ_1 and ϕ_2). For example, the Baumslag-Solitar group $BS_{m,n}$ can be written as $\mathbb{Z}*_{\mathbb{Z}}$, where $\phi_1(x) = mx$ and $\phi_2(x) = nx$.

The construction was introduced to help the authors prove embedding theorems. Among the most memorable is

Theorem 2.1 *[Higman et al. 1949]*
Any countable group is a subgroup of a two generator group.

We refer to an expression of the group G in either of the forms
$A *_C B$, where the cases $C = A$ or $C = B$ are excluded,
or $A*_C$,
as a *splitting* of G over C. The study of splittings of G is the main objective of this survey. A combinatorial analysis of words in the generators leads to

Proposition 2.2 *(i) If the maps from C to A and to B are injective, so are those from A or B to $A *_C B$.*

(ii) If the maps from C to A are injective, so is that from A to $A*_C$.

Sketch of proof (i) Choose transversals T_A, T_B, i.e. sets of coset representatives for A/C and B/C. Any product (word) of elements of A and B can be normalised as follows. If two consecutive letters of the word belong to the same subgroup (A or B), just replace them by their product. Continue doing this until elements of A and B alternate in the word, and none is in C (unless there is only one letter left). Now if the first two letters in the word are $a \in A$, $b \in B$, write $a = a'c_1$ with $1 \neq a' \in T_A$ and $c' \in C$, and replace a by a' and b by $b' = c_1 b$. Now perform the same operation with b' and the letter next following. At the end we have a word in which the final letter is in C; the rest consist of letters ($\neq 1$) in T_A alternating with ones in T_B. This we call a reduced word.

Conversely, if we take the set of reduced words and define a product by concatenation of words, followed by reduction as above, we can verify that we have a group. Hence this is the same as G, and each element is represented by a unique reduced word.

(ii) The argument here is similar, but a little messier. We have $C \subseteq A$, injective homomorphisms $\phi_1, \phi_2 : C \to A$, and an element t with $t^{-1}\phi_1(c)t = \phi_2(t)$ for all $c \in C$. Choose transversals T_1, T_2 for $A/\phi_1(C)$ and $A/\phi_2(C)$. A word is a sequence of elements of A and symbols t or t^{-1}. To normalise, we multiply together any two adjacent elements of A; also any adjacent pair tt^{-1} or $t^{-1}t$. If now the word begins with ata', write $a = b\phi_1(c)$ with $b \in T_1$ and $c \in C$ and replace a by b and a' by $\phi_2(c)a'$. If the word begins with $at^{-1}a'$, write $a = b\phi_2(c)$ with $b \in T_2$ and $c \in C$ and replace a by b and a' by $\phi_1(c)a'$.

Now perform any further cancelling. If this decreases the number of appearances of t and t^{-1}, we start again. If not, we leave bt or bt^{-1} alone and continue the reduction with the part of the word to the right of this. At the end we have a product in which elements of A alternate with appearances of t or t^{-1}; immediately preceding a t we have an element of T_1 and preceding a t^{-1} an element of T_2. Moreover, between neighbouring appearances of t and t^{-1} we must have an element $\neq 1$. This defines a reduced word. The argument continues in a similar way to case (i). □

2.2. Graphs of groups

We can represent a splitting $A *_C B$ pictorially by two vertices labelled A, B joined by an edge labelled C; and a splitting $A*_C$ by a vertex labelled A joined to itself by a loop labelled C. We can also refine splittings. If, for example, $G = A *_C B$ and $B = D *_F E$ with $C \subseteq D$, then also $G = (A *_C D) *_F E$ and we can write $G = A *_C D *_F E$, and represent this pictorially in the obvious way. To include more complicated arrangements, we next define a graph of groups.

We need to fix terminology for abstract graphs. A *graph* Γ consists of vertices and edges. We regard each edge e as oriented, and assign it its origin vertex $o(e)$ and terminal vertex $t(e)$. Each edge e has an opposite $\overline{e}$; we have $o(\overline{e}) = t(e)$, $t(\overline{e}) = o(e)$ and $\overline{\overline{e}} = e$, and also require $\overline{e} \neq e$. An edge e is called a *loop* if $o(e) = t(e)$.

A *graph* (Γ, G) *of groups* consists of a graph Γ; an assignment to each vertex v of Γ of a vertex group G_v, and to each edge e an edge group G_e where we must have $G_{\overline{e}} = G_e$; and injective homomorphisms $\phi_e : G_e \to G_{t(e)}$ (thus $\phi_{\overline{e}} : G_e \to G_{o(e)}$).

Generalising the definition of HNN group we define the fundamental group of a graph of groups. The neatest presentation is given in [Serre 1977]. We define the group $F(G)$ to be generated by the vertex groups G_v and abstract symbols t_e corresponding to the edges, subject to the relations $t_{\overline{e}} = t_e^{-1}$ and, for all e and all $x \in G_e$, $t_e \phi_e(x) t_e^{-1} = \phi_{\overline{e}}(x)$. Now choose a maximal tree $T \subseteq \Gamma$ and let $\pi_1(\Gamma, G)$ be the quotient of $F(G)$ by the group generated by those t_e with $e \in T$. There is even a formulation avoiding the choice of T. Define a map $\Gamma \to \Delta$ by identifying all vertices together; let I be the graph of groups over Γ with all vertex and edge groups trivial. We have a natural map $\pi : F(G) \to F(I)$ and an identification $F(I) \equiv \pi_1(\Delta)$; now if v is a vertex of Γ let $\pi_1(\Gamma, G) := \pi^{-1}(\operatorname{Im} \pi_1(\Gamma, v) \to \pi_1(\Delta))$.

If Γ has just one edge e, then according as e is a loop or not this reduces to the above construction of $A*_C$ or $A *_C B$. The combinatorial arguments with words generalise Proposition 2.2 to give

Theorem 2.3 *[Serre 1977, Theorem 11, p. 64] If (Γ, G) is a graph of groups, all the homomorphisms $G_v \to \pi_1(\Gamma, G)$ are injective.*

A completely different proof is given in [Bridson and Haefliger 1999, IIIC, 2.17].

2.3. Groups acting on trees

A crucial insight of [Serre 1977] was an alternative approach to the theory of graphs of groups. Let G be a group of automorphisms of a tree T. If, for $g \in G$ and e an edge of T, $g.e = \overline{e}$ we say we have an *inversion*. If G acts without inversions, the quotient of T by G has the natural structure of a graph Γ. Choose a maximal tree T_0 in Γ, and a lifting $\sigma : T_0 \to T$ to a subtree of T. Then to each vertex v of Γ we associate the stabiliser G_v of the vertex $\sigma(v)$ of T under the action of G. If e is an edge of T we define G_e to be the stabiliser of $\sigma(e)$ and ϕ_e to be the inclusion. For an edge not in T we must first choose one of e and $\overline{e}$, then define $\sigma(e)$ to be the edge of

Δ lying over e and with $o(\sigma(e)) := \sigma(o(e))$, and define G_e to be the stabiliser of $\sigma(e)$. Then we can take $\phi_{\overline{e}}$ to be the inclusion, but as $\sigma(\overline{e}) \neq \overline{\sigma(e)}$, we choose $t_e \in G$ with $t_e(\sigma(\overline{e})) = \overline{\sigma(e)}$; then ϕ_e is defined as conjugation by t_e. We have thus constructed a graph (Γ, G) of groups. Then

Theorem 2.4 *[Serre 1977, Theorem 13, p. 76] In this situation, there is a natural isomorphism $G \to \pi_1(\Gamma, G)$. The construction gives a bijection between*
(a) graphs of groups, and
(b) groups acting faithfully (without inversions) on trees.

This correspondence is usually referred to as 'Bass-Serre theory', since [Serre 1977] was based on notes by Bass on lectures by Serre. The lectures were given in 1968-9, and the year of publication gives a misleading impression of priority.

Any action of a group G on a tree T can be modified by subdividing each edge of T at its mid-point; the action on the subdivided tree then has no inversions. We will be somewhat cavalier on the question of inversions in some of the statements below.

There are numerous alternative presentations of these equivalent concepts. The article [Scott and Wall 1979] gives a topological approach. The graph of groups is 'fattened up' to a topological space $X(\Gamma, G)$ by replacing each vertex v of Γ by an Eilenberg-Maclane space $K(G_v, 1)$ and each edge e by a product $K(G_e, 1) \times [-1, 1]$. We must identify $K(G_{\overline{e}}, 1) \times [-1, 1]$ with $K(G_e, 1) \times [-1, 1]$ by changing sign in the second factor, and attach the ends by a map $K(G_e, 1) \times 1 \to K(G_{t(e)}, 1)$ realising $\phi(e)$.

In this version, the basic theorem states that $X(\Gamma, G)$ is contractible, and the fundamental group is simply defined by $\pi_1(\Gamma, G) := \pi_1(X(\Gamma, G))$. Construct a graph T by taking the universal cover of $X(\Gamma, G)$ and contracting each connected component of a preimage of a vertex space $K(G_v, 1)$ to a point, and each connected component of a preimage of an edge space $K(G_e, 1) \times [-1, 1]$ to an edge $[-1, 1]$. Thus there is a natural action of $\pi_1(\Gamma, G)$ on T. Since $X(\Gamma, G)$ is contractible, so is T, i.e. T is a tree.

Abusing notation, denote $\pi_1(\Gamma, G)$ by G. Since T is contractible, its chain groups lie in an exact sequence $0 \to C_1(T; k) \to C_0(T; k) \to k \to 0$. As the chains arise by lifting chains in Γ, we can rewrite this sequence as $0 \to \oplus_e (kG \otimes_{kG_e} k) \to \oplus_v (kG \otimes_{kG_v} k) \to k \to 0$. Tensoring over kG with any kG-module M, and taking the homology exact sequence of G for this sequence of coefficient modules reduces, using Shapiro's lemma, to

$$\ldots \to \oplus_e H_q(G_e; M) \to \oplus_v H_q(G_v; M) \to H_q(G; M) \to \oplus_e H_{q-1}(G_e; M) \to \ldots \quad (1)$$

Similarly, cochains of T give an exact sequence $0 \to k \to C^0(T; k) \to C^1(T; k) \to 0$, which can be identified with $0 \to k \to \oplus_v \mathrm{Coind}\,_{G_v}^{G} k \to \oplus_e \mathrm{Coind}\,_{G_e}^{G} k \to 0$, and tensoring over kG with M and taking the cohomology sequence of G with these coefficients gives

$$\ldots \to H^q(G; M) \to \oplus_v H^q(G_v; M) \to \oplus_e H^q(G_e; M) \to H^{q+1}(G; M) \to \ldots \quad (2)$$

These exact sequences were found by [Swan 1969] for the case $G = A *_C B$ and [Chiswell 1976c] in general.

2.4. Structure of G-trees

An edge e of a graph of groups is said to be *trivial* if it is not a loop and ϕ_e is an isomorphism. We can then define a new graph of groups by contracting e to a point E, setting $G'_E := G_{o(e)}$ and $G' = G$ on all remaining edges and vertices. The result has the same fundamental group as before. Equivalently, we take the corresponding G-tree T and identify each edge lying over e to a point to give a new G-tree T'. The G-map $T \to T'$ is called an *elementary collapse.* A graph of groups with no trivial edge is said to be *reduced.* Thus a finite graph of groups can always be reduced by collapse moves. Our definition of splitting can be rephrased: G splits over C if G is the fundamental group of a reduced graph of groups with just one edge e and $G_e = C$.

One application of Bass-Serre theory is to obtain subgroup theorems. For example, a free group is the fundamental group of a graph of groups with one vertex and all edge and vertex groups trivial; thus it acts freely on a tree. Any subgroup also acts freely, so is the fundamental group of a graph of groups with all edge and vertex groups trivial. Thus any edge which is not a loop is trivial; contracting such edges one by one (strictly speaking, we may have to contract infinitely many edges...) we arrive at a graph with just one vertex, so the group is free.

Similarly, a subgroup H of $G = A * B$ is the fundamental group of a graph of groups with all edge groups trivial and all vertex groups isomorphic to subgroups of A or B. Arguing as above leads to

Theorem 2.5 *[Kuroš 1937] Any subgroup of a free product $A * B$ is a free product of a free group and groups isomorphic (in fact, conjugate in G) to subgroups of A and B.*

More generally, consider a subgroup H of a group $G \cong A *_C B$ or $A*_C$. The decomposition of G corresponds to an action of G on a tree T. This action restricts to the subgroup H, and the theory now yields an isomorphism of H with the fundamental group of another graph of groups. Also, the edge (vertex) stabilisers of the action of H are subgroups of those of G. See e.g. [Scott and Wall 1979] for a number of further easy deductions of theorems about subgroups.

A *ray* ρ in a tree T is a sequence $\{V_i;\, i \in \mathbb{N}\}$ of distinct vertices with consecutive V_i joined by edges: thus $\rho \cong [0, \infty)$. A 2-ended ray, or *line*, is a similar sequence but with $i \in \mathbb{Z}$.

Lemma 2.1 *[Tits 1970] Let g be an automorphism of the tree T. Then either*

(i) g has an inversion, or the set of points of T fixed under g is a non-empty subtree T^g of T, or

(ii) there is a uniquely defined line $\{V_i;\, i \in \mathbb{Z}\}$ in T and a positive integer n such that, for all i, $gV_i = V_{i+n}$. Moreover, if w is a vertex of T whose distance from the nearest V_i is d, then $d(w, gw) = n + 2d$.

This dichotomy recalls the analysis of isometries of the hyperbolic plane. Accordingly, an element of type (i) is said to be *elliptic*, one of type (ii) *hyperbolic*, and in the latter case the line consisting of the V_i is called its *axis*.

Two rays define the same *end* of T if their intersection is infinite (hence another ray). Write ∂T for the set of ends of T.

A G-tree is said to be *minimal* if there is no proper G-invariant subtree. Any reduced G-tree is minimal, but not conversely.

One can now classify actions.

Lemma 2.2 *[Tits 1970] A group action on a tree with no inversions belongs to one of the following types:*

Elliptic *G has a fixed point. Every element of G is elliptic.*

Parabolic *G has no fixed point in T but has one in ∂T. Thus there is a ray ρ such that for every $g \in G$, $g\rho \cap \rho$ is a ray.*

Dihedral *G has no fixed point in $T \cup \partial T$, but there is an invariant pair of points in ∂T. They are joined by a G-invariant line on which G acts by translations and reflections via an epimorphism to the infinite dihedral group.*

Hyperbolic *There exist two hyperbolic elements of G such that the intersection of their axes is compact. In this case, sufficiently high powers of these elements freely generate a free subgroup of G.*

There are variant versions of this; see also [Bass 1976b], [Roller 1993]. Proposition 8.1 below gives a generalisation; we give further details there.

One can be more precise in the parabolic case. There is a homomorphism $\omega : G \to \mathbb{Z}$ such that for each $g \in G$ and all large enough n, $gV_n = V_{n+\omega(g)}$. Thus $g \in G$ is elliptic if $\omega(g) = 0$, hyperbolic if $\omega(g) \neq 0$. If $g(V_n) = V_n$ then also $g(V_N) = V_N$ for all $N > n$, so the stabiliser subgroups G_n of V_n form an increasing sequence, whose union is the subgroup $G^0 = \operatorname{Ker} \omega$ of elliptic elements. Thus either the sequence G_n is eventually constant or G^0 is a union of proper subgroups, hence is not f.g.

If $\omega = 0$ then $G = G^0$. Were $G = G_n$ for some n, the action would be elliptic. So G is not f.g.; for any n there are elements not fixing V_n, so in particular there is no invariant line.

If $\omega(g) = k > 0$, we can take the line $\{V_n\}$ for $n \in \mathbb{Z}$ as the axis of g. Then $gG_ng^{-1} = G_{n+k}$ for all n, so G has an HNN splitting $G_n *_{G_n}$. If the sequence G_n stabilises it must be constant, equal to G^0, so the line is stable under all of G.

[Serre 1977] defines a group G to have *property (FA)* if every action of G on a tree has a fixed point. In view of the theory, this implies that G has no non-trivial splittings.

If an action does not give a splitting, it must be elliptic or parabolic, and moreover in the latter case $G = G^0$, hence is not f.g. Conversely if G is not f.g., it is the union of a strictly increasing sequence of subgroups G_n: hence is the fundamental group of the graph of groups with underlying graph the positive real axis (with vertices at the integers) and $G_{[i,i+1]} = G_i$, and the corresponding action on a tree has a fixed end but no fixed point. Thus if G is f.g., and has no non-trivial splitting, it has (FA) [Serre 1977, Theorem 15, p 81].

Thus (FA) is the class of groups about which the theory will yield no positive information. Serre proves that every subgroup of finite index in $SL_3(\mathbb{Z})$ satisfies (FA), and gives references for the extension of this assertion to any Chevalley group of rank ≥ 2 over a ring of integers (or, more generally, S-integers) in an algebraic number field.

An f.g. group G is said to be *small* if it admits no hyperbolic action on a tree. Thus if G has no free subgroup of rank 2, it is small.

Following [Culler and Vogtmann 1996] we define G to have (AR) if every action of G on a tree has a fixed point or an invariant line. This excludes just the hyperbolic actions, and those parabolic actions where G_n does not stabilise. If every subgroup of G satisfies (AR) — such groups G are sometimes called *slender* — it follows from the above example that every subgroup is f.g., so G is noetherian. Conversely, if G is noetherian, and the subgroup H acts on a tree, the action cannot be hyperbolic, else H would contain a free subgroup of rank 2, and hence one of infinite rank. If the action is parabolic, then since H^0 is f.g., the H_n stabilise, so by the above, there is an invariant line. And if the action is elliptic or dihedral we have an invariant point or line respectively. Thus every subgroup H of G satisfies (AR) and G is slender.

2.5. SL_2 for local fields

The second half of [Serre 1977] contains an application to groups $SL_2(K)$ and their subgroups. Suppose K is a field (which for convenience I will suppose commutative) with a discrete rank 1 valuation v: thus $v : K^* \to \mathbb{Z}$ is a homomorphism, $v(0) = \infty$, and $v(x+y) \geq \min(v(x), v(y))$ for all $x, y \in K$. Write $\mathcal{O}_v := \{x \in K \,|\, v(x) \geq 0\}$. Choose an element π of K with $v(\pi) = 1$, and write $\mathfrak{m}_v$ for the ideal $\pi\mathcal{O}_v = \{x \in K \,|\, v(x) > 0\}$ and k for the residue field $\mathcal{O}_v/\mathfrak{m}_v$.

Let V be a 2 dimensional vector space over K. A *lattice* in V is a free $\mathcal{O}_v$-module of rank 2 spanning V. We say two lattices L, L' are *homothetic* if $L' = xL$ for some $x \in K$. Given two lattices L, L', there exists α such that $L'' = \alpha L' \subseteq L$ and L/L'' is a cyclic $\mathcal{O}_v$-module: this L'' is uniquely determined, so if we write $L/L'' \cong \mathcal{O}/\beta\mathcal{O}$ then $v(\beta)$ is uniquely determined by L and L'. Indeed, we can choose a basis e_1, e_2 for V such that L is the $\mathcal{O}_v$-module spanned by e_1 and e_2 and L'' is spanned by βe_1 and e_2. If we replace L, L' by $xL, x'L'$ then we can replace α by $x\alpha x'^{-1}$, so $v(\beta)$ is unaltered.

We now define a tree whose vertices are homothety classes $\{L\}$ of lattices in V. The distance of two classes is defined by $d(\{L\}, \{L'\}) := v(\beta)$: if this is 0, the classes coincide. Since L is homothetic to βL, $\beta L \subset L''$ and $L''/\beta L \cong \mathcal{O}/\beta\mathcal{O}$, the distance is symmetric. Join two classes at distance 1 by an edge. The result is connected, any two classes at distance d may be represented by the lattices $\langle e_1, e_2\rangle$ and $\langle \pi^d e_1, e_2\rangle$, and are then joined by edges via the lattices $\langle \pi^r e_1, e_2\rangle$ with $1 \leq r < d$. It is easy to see that this chain joining them is unique, so the graph is a tree.

The group $GL(V)$ acts on this tree in an natural way, and is transitive on vertices and on edges, but this action contains inversions: if we restrict to elements $A \in GL(V)$ such that $v(\det(A))$ is even, inversions do not occur. We thus restrict to the subgroup $SL(V)$. There are then two classes of vertices (two vertices are in the same class if and only if their distance is even) and one class of edges; hence we obtain a splitting of $SL_2(K)$ as amalgamated free product.

This action has other elegant properties which are elucidated by Serre. Let us write $A = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ for an element of $SL_2(K)$; then (up to conjugacy)

(a) the stabiliser of a vertex is $\{A \in SL_2(\mathcal{O}_v)\}$, in fact the stabiliser of the vertex corresponding to the lattice L is equal to $SL(L)$,

(b) the stabiliser of an edge is $\{A \in SL_2(\mathcal{O}_v) \mid c \in \mathfrak{m}_v\}$,

(c) the stabiliser of a ray is $\{A \in SL_2(\mathcal{O}_v) \mid c = 0\}$,

(d) the stabiliser of an end is (at least if K is complete) a Borel subgroup $\{A \in SL_2(K) \mid c = 0\}$, and

(e) the stabiliser of a line is a Cartan subgroup $\{A \in SL_2(K) \mid b = c = 0\}$.

A subgroup stabilises a vertex if and only if it is bounded in the v-adic topology. Hence a subgroup G of $GL(V)$ containing no bounded subgroup acts freely on the tree, so is free. In particular, this applies if K is the field $\widehat{\mathbb{Q}}_p$ of p-adic numbers and the subgroup G is discrete (in the p-adic topology) and torsion-free, giving a beautiful proof of a theorem of Ihara.

Let G be an f.g. subgroup of $SL_2(\mathbb{C})$. Now $\mathbb{C}$ is abstractly isomorphic to the algebraic closure $\overline{\mathbb{Q}_p}$ of the field $\mathbb{Q}_p$ of p-adic numbers, so $G \subset SL_2(\overline{\mathbb{Q}_p})$. The field K generated by the components of the matrices corresponding to a finite set of generators of G is finite over $\mathbb{Q}_p$, hence has a discrete valuation. We thus have an action of $SL_2(K)$, and hence of G, on a tree. Thus either there is a non-trivial splitting of G or the action of G is elliptic or parabolic. In the first case, G is conjugate into a vertex stabiliser $SL_2(\mathcal{O})$; in the latter to the group (of upper triangular matrices) stabilising an end. This argument is due to [Bass 1984].

The splitting of $SL_2(K)$ generalises to a result corresponding to any Tits system. A more elaborate construction of an action on a tree yields a splitting $GL_2(k[t]) = GL_2(k) *_{B(k)} B(k[t])$.

2.6. Length functions

The concept of a length function was introduced by [Lyndon 1963] to allow a more systematic treatment of cancellation arguments. A *length function* on G is a mapping $\ell : G \to \mathbb{R}$ such that, if we define $\delta_\ell : G \times G \to \mathbb{R}$ by

$$\delta_\ell(g,h) = \tfrac{1}{2}(\ell(g) + \ell(h^{-1}) - \ell(gh^{-1})),$$

we have

(L0) $\ell(1) = 0$,

(L1) for all $g \in G$, $\ell(g^{-1}) = \ell(g)$, (L2) if $\delta_\ell(g,g') > \delta_\ell(g,g'')$ then $\delta_\ell(g,g'') = \delta_\ell(g',g'')$.

An equivalent formulation of (L2) is that the two smallest of the three numbers $\delta_\ell(g,g')$, $\delta_\ell(g,g'')$ and $\delta_\ell(g',g'')$ are equal. As noted in [Chiswell 1976b], (L0)–(L2) already imply that, for all g, g', $\delta_\ell(g,g') \geq 0$. Hence $\ell(g) \geq 0$ for all g, and $d(g,h) := \ell(gh^{-1})$ satisfies the triangle inequality; if also $\ell(g) > 0$ for all $g \neq 1$, d gives a metric on G.

Chiswell relates length functions to group actions on trees by proving

Proposition 2.6 *(i) If (LZ0) ℓ is $\mathbb{Z}$-valued and*

(LZ0$^+$) for all $g, g' \in G$, $\delta_\ell(g,g') \in \mathbb{Z}$,

then there exist a tree T, a vertex V of T and an action of G on T such that, for any $g \in G$, $\ell(g)$ is the distance in T from V to gV.

(ii) The edge stabilisers of this action are trivial if and only if

(LM3) if $\delta_\ell(g,g') + \delta_\ell(g^{-1},g'^{-1}) > \ell(g) = \ell(g')$, then $g = g'$.

(iii) The action is free if and only if

(LM1) for all $g \neq 1 \in G$, $\ell(g^2) > \ell(g)$.

Further results of [Chiswell 1979, 1981] are

Proposition 2.7 *(i) There is an embedding of G in a group $A*_C$ such that for any $g \in G$, $\ell(g)$ is the number of occurrences of t and t^{-1} in a minimal word representing g if and only if (LZ0$^+$) holds and*

(LM1^{odd}) if $\ell(g)$ is odd, $\ell(g^2) > \ell(g)$.

(ii) There is an embedding of G in a free product of subgroups A_i with the common subgroup C amalgamated such that for any $g \in G$, $\ell(g)$ is the length of a minimal word representing g if and only if

(LM1^{even}) if $\ell(g)$ is even and $\neq 0$ then $\ell(g^2) > \ell(g)$ and

(LM2) for no g is $\ell(g^2) = 1 + \ell(g)$.

(iii) The group G itself has such a decomposition if and only if we further have

(LM4) G is generated by $\{g \in G \mid \ell(g) \leq 1\}$.

Many of the arguments concerning length functions go through without the restriction to integer-valued lengths: we will return to this in Chapter 8.

2.7. Alternative constructions

A number of authors have considered extensions of the theory from trees to complexes of higher dimension: we refer to [Bridson and Haefliger 1999] for a full account and earlier references, and content ourselves here with a brief outline.

Suppose given an action of a group G on a simplicial complex, or more generally a complex Y of polyhedral cells, such that if $g \in G$ preserves a cell σ then it fixes σ pointwise. We can form the quotient $G\backslash Y$, which is also a complex of cells, perhaps with some identifications on their boundaries. Suppose first that there is a subcomplex X such that the projection $X \to G\backslash Y$ is an isomorphism. Then for each cell σ of

X we have the stabilising subgroup G_σ and for each inclusion $\sigma \subseteq \tau$ an inclusion $G_\tau \subseteq G_\sigma$. We can call such an arrangement of groups and inclusions a *complex of groups*. As above, there is a natural definition of a universal group $\hat{G}$ for the complex. There is also a natural construction of a $\hat{G}$-complex Y giving rise to the complex of groups in the way just described. We would like an analogue of Theorem 2.3 stating that the maps $G_\sigma \to \hat{G}$ are injective. If this holds, the complex $(X, \{G_\sigma\})$ is said to be *developable*. However, examples show that this is not true in all cases.

Again let G act on Y, but drop the requirement of the existence of an X. Then for each cell σ of $G\backslash Y$ we can choose a preimage cell σ' of Y and define G_σ to be the stabiliser of σ': a different choice will change this to a conjugate subgroup. Thus for an inclusion $\sigma \subseteq \tau$ a map $G_\tau \to G_\sigma$ can only be defined after composing the inclusion in G with an inner automorphism. Thus if $\rho \subset \sigma \subset \tau$, the composite of the maps $G_\rho \to G_\sigma \to G_\tau$ will differ from the chosen map $G_\rho \to G_\tau$ by conjugation by some $g_{\rho.\sigma,\tau} \in G_\tau$. These elements satisfy certain compatibility conditions. The theory of complexes $(X, \{G_\sigma\})$ of groups is now elaborated with this additional complication; again there is a concept of a complex being developable.

One result which holds in general is that if we restrict the complex to (the groups, morphisms and conjugating elements for) the *star* $st(\sigma)$ of a given cell σ, i.e. the cells containing σ in their closure, the resulting local complex is always developable. Its universal space may be denoted $\tilde{X}(st(\sigma))$. The best general result is Theorem 7.1 below.

We now give a brief discussion of pregroups, as defined by [Stallings 1971]. A pregroup consists of a set P, an element $1 \in P$, an involution i of P, denoted $i(x) = \overline{x}$, a subset $D \subseteq P \times P$ and a map $m : D \to P$, denoted $m(x, y) = xy$, satisfying

(P1) For all $x \in P$, $(x, 1), (1, x) \in D$ and $x1 = 1x = x$.

(P2) For all $x \in P$, $(x, \overline{x}), (\overline{x}, x) \in D$ and $x\overline{x} = \overline{x}x = 1$.

(P3) If $(x, y) \in D$ then $(\overline{y}, \overline{x}) \in D$ and $\overline{y}\overline{x} = \overline{xy}$.

(P4) If $(x, y) \in D$ and $(y, z) \in D$ then $(xy, z) \in D$ if and only if $(x, yz) \in D$, and then $x(yz) = (xy)z$.

(P5) If $(w, x) \in D$, $(x, y) \in D$ and $(y, z) \in D$ then either $(w, xy) \in D$ or $(xy, z) \in D$.

We can define a group $U(P)$ whose generators are the elements of P and with relators expressing the conditions that 1 is the unit element, $\overline{x}$ is the inverse of the element x, and whenever $(x, y) \in D$, xy is the product of the elements x and y. It is shown in [Stallings 1971] that when P is a pregroup the map from P to $U(P)$ is injective.

In [Stallings 1973] a CW complex of type $K(U(P), 1)$ and its universal covering complex are constructed using only P. A subgroup theorem, exhibiting any subgroup of $U(P)$ as the universal group of a pregroup constructed in a natural way, is given in [Rimlinger 1987a]. See also [Rimlinger 1987b].

3. Stallings' theorem

3.1. Ends

We next recall the theory of ends. In [Freudenthal 1931] the end point compactification was constructed, and ends defined for a peripherally compact (e.g. locally compact) topological space X. The application to groups was initiated by [Hopf 1943]. The account may be simplified by assuming X a simplicial complex, and it is then easy to see that only the 1-dimensional skeleton of X is required.

Let X be a locally finite simplicial complex. For any finite subcomplex Y we can remove Y from X and count the number $n(Y)$ of infinite connected components of the complement $Y - X$. The *number of ends* $e(X)$ is defined to be the least upper bound of the integers $n(L)$.

Write $C^*(X)$ for the chain complex consisting of (infinite) cochains of X, where the coefficient group k is a field; and $C^*_f(X)$ for the subcomplex of finite cochains (i.e. taking a non-zero value on only finitely many simplices). Write $C^*_e(X)$ for the quotient complex, so there is a short exact sequence $0 \to C^*_f(X) \to C^*(X) \to C^*_e(X) \to 0$, with an exact homology sequence

$$H^0_f(X) \to H^0(X) \to H^0_e(X) \to H^1_f(X) \to H^1(X).$$

Then $e(X) = \dim_k H^0_e(X)$. For if Y is a subcomplex, there are $n(Y)$ different 0-cochains, each equal to 1 on one component of $X - Y$ and zero on the others. Each has coboundary supported in Y, hence finite, and they are independent modulo finite cochains, so $\dim H^0_e(X) \geq n(Y)$. Conversely, given n 0-cochains with finite boundaries which are independent modulo finite cochains, take Y to be a finite subcomplex containing all the boundaries; then it follows that $n(Y) \geq n$.

For G an abstract group, and k a field, write $\overline{kG}$ for the group of maps $G \to k$, and kG for the subgroup of those with finite support. Say that $A, B \in \overline{kG}$ are *almost equal*, and write $A =^a B$, if $A - B$ is finite, i.e. $A - B \in kG$. An element $A \in \overline{kG}$ is said to be *almost invariant* if, for all $g \in G$, $Ag =^a A$. This invaluable notion was introduced by [Cohen 1970].

We have a short exact sequence $0 \to kG \to \overline{kG} \to k_eG \to 0$; we can regard the terms as G-modules. We can define $e(G) := \dim_k H^0(G; k_eG)$; as above, this is independent of the field k. Note that if G is finite, $k_eG = 0$ so $e(G) = 0$.

The sequence of coefficient groups has an exact cohomology sequence. By Shapiro's lemma, $H^1(G; \overline{kG}) = 0$ and $H^0(G; \overline{kG}) = k$. If G is infinite, we see directly that $H^0(G; kG) = 0$. Thus the cohomology sequence reduces to

$$0 \to k \to H^0(G; k_eG) \to H^1(G; kG) \to 0. \tag{1}$$

Hence if G is infinite, $e(G) = 1 + \dim_k H^1(G; kG)$.

If $k = \mathbb{F}_2$, we can identify $\overline{kG}$ and kG with the groups of all subsets of G (under Boolean addition, i.e. symmetric difference) and of finite subsets. We say that an almost invariant $A \subseteq G$ is *proper* if both A and its complement $A^* := A + G$ are infinite. Write QG for the subgroup of $\overline{\mathbb{F}_2 G}$ consisting of almost invariant sets. Since this is the set of elements of $\overline{\mathbb{F}_2 G}$ whose image in $k_e G$ is G-invariant, we have $e(G) = \dim_{\mathbb{F}_2}(QG/\mathbb{F}_2 G)$.

For f.g. groups we can relate the two definitions. Choose a finite set S of generators of G and form the Cayley graph $X := \Gamma(G; S)$. We can identify $C^0(X)$ with $\overline{kG}$ and $C^0_f(X)$ with kG. To calculate ends we need to study 0-cochains whose coboundary is a finite 1-cochain. But the coboundary of A is the set of edges (g, s) such that just one of g, gs lies in A, and since S is finite, this set is finite if and only if, for each $s \in S$, $A + As$ is finite; i.e. A is almost invariant. Thus $e(G) = e(\Gamma(G; S))$. More generally, the theory of Freudenthal and Hopf shows that if G acts freely (or indeed properly) and uniformly on a connected X, then $e(G) = e(X)$. Thus if H has finite index in G, $e(H) = e(G)$ and if $K \lhd G$ is a finite normal subgroup, $e(G) = e(G/K)$.

Theorem 3.1 *[Hopf 1943] (i) If G is f.g., then $e(G)$ is equal to $0, 1, 2$ or ∞.*

(ii) We have $e(G) = 2$ if and only if G has an infinite cyclic subgroup of finite index.

Sketch of proof Suppose G infinite and that $e(G) = n < \infty$. Let S generate G. Let L be a finite connected subgraph of $\Gamma(G, S)$ such that $\Gamma(G, S) - L$ consists of n infinite components $V_1, \ldots, V_n$. As G is infinite, we can find $g \in G$ with $gL \cap L = \emptyset$, so $gL \subset V_1$, say. Since $\Gamma(G, S) - (L \cup gL)$ has only n infinite components, only one of the components of $V_1 - gL$ is infinite. But $L \cup V_2 \cup \ldots \cup V_n$ is connected, so $\Gamma(G, S) - gL$ has at most two infinite components, and $n \leq 2$.

If $n = 2$, we have $gV_1 \subset V_1$, the difference is finite, and maps onto the quotient of $\Gamma(G, S)$ by the subgroup $\langle g \rangle$ generated by g. Hence this quotient is compact, and $\langle g \rangle$ has finite index in G. ☐

It follows that $e(G) = 2$ if and only if G has a finite normal subgroup F with G/F isomorphic to $\mathbb{Z}$ or to the infinite dihedral group $\mathbb{Z}_2 * \mathbb{Z}_2$ (and hence, in particular, G splits over F). This was first shown in [Wall 1967]; it follows more directly from the theory below.

Another characterisation, which anticipates later generalisations, is given in [Scott and Wall 1979, 5.8]: if H is an infinite subgroup of G, and A a proper almost invariant set such that $hA =^a A$ for all $h \in H$, then H has finite index in G.

3.2. Stallings' theorem

This picture is rounded off by a major result of Stallings. The original statement was

Theorem 3.2 *[Stallings 1968] Let G be a finitely presented torsion-free group with $e(G) > 1$. Then G splits as a free product.*

Stallings was led to his argument by considering the proofs of the Sphere Theorem for 3 dimensional manifolds: see [Papakyriakopoulos 1957] and [Whitehead 1958]. We now sketch his proof.

Sketch of proof First define a *bipolar structure* on a group G to be a partition of $G - \{1\}$ into disjoint sets denoted AA, AB, BA and BB, at least two of which are non-empty, such that

(i) if $X \neq Y$, then $WX.YZ \subseteq WZ$,

(ii) $(XY)^{-1} = YX$,

where $W, X, \ldots$ are variables taking the values A, B. Define a *broken word* of length n to be a sequence $g_1, \ldots, g_n \in G - \{1\}$, with $g_i \in X_iY_i$ say, such that $X_{i+1} \neq Y_i$ for $1 \leq i < n$. This sequence represents the element $g = g_1g_2 \cdots g_n \in X_1Y_n \subset G$. We also require

(iii) For each $g \in G - \{1\}$ there is $N(g) \in \mathbb{N}$ such that no broken word representing g has length $> N(g)$.

An element $g \in G - \{1\}$ is called *irreducible* if we can take $N(g) = 1$. A short combinatorial argument shows that every element of G can be uniquely expressed as a broken word in irreducible elements.

Write A for the set consisting of 1 and the irreducible elements of AA; similarly B: then A and B are subgroups of G. If AB contains no irreducible element, $G = A * B$; if $t \in AB$ is irreducible, then $G = A*_{\{1\}} = A * \mathbb{Z}$, with t generating the second free factor.

Next let L be a connected locally finite simplicial complex with $H^1(L) = 0$. Call a 1-cocyle connected if it cannot be written as the sum of two disjoint cocycles. We regard a 0-cochain E as a subset of the set of vertices of L; we call E connected if any two of its vertices can be joined by a path consisting of edges with both ends in E. We denote the complement of E by E^*; it will often be convenient below to write $E^?$ to denote either E or E^*. Then

(i) If P is a connected 1-cocycle, and $\delta(E) = P$, then both 0-cochains $E^?$ are connected.

(ii) If P and Q are disjoint connected 1-cocycles, $\delta(E) = P$ and $\delta(F) = Q$, then at least one of the four cocycles $E^? \cap F^?$ is zero.

If $H^1_f(L) \neq 0$ there is a finite 1-cocyle $\delta(E) = P$ such that both $E^?$ are infinite. Among such P choose a minimal one, i.e. one whose support consists of the smallest possible number of 1-simplices.

(iii) Let $P = \delta(E)$, $Q = \delta(F)$ be minimal finite 1-cocycles. Then P is connected, both of $E^?$ are connected and infinite. Moreover, at least one of $E^? \cap F^?$ is finite.

Now apply this to the case where K is a finite simplicial complex with fundamental group G and L its universal cover, so G acts freely on L and $H^1(L) = 0$. Recall that $e(G) = 1 + \dim H^1_f(L)$. The case with 2 ends is characterised by

(iv) Let $P = \delta(E)$ be a minimal finite 1-cocycle and $g \in G$ of infinite order. If $P + gP$ bounds a finite 0-cocycle, $\dim_{\mathbb{F}_2} H^1_f(L) = 1$. Hence

(v) If $e(G) = \infty$ and $P = \delta(E)$ is a minimal finite 1-cocycle, then for every $g \in G$ of infinite order exactly one of the four sets $E^? \cap gE^?$ is finite.

Now if G is torsion free, define g to belong to AA, AB, BA or BB respectively according as $E \cap gE, E \cap gE^*, E^* \cap gE$ or $E^* \cap gE^*$ is finite. One can verify that this defines a bipolar structure, and hence G is a free product. □

Stallings' paper was immediately followed by [Bergman 1968], extending the result to all f.g. groups. Bergman's key construction is to take a locally finite graph X on which G acts so that $G\backslash X$ is a finite graph; and form L by adjoining to X an edge vw for each arc $v = v_0, v_1, \ldots, v_n = w$ in X joining the vertices. We have coboundary maps $\delta_X : C^0(X) \to C^1(X)$ and $\delta_L : C^0(X) = C^0(L) \to C^1(L)$ and a natural map $\eta : C^1(X) \to C^1(L)$ such that the value of $\eta(c)$ on an edge of L is the sum of the values of c on the corresponding edges of X; then $\delta_L = \eta \circ \delta_X$. If $c \in C^1(X)$ is a cochain with finite support, let $\mu(c) := (\mu_1, \mu_2, \ldots)$, where $\mu_k(c)$ is the number of edges in the support of $\eta(c)$ corresponding to arcs of length k. A key lemma states that the lexicographic ordering of sequences $\mu(c)$, $c \in C^1_f(X)$ is a well-ordering, so that any subset has a least element.

Now write $B(X) := \{c \in C^0(X) \,|\, \delta_X(c) \in C^1_f(X)\}$, and say that $c \in B(X)$ is *minimal* if c and c^* are infinite and $\mu(\delta_X(c))$ has the least possible value subject to this. Then if c and c' are both minimal, one of $c^? \cap c'^?$ is empty. One can now tie in with Stallings' arguments.

Geometrically, one can think in terms of cuts of the Cayley graph — i.e. finite sets of edges which separate the graph into more than one infinite subset. Then μ is a sort of norm defined on the set of such cuts.

Alternative arguments were given by [Dunwoody 1969] and [Cohen 1970]. Dunwoody defined minimality of a cut by simply counting the number of edges in $\delta_X(c)$; as this gives a weaker notion, an additional argument is required.

A very minor extension to the arguments (due to Stallings) removes the restriction that G is torsion free; the full result is given in [Wall 1971] and in [Stallings 1971].

Theorem 3.3 *Let G be an f.g. group. Then $e(G) > 1$ if and only if G splits over a finite subgroup.*

3.3. Virtually free groups and groups with c.d. 1

Part of the motivation for Stallings' original paper was a question of Serre about virtually free groups G. If G is also f.g. and torsion free, then if the free subgroup has rank at least 2 it, and hence also G, has infinitely many ends, so G is a free product. Each factor also has a free subgroup of finite index. Since, by Gruško's theorem, free product decompositions must terminate, we reduce to the case when the free subgroup has rank 1. But then G also must be infinite cyclic. Hence

Theorem 3.4 *[Stallings 1968] If G is f.g., torsion free, and virtually free, then G is free.*

More generally, we have

Theorem 3.5 *For any G, the following are equivalent:*
(a) There exists a G-tree such that the orders of the vertex stabilisers are bounded.
(b) G is the fundamental group of a graph of groups of bounded orders.
(c) G is virtually free.

This was proved by [Karrass et al. 1973], using Stallings' results, for the case when G is f.g. It was extended to the countable case by [Cohen 1973], using ideas of [Swan 1969], and Cohen also reduced the general case to an embedding theorem which was then proved by [Scott 1974]. We mention a more direct approach below.

A related application concerns groups of c.d. 1. It is easy to show that if c.d. $G = 1$ then G has more than one end. An induction as above now shows, if G is f.g., that it is free. It was shown by Swan how to remove the finiteness assumption:

Theorem 3.6 *[Swan 1969]; see also [Cohen 1972a] Any group of cohomological dimension 1 is free.*

The case c.d.$_k G = 1$ is considered in Theorem 4.4.

4. Bounds on splittings

4.1. Early results on accessibility

In the case of free product decompositions, Gruško's Theorem 1.1 (a topological proof of which was given by [Stallings 1965] and a proof using Bass-Serre theory by [Chiswell 1976]) tells us that, in the case of f.g. groups, the process of decomposing a group, then its factors, and so on must terminate. Is there a corresponding result for free products with amalgamation, at least in the case when the amalgamation takes place over finite subgroups? The question first arose as follows.

Let (Y, X) be a CW pair such that Y obtained from X by attaching cells of dimension ≤ 1, then (i) c.d.$(Y, X) \leq 1$ and (ii) for each $x \in X$, $\pi_1(X, x) \to \pi_1(Y, x)$ is injective. (The last clause gives a condition for each component of X.) Say that (Y, X) satisfies (D1) if these conditions hold. In 1971 I conjectured

Conjecture 4.1 *If (Y, X) satisfies (D1), there exist a CW-complex Z obtained from X by attaching cells of dimension ≤ 1 and a homotopy equivalence $Z \to Y$ (rel X).*

We may suppose Y connected. If $\pi_1(Y) = G$, and X has components X_i, with respective fundamental groups H_i, the question depends only on G and the subgroups H_i. The condition is equivalent to the kernel of the natural map $\bigoplus_i (\mathbb{Z}G \otimes_{\mathbb{Z}H_i} \mathbb{Z}) \to \mathbb{Z}$ being a projective $\mathbb{Z}G$-module. One can show that if $(G; \{H_i\})$ satisfies (D1) and G has one end, then one of the H_i is equal to G and the rest are trivial; the cases when G has 0 or 2 ends can also be handled directly. Now if G satisfies

(P) G has infinitely many ends and is not a free product,
then G is of the form $A *_C B$ or $A *_C$ with C finite and non-trivial, and A and B

satisfy (P). One can show that if Conjecture 4.1 holds for A and B, then it holds for G.

In [Wall 1971] I defined an f.g. group G to be 0-accessible if it has at most one end, and n-accessible if it splits as $A *_C B$ or $A *_C$ with C finite and A and B $(n-1)$-accessible; say G is *accessible* if it is n-accessible for some n. Then if all f.g. groups are accessible, Conjecture 4.1 holds if G is f.g.

Accessibility of G is equivalent to G being the fundamental group of a finite graph of groups with finite edge groups and vertex groups having at most one end. We argue by induction on n. If G is 0-accessible, we take the graph with a single vertex. If G is $A *_C B$ or $A *_C$ with C finite and A and B coming from graphs of groups, then the finite subgroup C is conjugate into a vertex group for each graph, so adding an edge with group C yields a graph of groups with fundamental group G. Conversely, suppose G is the fundamental group of a graph Γ of groups with n edges, finite edge groups and vertex groups having at most one end. Each edge determines a splitting of G, and the graph for A (or the union of those for A and B) has only $(n-1)$ edges. It follows by induction that G is accessible.

For any right $\mathbb{Z}G$-module M, a (right) *derivation* $d : G \to M$ is a mapping satisfying the rule $((xy)d) = (xd)y + (yd)$ for all $x, y \in G$; the derivation is *inner* if, for some $m \in M$, $xd = m + mx$ for all $x \in G$. The set $\mathrm{Der}(G, M)$ of all derivations is a left G-module; inner derivations a submodule $\mathrm{Inn}(G, M)$; the quotient is $H^1(G; M) = \mathrm{Der}\,(G, M)/\mathrm{Inn}\,(G, M)$.

Set $\mathfrak{D}G := \mathbb{Z} \otimes_{\mathbb{Z}G} Der(G, \mathbb{Z}G)$.

Theorem 4.1 *[Bamford and Dunwoody 1976]*
(i) $\mathfrak{D}$ is a contravariant functor from the category of f.g. groups and injective homomorphisms to the category of abelian groups.
(ii) If G is finite of order N, then $\mathfrak{D}G \cong \mathbb{Z}/N$.
(iii) If G has one end, then $\mathfrak{D}G \cong \mathbb{Z}$.
(iv) If $G = A *_C B$ with C finite, then $\mathfrak{D}G$ is the pullback of $\mathfrak{D}A \to \mathfrak{D}C \leftarrow \mathfrak{D}B$.
(v) If $G = A *_C$ with C finite, then there is a pullback diagram $\mathfrak{D}A \to \mathfrak{D}C \leftarrow \mathbb{Z}$.

It follows at once that if G is accessible, then $\mathfrak{D}G$ is finitely generated, and that if so, G is the fundamental group of a graph of groups with vertex groups having at most one end and edge stabilisers finite, and the number of edges in the graph is 1 less than the rank of $\mathfrak{D}G$. With rather more effort, the authors are able to show that if G is a.f.p. and $\mathfrak{D}G$ is finitely generated, then G is accessible.

Using some of the ideas of [Cohen 1972a], [Dunwoody 1979] gave a complete reworking of Stallings' theory leading to stronger conclusions. Taking the viewpoint of Bass-Serre theory, he aims at a direct construction of a tree on which G acts. First he gives a new axiomatic approach to trees.

If e, f are distinct (oriented) edges of a tree T, define $e < f$: if the geodesic from

$o(e)$ to $t(f)$ has e as its first edge and f as its last. Then it is easy to verify that this is a partial order satisfying:

(T1) If $e \leq f$ then $\overline{f} \leq \overline{e}$;

(T2) If $e \leq f$ there are only finitely many g with $e \leq g \leq f$;

(T3) For any e, f at least one of $e \leq f$, $e \leq \overline{f}$, $\overline{e} \leq f$ and $\overline{e} \leq \overline{f}$ holds.

(T4) For no e, f is $e \leq f$ and $e \leq \overline{f}$.

(T5) For no e, f is $e \leq f$ and $\overline{e} \leq f$.

Conversely, if these hold, one can uniquely construct a tree with these edges and this ordering.

The idea is that the edges e will be chosen from the set QG of proper almost invariant subsets E of G: more precisely, from equivalence classes of these subsets, where

$$E \sim F \text{ if } E =^a F \text{ and } |E - F| = |F - E|.$$

Define also $\overline{e}$ to be the class of E^* and $e \leq f$ if there are representative $E, F \in QG$ with $E \subset F$ and $F - E$ finite. This ensures that (T2) holds, and (T1), (T4) and (T5) are automatic. To ensure we get a G-tree we must pick a collection invariant under (the left action of) G. The key is thus (T3).

We say that two subsets A and B of a G-set X are *nested* if one (at least) of the four sets $A^? \cap B^?$ is empty, and *almost nested* if one is finite. Condition (T3) thus translates as saying that the chosen collection of almost invariant sets is almost nested.

Choose a finite set S of generators of G, and form the Cayley graph $\Gamma(G; S)$. We regard subsets B of G as 0-cochains in Γ; recall that B is almost invariant if and only if δB is finite, considered as a set of edges. Stallings' concepts of connectedness and minimality now play a role.

Say that $B \subset G$ is *connected* if $C \subseteq B$, $\delta C \subseteq \delta B$ implies $C = B$. Then any $B \in QG$ has finitely many components, each $\in QG$.

If $B, C \in QG$, $B =^a \{g | gC \subset B \text{ or } gC^* \subset B\}$.

For B, C subsets of G, define $B \succeq C$ if either δB has more edges than δC, or both have the same number and $B \supseteq C$.

A decreasing sequence of sets $B_n \in QG$, with non-empty intersection, and with $|\delta B_n|$ constant, is eventually constant. Further, a sequence $\{B_n\}$ of connected sets, with non-empty intersection, and such that for each n, $B_n \succeq B_{n+1}$ is eventually constant.

If $B, C \in QG$ and $|\delta B| = |\delta C| = c$ then either all $|\delta(B^? \cap C^?)| = c$ or one of them is $< c$.

Now we can find a minimal connected element B of QG containing 1. It follows that the sets gB are almost nested. Thus if we take their classes as the edges, all of (T1)-(T5) are satisfied; we have a graph on which G acts; the edge groups are all conjugate and are finite. This proves Theorem 3.3.

An abstraction of this argument gives the

Theorem 4.2 *[Dicks and Dunwoody 1989] Let G be a group, X a G-set with finite stabilisers, $A \neq \emptyset$. Then if V is a G-invariant almost-equality class in the set of maps $X \to A$, there exists a G-tree with vertex set V and finite edge stabilisers.*

Dicks and Dunwoody use this 'almost stability theorem' as the key result for deducing the main applications of Stallings' theorem.

Theorem 4.3 *[Dunwoody 1979] Suppose G a.f.p. Then G is accessible if and only if $H^1(G; kG)$ is finitely generated as G-module.*

Sketch of proof That accessibility implies finite generation follows by a similar argument to Proposition 4.1.

Next suppose $H^1(G; kG)$ finitely generated. Then also $M(G) := \{\alpha \in \overline{kG} \mid \alpha g =^a \alpha \text{ for all } g \in G\}$ is finitely generated. But $M(G)$ is generated by the characteristic functions of connected almost invariant sets.

For $B \in QG$ connected, there is a finite set Ψ_B of connected sets C such that $B \succeq C$ and $B \neq C$, such that for all $g \in G$, at least one of $B^? \cap gB^?$ has every component equal to xC for some $x \in G$ and some $C \in \Psi$. We can choose a finite set Φ of connected almost invariant sets which generates $M(G)$ and is such that for any $B \in \Phi$, $\Psi_B \subset G\Phi$.

If the classes of $G.\Phi$ are taken as the edge set of a graph, it can be shown that, for each vertex group G_v each $B \in \Phi$ and each $g \in G$, either $gG_v \subset^a B$ or $gG_v \subset^a B^*$. It follows that the restriction of the element $D_B \in Der(G, kG)$ corresponding to B is an inner derivation in Der (G_v, kG). Since the classes of the D_B generate $H^1(G, kG)$, the restriction map to $H^1(G_v; kG)$ is zero. We have the exact sequence (2) $H^1(G; kG) \to \oplus_v H^1(G_v; kG) \to \oplus_e H^1(G_e; kG)$. Since the edge groups are finite, $H^1(G_e; kG)$ is zero. Hence also each $H^1(G_v; kG)$, and hence also $H^1(G_v; kG_v)$ vanishes. So each vertex group has one end, and G is accessible. □

Now if c.d.$_k G = 1$, there is an exact sequence $0 \to P_1 \to P_0 \to k \to 0$ with P_0 and P_1 projective kG-modules which, if G is a.f.p., may be taken finitely generated. Thus $H^1(G; kG)$ is a quotient of Hom$_{kG}(P_1, kG)$, which is finitely generated. Hence G is accessible. Each vertex group has $H^1(G_v; kG_v) = 0$; this together with c.d.$_k G_v \leq 1$ implies c.d.$_k G_v = 0$. Hence G_v is finite and k-torsion free.

If G is f.g. and has a free subgroup F of index N, then taking $k = \mathbb{Z}[N^{-1}]$ we see that c.d.$_k G = 1$, so the above applies.

Arguments as in the torsion free case allow the relaxation of the finiteness conditions, and so yield

Theorem 4.4 *[Dunwoody 1979] If k is a non-zero ring, the following are equivalent:*

(a) c.d.$_k$ $G \leq 1$;

(b) G is the fundamental group of a graph of groups with orders invertible in k.

(c) There exists a G-tree such that the vertex stabilisers are finite groups of orders invertible in k.

If also G is f.g., it has a free subgroup of finite index invertible in k.

The paper [Dunwoody 1979] also contains a relativised form of these results. If H is a subgroup of G we say that a splitting of G is *adapted* to H if H is conjugate into a vertex group, or equivalently, fixes a vertex of the G-tree corresponding to the splitting. For example, if X is an almost invariant set whose left translates gX are almost nested, then if $XH = H$ the splitting given by taking the almost equality classes of the gX as edges is adapted to H.

To study adapted splittings, we introduce the kernel $K(G;H)$ of the restriction map $\rho : H^1(G;\mathbb{Z}G) \to H^1(H;\mathbb{Z}G)$. More generally, if $\mathbf{S} = \{S_i\}$ is a collection of subgroups of G, a splitting is adapted to $\mathbf{S}$ if each S_i fixes a vertex of the G-tree. Write $\rho_i : H^1(G;\mathbb{Z}G) \to H^1(S_i;\mathbb{Z}G)$ for the restriction maps, and $K(G;\mathbf{S})$ for the intersection of their kernels.

A first relativisation of Stallings' theorem is due to Swarup.

Theorem 4.5 *[Swarup 1977] Let G be an f.g. group and $\mathbf{S}$ a set of subgroups. If $K(G;\mathbf{S})$ is non-zero, then there is a splitting $G = G_1 *_F G_2$ or $G = G_1 *_F$ of G over a finite group which is adapted to $\mathbf{S}$.*

In the case when there is only one subgroup, the result was proved by [Swan 1969] if G is torsion free, and by [Swarup 1975] for any f.g. G. A key argument in all these proofs is to embed H in an f.g. group H_1, set $K := H_1 \times \mathbb{Z} \times \mathbb{Z}$ and $L := K *_H G$. Then L has more than one end, so Theorem 3.3 gives a splitting over a finite group. We can now argue on such a splitting.

We now have versions of Theorems 4.1, 4.3 adapted to a subgroup H. Say that a pair (G,H) with $H \subseteq G$ is finitely generated if G is generated by H and a finite set. Say that (G,H) is *accessible* if G is the fundamental group of a finite graph of groups with finite edge groups and vertex groups having at most one end except for one vertex group which has $H \subseteq G_v$ and $K(G_v,H) = 0$. We are led to consider the almost invariant subsets X of G with $XH = H$.

Define $\mathfrak{D}(G,H) := \mathbb{Z} \otimes_{\mathbb{Z}G} D_1(G,H)$, where $D_1(G,H)$ is the kernel of the composite $\mathrm{Der}\,(G,\mathbb{Z}G) \to H^1(G;\mathbb{Z}G) \xrightarrow{res} H^1(H;\mathbb{Z}G)$. In particular, if H is f.g. with at most one end, $\mathfrak{D}(G,H) = \mathfrak{D}(G)$.

Theorem 4.6 *[Dunwoody 1979] Let (G,H) be a finitely generated pair. Then*

(i) $K(G,H) \neq 0$ if and only if G has a split over a finite group, adapted to H.

(ii) (G,H) is accessible if and only if $K(G,H)$ is a finitely generated kG-module if and only if $\mathfrak{D}(G,H)$ is finitely generated over $\mathbb{Z}$

If the pair (G,H) also satisfies D1, finite generation of $\mathfrak{D}(G,H)$ follows as in the absolute case, so this result proves Conjecture 4.1 in the f.g. case.

The (torsion-free) rank of $\mathfrak{D}(G)$ does not provide enough information to lead to the most general results about accessibility. The next result was obtained by Linnell, using traces. We will not develop notions of trace for modules over group rings here, but refer to [Bass 1976a] for a general study and applications. If M is an k-module, write $d_k(M)$ for the minimum number of generators of M as k-module. For any group G we have the augmentation ideals $\mathfrak{G} \lhd \mathbb{Z}G$ and $\mathfrak{G}_\mathbb{Q} := \mathfrak{G} \otimes \mathbb{Q} = \mathrm{Ker}\,(\epsilon : \mathbb{Q}G \to \mathbb{Q})$.

Theorem 4.7 *[Linnell 1983] Suppose $G \neq \{1\}$ is the fundamental group of a finite reduced graph of groups with finite edge groups G_e. Then*

$$2d_{\mathbb{Q}G}(\mathfrak{G}_\mathbb{Q}) \geq 1 + \sum_e |G_e|^{-1}.$$

Corollary 4.1 *An f.g. group whose finite subgroups have bounded order is accessible.*

The idea is that if G is not accessible, then for any presentation as a finite graph of groups with finite edge groups, at least one of the vertex groups must have more than one end, hence split. We thus have presentations with arbitrarily many edges, and if the orders of the edge groups are bounded, this contradicts the estimate of Theorem 4.7. (Some care is also needed to allow for trivial edges.)

The idea of proof of Theorem 4.7 is to decompose the augmentation module of G. For any finite subgroup F of G, write ι_F for the idempotent $|F|^{-1}\sum\{g \in F\}$. Now if $G = A *_C B$, $\mathfrak{G}_\mathbb{Q} \cong \mathfrak{A}\mathbb{Q}G \oplus \iota_C\mathfrak{B}\mathbb{Q}G \cong (\mathfrak{A}\mathbb{Q}G \oplus \mathfrak{B}\mathbb{Q}G)/\mathfrak{C}\mathbb{Q}G$, and if $G = A*_C$, $\mathfrak{G}_\mathbb{Q} = \mathfrak{A}\mathbb{Q}G \oplus \iota_C\mathbb{Q}G$. Iterating this gives a decomposition whenever G is expressed as the fundamental group of a (reduced) graph of groups with finite edge groups. Linnell next considers the weak closure $W(G)$ of the group ring $\mathbb{C}G$ in the right regular representation of G on $L^2(G)$. He shows that any projection p in WG can be arbitrarily closely approximated by a projection in $p\mathfrak{G}W(G)$. Now the trace of any idempotent endomorphism of the ring of $d \times d$ matrices over $W(G)$ can be shown to lie in the interval $[0, d]$; since G admits d generators as group, $\mathfrak{G}$ admits d generators as $\mathbb{Z}G$-module, so the same bound on traces follows for endomorphisms of $\mathfrak{G}W(G)$. Approximating the projections giving the decomposition of $\mathfrak{G}_\mathbb{Q}$ by endomorphisms of $\mathfrak{G}W(G)$ and taking traces now yields the result.

4.2. Accessibility of f.p. groups

The original accessibility conjecture was solved by [Dunwoody 1985a], with a new geometrical idea: instead of studying actions of groups on 1 dimensional objects (graphs and trees), we move up to 2-dimensional simplicial complexes L, and tracks in them. This parallels the use of normal surfaces in 3-dimensional topology, and a piecewise linear version of minimal surface arguments, as in [Dunwoody 1985b].

Following a suggestion of the referee, we mention in parenthesis a geometrical approach to a proof of Stallings' theorem. Let G be a finitely presented group with infinitely many ends. Choose a closed smooth manifold M with fundamental group G, and give it a Riemannian metric. Then G acts freely and isometrically on the universal cover $\tilde{M}$. Choose two ends of $\tilde{M}$, and let S be a hypersurface of least possible area which separates them (it can be shown that such exist). By a fundamental result of minimal surface theory, S is smooth except on a subset of codimension 2.

Suppose $g \in G$ such that $gS \neq S$, but gS intersects S. Then a cut and paste argument on $S \cup gS$ will replace S, gS by another pair of hypersurfaces of the same total area, each separating the same two ends. If one of these has less area than S this contradicts the definition of S; if not, by rounding the corner introduced by the cutting and pasting we decrease the area and reach the same contradiction. Thus for each $g \in G$ either $gS = S$ or $gS \cap S = \emptyset$. Now we can construct the dual tree with finite edge stabilisers using Dunwoody's argument.

We return to the question of accessibility, and seek to represent many different splittings by subspaces of the same 2-dimensional complex L. Suppose for simplicity that each 1-simplex of L is a face of at least one 2-simplex. A *slice* is a subset S of L such that, for each 2-simplex σ of L, $S \cap |\sigma|$ is a union of finitely many disjoint straight lines, each joining distinct edges of σ. A *track* is a connected slice. Any slice is a disjoint union of tracks.

A slice S meets the boundary of a 2-simplex σ in an even number of points, none at a vertex, and such that the sum of the numbers on any two edges is no less than the number on the third. This set of points determines $S \cap \sigma$ uniquely; indeed, a collection of points on the edges of L, whose intersection with each 2-simplex satisfies this condition, determines a unique slice.

A *band* is a connected subset B of L whose intersection with any 2-simplex σ of L is a union of finitely many components each of which is the convex hull of two closed intervals in the interior of distinct faces of σ.

A band B defines a track $m(B)$ by replacing, for each part which is the convex hull of two intervals, the line joining the mid-points of those intervals. We can retract B on these mid-points: it can be identified with a bundle over $m(B)$ with fibre $[-1, 1]$, so its boundary ∂B is a double covering of $m(B)$: we say B is twisted if this bundle (and hence this double covering) is non-trivial. Otherwise ∂B consists of two tracks, which we say are parallel. Any track is of the form $m(B)$ for some band B: we call it twisted or not according as B is.

A track S defines a 1-cocycle $z(S)$ on L with coefficient group $\mathbb{F}_2$ by setting, for any 1-simplex γ of L, $z(S)(\gamma) := \#(S \cap |\gamma|)$. This is always a cocycle; it is a coboundary if and only if S separates L, which implies S untwisted. More precisely,

Lemma 4.1 *Suppose $\beta_1(L) < \infty$. Let $T = \{t_1, \ldots, t_n\}$ be a set of disjoint tracks. Then $|L| - \bigcup T$ has at least $n - \beta_1(L)$ components, and T contains at most $\beta_1(L)$ twisted tracks.*

It also follows easily that in any collection of more than $2\beta_1(L) + \alpha_0(L) + \alpha_2(L)$ tracks (where $\alpha_i(L)$ denotes the number of i-simplices of L), at least two are parallel.

¿From now on assume $H^1(L) = 0$, so every track t separates L and is untwisted. Write $||t||$ for the number $\#(t \cap L^1)$ of points of t on the edges of L. Say that t is *minimal* if

(i) $||t||$ is finite.

(ii) The two components of $|L| - t$ each contain infinitely many vertices of L.

(iii) If the track t_1 satisfies (i) and (ii), then $||t_1|| \geq ||t||$.
If a minimal track t exists, set $\nu(L) := ||t||$.

For any set U of vertices of L, write $U^* := L_0 - U$. Let $w(U)$ denote the number of 1-simplices of L with one end in U and the other in U^*. If minimal tracks exist, there is a subset $U \subset L_0$ with U and U^* infinite and $w(U)$ finite. If such a U exists, there are minimal tracks t, $\nu(L) \leq w(U)$, and if $\nu(L) = w(U)$ there is a minimal track t such that U is the set of vertices in one component of $|L| - t$.

If s and t are tracks with $s \cap t \cap L^1 = \emptyset$, then $(s \cup t) \cap L^1$ determines a unique slice. If s and t are minimal, this must be the union of two minimal tracks (the non-trivial part of this assertion is that each component of the slice satisfies (ii)). Extending this, a collection of n minimal tracks, such that no two intersect on L^1, determines a union of n minimal tracks.

Proposition 4.8 *Let L be a connected 2-dimensional simplicial complex such that $H^1(L) = 0$, G acts freely on L, and there is a minimal track t in L. Then L has a minimal track s such that, for each $g \in G$, either $gs = s$ or $gs \cap s = \emptyset$.*

Sketch of proof For each $g \in G$, gt is a minimal track. Moving the intersections with the edges into general position, we can choose new minimal tracks t_g such that

(i) For each 1-simplex γ of L, $\#(t_g \cap |\gamma|) = \#(gt \cap |\gamma|)$.

(ii) If $g_1 \neq g_2$ then $t_{g_1} \cap t_{g_2} \cap |L^1| = \emptyset$.

For each 1-simplex γ of L,

$$\#\left(|\gamma| \cap \bigcup_{g \in G} t_g\right) = \sum_{g \in G} \#(|\gamma| \cap gt) = \sum_{g \in G} \#(g^{-1}|\gamma| \cap t).$$

This number is thus finite, and is the same for edges in the G-orbit of γ. Hence the tracks t_g can be chosen so that $(L^1 \cap \bigcup_{g \in G} t_g)$ is a G-set. This set determines a unique slice, which also is G-invariant. It is a union of tracks. Extending the above argument, one shows that these are minimal. □

Theorem 4.9 *[Dunwoody 1985a] Every a.f.p. group is accessible.*

Sketch of proof By hypothesis, there exists a connected 2-dimensional simplicial complex L such that $H^1(L) = 0$, G acts freely on L, and $G\backslash L$ is a finite 2-complex. If $e(G) \leq 1$ there is nothing to prove; otherwise there exists $U \subset L^0$ such that U and U^* are infinite and $w(U)$ is finite. Hence $|L|$ contains a minimal track t. By the proposition, L has a minimal track s such that, for each $g \in G$, either $gs = s$ or $gs \cap s = \emptyset$. Write $A := \bigcup_{g \in G} gs$.

We now define a tree. It has vertices v corresponding to components C_v of $|L| - A$, and edges $E\Gamma$ corresponding to the distinct tracks gs. For each of these tracks t there are just two components C_v whose closures contain t: we define the corresponding vertices to be the end points of the edge in Γ. Since removing a track gs disconnects L, removing any edge disconnects Γ, so Γ is a tree, and we have an induced action of G on it.

If, for each $v \in V\Gamma$, G_v has at most one end, G is accessible. Suppose then that G_v has at least 2 ends: to avoid confusion, write C for v considered as a component of $|L| - A$. Let K be the subcomplex of L consisting of those simplexes that intersect C. Dunwoody shows that Proposition 4.8 can be applied using K in place of L, G_v in place of G, and using only tracks contained in C. This yields a track $s' \subset C$ such that $|K| - s'$ has two components each with infinitely many vertices and, for each $h \in G_v$ (and hence also for each $h \in G$), hs' is either equal to or disjoint from s'. Clearly, s' is not parallel to any track gs.

Since the number of non-parallel disjoint tracks is bounded, this process cannot be repeated indefinitely, and the result follows. □

The paper also contains a new proof of Theorem 3.3, which suggests the following formulation.

Theorem 4.10 *If G is a group acting properly discontinuously on a locally finite 2-complex X with $H^1(X;\mathbb{F}_2) = 0 \neq H^1_c(X;\mathbb{F}_2)$, then G splits over a finite subgroup.*

Finally in [Dunwoody 1993], there is an example of a group G which is f.g. but inaccessible. As we see from Linnell's Theorem 4.1, G must have torsion subgroups of arbitrarily high orders, so it is only to be expected that an infinite procedure is required to construct the group.

4.3. Bounds for splittings

Generalising the question of accessibility, one can seek bounds on more general decompositions as graphs of groups. These cannot be expected without restrictions: [Stallings 1965, Remark 4.4] gave an example of a decomposition $G = A *_C B$ where all four groups are free of rank 2; another striking example of 'bad' decomposition is that given by [Dunwoody and Jones 1999] of an f.g. group G and an isomorphism $G \cong G *_{\mathbb{Z}} G$. The construction involves an infinite 'folding' process and will not be repeated here. Another interesting example is given by [Bestvina and Feighn 1991b].

The best general result currently known is due to [Bestvina and Feighn 1991a]. We seek a bound on the size — it is convenient to measure this by counting the number of vertices — of the underlying graph Γ of a graph of groups with fundamental group the given group G, corresponding to an action of G on a tree T. It is natural to require T to be reduced; instead they use a weaker condition, which we call weakly reduced, that T is minimal and that Γ has no vertex v of valence 2, $v = t(e) = t(e')$ with $e \neq e'$ and $\phi_e, \phi_{e'}$ isomorphisms.

Say that an f.g. group E is *small* if it does not admit a hyperbolic action on a tree, and that an arbitrary group is small if it has no action on a tree such that some f.g. subgroup acts hyperbolically. In particular, if E has no subgroup which is a free group of rank 2, then E is small. Then the following holds.

Theorem 4.11 *[Bestvina and Feighn 1991a] Let G be a.f.p. Then there is an integer $\gamma(G)$ such that for any weakly reduced G-tree T with small edge stabilisers, the number of vertices in T/G is bounded by $\gamma(G)$.*

Sketch of proof The first step is a 'resolution theorem' obtained in [Dunwoody and Fenn 1987]. Suppose G is a.f.p., then there exist a 2-complex K with $H^1(K;\mathbb{F}_2)=0$ and a free action of G on K with compact quotient L. Define $\delta(G)$ to be the least value of $\alpha_0(L)+\alpha_2(L)+2\beta_1(L)$ over all such actions, as in the remark following Lemma 4.1. Then if T is a minimal G-tree, there exist a minimal G-tree T' and a G-map $\alpha: T'\to T$ such that T' has at most $\delta(G)$ orbits of vertices. The G-map α need not be simplicial, but can be made so by subdividing T'. The map α can be expressed as the composite of a sequence of folding moves.

Fix the G-tree T with finite edge stabilisers and the resolution α. There is a rather technical division of vertices of T into live and dead, and one sees easily that there are at most $\delta(G)+\beta_1(G)$ live vertices. The rest of the argument can be illustrated by the easiest special case: when the edge stabilisers satisfy FA. It is shown successively that in T/G each valence 1 vertex is live; if v is a dead vertex of valence 2, with adjacent edges e_1, e_2, then $G_v = G_{e_1} *_{(G_{e_1}\cap G_{e_2})} G_{e_2}$; and T/G has no two adjacent dead valence 2 vertices. A simple counting argument now shows that T/G has at most $4\delta(G)+9\beta_1(G)-5$ vertices. □

There are several other accessibility results in the literature. Some are contained in particular splitting theorems; others are of general applicability.

[Delzant 1996] first defines $T(G)$ as the minimum number of relations for a presentation of G with each relator a word of length 3. This equals the minimum, over all presentations, $\langle g_i \mid R_j\rangle$, of $\sum_j(|R_j|-2)$. He then proves $T(A*B)=T(A)+T(B)$.

To study amalgamated free products Delzant requires a relativised definition: given an f.g. group G and a finite set $H=\{H_i\}$ of f.g. subgroups, we say that $T(G;H)\le k$ if there exist a simply-connected 2-dimensional simplicial complex P and an action of G on it such that there are k G-orbits of 2-simplices; the vertex stabilisers are conjugate to the subgroups H_i; and each subgroup H_i is a vertex stabiliser.

Say that a decomposition $G=A*_C B$ of G adapted to H is *rigid* if for any G-tree adapted to H, if C stabilises an edge, then C fixes a vertex; and *semi-rigid* if for any G-tree adapted to H, if C stabilises an edge, then C fixes a vertex or an end.

Theorem 4.12 *[Delzant 1996] Let $G=A*_C B$ be a semi-rigid splitting adapted to H; write H_A, H_B for the subsets of H of subgroups conjugated into A resp. B.*

If the splitting is reduced, $T(A;C,H_A)+T(B;C,H_B)\le T(G;H)$; if not, there is a set K of subgroups of C with either $T(A;H_A)+T(B;K,H_B)\le T(G;H)$ or $T(A;K,H_A)+T(B;H_B)\le T(G;H)$. Hence

(i) For any splitting, $T(A;C)+T(B;C)=T(G;C)$.

(ii) If $G=A*_C B$ is a semi-rigid splitting, $r(A)+r(B)\le 2r(C)+12T(G)+\beta_1(G)$.

(iii) For any semi-rigid decomposition of G as a graph of groups without trivial edges, adapted to H, the number of vertices is at most $4T(G;H)+2\beta_1(G;H)+\#(H)$.

In [Delzant and Potyagailo 2001] a bound is obtained on the extent of decompositions of a group G, assumed to have no element of order 2, with edge stabilisers belonging to a prescribed class $\mathcal{C}$ of small subgroups of G subjected to some axioms.

[Weidmann 2002] defines (cf. [Sela 1997b]) an action of G in a tree T to be *k-acylindrical* if no non-trivial element of G fixes any segment of length $\geq k$ in T. He then shows that if G admits g generators, and does not split over the trivial group, then for any k-acylindrical action on a tree T, $G\backslash T$ has at most $1+2k(g-1)$ vertices.

4.4. Uniqueness of decompositions

If T and T' are trees, a (simplicial) map $T \to T'$ is a continuous map taking each vertex to a vertex and each edge linearly to an edge. If this is equivariant with respect to an action of G we call it a G-map. The (rather messy) formulation of this notion in the language of graphs of groups is given by [Bass 1993] (Bass also includes a homomorphism $G \to G'$ in his definition of morphism).

We have defined an elementary collapse of G-trees; *elementary deformation* is the equivalence relation generated by elementary collapses. An elementary collapse of T to T' defines a G-map $T \to T'$. There is no inverse map, but there exist a subdivision of T' and a G-map $\sigma : T' \to T$, simplicial with respect to this subdivision, which is almost inverse to σ: the idea is that if the vertex V and incident edge e of T have the same stabiliser C, and are collapsed to E, we subdivide the images in T' of each edge ($\neq e$) incident to V. Then the half-edges next to V are mapped to e and the other halves to their corresponding edges in T. Thus if T_2 is an elementary deformation of T_1, there exist simplicial subdivisions T_1', T_2' and proper simplicial G-maps $T_1' \to T_2$, $T_2' \to T_1$ whose composite in either order is properly homotopic to the identity. Moreover, T_1 and T_2 have the same vertex groups.

If e and f are adjacent edges, with $o(e) = o(f)$, $f \notin Ge \cup G\overline{e}$, and $G_f \subsetneq G_e$, we can slide f over e by detaching it from $o(e)$ and re-defining $o(f) := t(e)$, and correspondingly for all edges in Gf. For example, sliding gives the relation $(A *_D B *_E C) \cong (A *_D C *_E B)$ which holds when $D \subseteq E$. A slide move can be expressed as the composite of the inverse of a collapse with another collapse, so is an elementary deformation. So is subdivision of an edge e (e.g. $A *_C B \cong A *_C C *_C B$), which is the inverse of a collapse.

Theorem 4.13 *[Forester 2002] Let G be a group and T_1, T_2 be G-trees, with $G\backslash T_1$ and $G\backslash T_2$ finite graphs. Then T_1 and T_2 are related by an elementary deformation if and only if they have the same elliptic subgroups.*

An equivalent statement is with 'the same maximal vertex groups'.

The proof is based on the method of folding G-trees. This was first used (not by this name) in [Chiswell 1976a] and developed by [Stallings 1983] and several others. To perform a fold, one chooses edges e and f with $o(e) = o(f)$ and identifies them together; also $ge = gf$ for every $g \in G$. It is not difficult to show that the result is a G-tree. The effect of a fold on the quotient graph depends on how the edges and vertices involved meet the G-orbits: for example, the fold is said to be of type B if e

or f projects to a loop in $G\backslash T$ and of type A otherwise. For example, if $C \subset C_1$ and $B_1 = C_1 *_C B$, the splitting $G = A *_{C_1} B_1$ is obtained by *folding* from the splitting $G = A *_C B$.

Forester also introduces a parabolic fold, which may be regarded as an infinite sequence of folds performed along a ray converging to an end ϵ: various rays with end ϵ get identified together. The proof proceeds by showing that a G-map $T_1 \to T_2$ can be factored as a sequence of folds, restricting the possibilities for such a factorisation, and reducing the theorem to a special case which can be handled directly.

A G-tree is *slide-free* if it is minimal and has no slide moves, i.e. for any two adjacent edges e, f, with $o(e) = o(f)$ and $G_f \subseteq G_e$, we have $f \in Ge \cup G\overline{e}$; it is strongly slide free if this hypothesis implies $f \in Ge$. Forester also shows that under the hypothesis of the theorem, if T_1 is strongly slide-free, there is a unique isomorphism $T_1 \to T_2$ of G-trees.

Corollary 4.2 *Any f.p. group G is the fundamental group of a minimal graph of groups with finite edge groups and vertex groups having at most one end; this is unique up to elementary deformation.*

The accessibility theorem 4.9 shows that any f.p. group G is the fundamental group of a graph of groups with finite edge groups and vertex groups having at most one end; we may suppose the graph minimal. For any G-tree T with finite edge stabilisers, each 0- or 1-ended subgroup H of G must fix a point of T. Thus the maximal vertex groups are uniquely determined, and the result follows from the Theorem.

5. Poincaré duality groups in dimension 2

5.1. Definition of duality groups

In [Wall 1967] I defined Poincaré complexes as CW-complexes satisfying the strongest form of the Poincaré duality theorem that holds for manifolds. To simplify this account, let us restrict to the orientable case. We say that a connected CW-complex X, dominated by a finite complex, is a PD^n complex if there is a fundamental class $[X] \in H_n(X;\mathbb{Z})$ such that, for any coefficient module M over $\mathbb{Z}\pi_1(X)$, cap product induces isomorphisms $H^i(X;M) \to H_{n-i}(X;M)$. This formulation supposes X connected; if not, we require that X have a finite number of components, each satisfying the condition. It can be shown fairly easily if $n \geq 3$ that X is homotopy equivalent to a complex of dimension n. We say that the group G is a PD^n group if $K(G,1)$ is a PD^n complex: we then have c.d. $G = n$. This condition requires G to be f.p., though many of the results on PD groups are obtained without this hypothesis. In both cases we can restrict the coefficients to be modules over a commutative ring k, and have PD^n_k complexes and groups.

The following result is often useful.

Proposition 5.1 *[Strebel 1977] If H is a subgroup of infinite index in a PD^n_k group, then c.d.$_k H \leq n-1$.*

Corollary 5.1 *If G is a PD^n group and H a subgroup with c.d. $H < n-1$, then G does not split over H.*

Proof Suppose there is a splitting $G = A *_H B$. Since A and B have infinite index in G, c.d. $A \leq n-1$ and c.d. $B \leq n-1$. The exact cohomology sequence (2) now reduces to $H^{n-1}(H; M) \to H^n(G; M) \to 0$. Choosing M so that $H^n(G; M) \neq 0$ now gives a contradiction. □

There is also a more general notion which has proved useful, and we now digress to discuss it. [Bieri and Eckmann 1973] defined G to be a *duality group* if there exist a G-module C, an integer n, and an element $e \in H_n(G; C)$, cap product with which induces isomorphisms $H^i(G; M) \cong H_{n-i}(G; C \otimes A)$ for all integers i and $\mathbb{Z}G$-modules A. They proved that any duality group G is f.g., $0 \neq C \cong H^n(G; \mathbb{Z}G)$; c.d. G = h.d. $G = n$; $H_n(G; C)$ is infinite cyclic, generated by e.

Duality may be characterised by the following conditions: $H^i(G; F) = 0$ for all $i \neq n$ and all free $\mathbb{Z}G$-modules F; $C := H^n(G; \mathbb{Z}G)$ is torsion-free; c.d. $G < \infty$, and there exists $e \in H_n(G; C)$, cap product with which induces isomorphisms $H^n(G; F) \cong C \otimes F$ for all free $\mathbb{Z}G$-modules F. If we strengthen the finiteness condition, there is a much simpler characterisation: if G satisfies (FP), then it is a duality group of dimension n if and only if $H^i(G; \mathbb{Z}G)$ vanishes for $i \neq n$ and is torsion-free for $i = n$. In particular, a torsion free one-relator group with one end (e.g. a Baumslag-Solitar group) is a duality group of dimension 2.

A group G is a duality group of dimension 1 if and only if it is f.g. free.

Let G be an f.g. group with c.d. $G = 2$. If $H^1(G; \mathbb{Z}G) \neq 0$ then also $H^1(G; \mathbb{F}_2 G) \cong H^1(G; \mathbb{Z}G) \otimes \mathbb{F}_2 \neq 0$, so G has more than one end. Now G is torsion free, so it follows that either $G \cong \mathbb{Z}$ or G is a free product. If also G is f.p., then it has type (FP). If G is neither $\mathbb{Z}$ nor a free product, then $H^1(G; \mathbb{Z}G) = 0$. A short argument now shows that $H^2(G; \mathbb{Z}G)$ is torsion free, and so G is a duality group of dimension 2. Thus an f.p. group G with c.d. $G = 2$ is a free product of duality groups of dimensions 1 and 2. This seems to be the only theorem known about the structure of groups of c.d. 2 in general; although there are many examples of such groups, one can hope for further structure theorems.

Geometrically, if there is a compact m-manifold X which is an Eilenberg Maclane complex $K(G, 1)$ for whose universal cover $\tilde{X}$, $H_k(\partial \tilde{X})$ vanishes for $k \neq q$ and is torsion-free for $k = q$, then G is a duality group of dimension $n = m - q - 1$, with dualising module $C = H^n(G; \mathbb{Z}G) \cong H_q(\partial \tilde{X})$. The closed complements of knots in S^3 provide examples. So do torsion free arithmetic subgroups of semisimple algebraic groups defined over $\mathbb{Q}$, according to [Borel and Serre 1973]. In a generalisation of the argument, using buildings, [Borel and Serre 1976] extend the conclusion to S-arithmetic groups (where S is a finite set of primes).

Let G be an n dimensional duality group over k with dualising module $D = H^n(G; kG)$. It was shown by [Farrell 1975] that if D is finitely generated over k then it is isomorphic to k and G is a PD^n_k group.

5.2. Poincaré duality pairs

Corresponding to a manifold with boundary, a PD^n pair is a CW pair (Y, X) with a fundamental class $[Y] \in H_n(Y, X; \mathbb{Z})$, such that cap product with $[Y]$ induces isomorphisms $H^i(Y; M) \to H_{n-i}(Y, X; M)$, $H^i(Y, X; M) \to H_{n-i}(Y; M)$ with respect to any coefficient bundle M and X is a PD^{n-1} complex with fundamental class $\partial_*[Y]$. If $X = \emptyset$ this reduces to the definition of PD^n complex. As before, we can define PD^n_k pairs by restricting coefficients to $k\pi_1(Y)$-modules.

A preliminary classification of low dimensional cases of Poincaré duality is

Lemma 5.1 *Suppose (Y, X) a connected orientable PD^n pair, with $\pi_1(Y) = G$.*

If $n = 0$, $(Y, X) \simeq (D^0, \emptyset)$.

If $n = 1$, then $(Y, X) \simeq (D^1, S^0)$ or $(S^1, \emptyset)$.

If $n = 2$, either $(Y, X) \simeq (S^2, \emptyset)$ or X is homotopy equivalent to a union of circles, Y is a $K(G, 1)$, and if $X \neq \emptyset$, G is free.

Sketch of proof If $n = 0$, Y is connected and dominated by a 0-dimensional complex, so is contractible, and X is empty.

If $n = 1$, $\tilde{Y}$ is connected and simply-connected, and for $i > 1$, $H_i(\tilde{Y}) \cong H^{1-i}(\tilde{Y}, \tilde{X}) = 0$, so $\tilde{Y}$ is contractible and Y a $K(G, 1)$. For $i \geq 2$ and M a G-module, $H^i(G; M) = H^i(Y; M) \cong H_{1-i}(Y, X; M) = 0$, so c.d.$G \leq 1$ and hence by Theorem 3.6 G is free. Now $H_1(Y; \mathbb{Z}) \cong H^0(Y, X; \mathbb{Z})$ is infinite cyclic if $X = \emptyset$ and trivial otherwise; the result follows.

If $n = 2$ it follows that, up to homotopy, X is a disjoint union of circles. As above, all reduced homology groups of $\tilde{Y}$ vanish except perhaps $H_2(\tilde{Y}; \mathbb{Z}) \cong H^0(\tilde{Y}, \tilde{X}; \mathbb{Z})$, which vanishes unless $X = \emptyset$ and G is finite. In this case, $\tilde{Y} \simeq S^2$ and G acts freely, preserving the orientation; the Lefschetz fixed point theorem implies that G is trivial. Otherwise, $\tilde{Y}$ is contractible, and so Y a $K(G, 1)$. Finally if $X \neq \emptyset$, for $i \geq 2$ and M a G-module, $H^i(G; M) = H^i(Y; M) \cong H_{2-i}(Y, X; M) = 0$, so again c.d.$G \leq 1$ and G is free. □

We will find more precise results when $n = 2$, even for the case of PD^2_k groups. At the end of the article we will study the case $n = 3$.

We define a *group pair* $(G, \mathbf{H})$ to consist of a group G and a list (usually finite, and possibly with repetitions) $\mathbf{H} = \{H_i\}$ of subgroups of G. There is an alternative, somewhat more intrinsic notation: let Ω be a G-set isomorphic to the disjoint union of coset sets G/H_i: we may also call (G, Ω) a group pair. We say $(G, \mathbf{H})$ is a PD^n pair if $(K(G, 1), \dot{\bigcup} K(H_i, 1))$ is a PD^n pair in the above sense. It is easy to show that if $\mathbf{H} \neq \emptyset$ then c.d. $G = n - 1$.

If M is a compact orientable surface with $\chi(M) \leq 0$ and with boundary components T_i, we may regard $H_i = \pi_1(T_i)$ as subgroups of $G = \pi_1(M)$; such a group pair we call a *surface pair*. More generally, we can define a *2-orbifold pair* to consist of the fundamental group of a 2-orbifold M (with $\chi_{orb}(M) \leq 0$) and those of its boundary components; if $\chi_{orb}(M) < 0$ we say we have a *Fuchsian pair*. In these cases there is an action of G on $\mathbb{H}^2$ with the conjugates of the H_i being the stabilisers of the limit points in $\partial\mathbb{H}^2$.

If $(G, \mathbf{H})$ is a group pair, and $K \lhd G$ a normal subgroup of G with $K \subseteq \bigcap_i H_i$, we can form the quotient pair $(G/K, \mathbf{H}/K)$ with $\mathbf{H}/K = \{H_i/K\}$. If $K \in \mathcal{K}$ and $(G/K, \mathbf{H}/K)$ is a surface pair, we say $(G, \mathbf{H})$ is a $\mathcal{K}$-by surface pair. Similarly for 2-orbifold and Fuchsian pairs. A pair is *inessential* if it is isomorphic to $(G; G, G)$ for some G.

Suppose given an expression of G as the fundamental group of a graph (Γ, G) of groups. For each vertex v of Γ, we define the *vertex pair* to be $(G_v, \mathbf{H}_v)$ where $\mathbf{H}_v$ is indexed by the edges e of Γ with $t(e) = v$ and the group H_e is the image of $\phi_e : G_e \to G_v$.

The analysis of PD^2 groups and pairs is due to Eckmann in various collaborations: see [Eckmann and Müller 1980] and [Eckmann and Linnell 1983], with a particular breakthrough due to [Müller 1981]. A full and coherent account is given in [Dicks and Dunwoody 1989, Chapter V]; we give an outline in the following section. The main result is

Theorem 5.2 *Any PD^2 group or pair with G infinite is a surface group or pair.*

The first observation corresponds to decomposing a manifold by cutting along a submanifold of codimension 1.

Lemma 5.2 *[Bieri and Eckmann 1978]; see also [Dicks and Dunwoody 1989, V.8.2] Let T be a G-tree with no trivial edge. If G is a PD^n_k group, and each edge group G_e is a PD^{n-1}_k group, then each vertex pair is a PD^n_k pair. Conversely, if each vertex pair is a PD^n_k pair, G is a PD^n_k group.*

There is a corresponding statement starting with a PD^n_k pair.

If $(G, \mathbf{H})$ is a group pair, we define the *double* $D(G, \mathbf{H})$ to be the fundamental group of the graph of groups having two vertices, each with vertex group G, and for each $H_i \in \mathbf{H}$ an edge joining the two vertices, with edge group H_i. It follows from the lemma that if $(G, \mathbf{H})$ is a PD^n_k pair, then $D(G, \mathbf{H})$ is a PD^n_k group. Note that the graph of groups has an obvious involution interchanging the vertices: this induces an automorphism σ of $D(G, \mathbf{H})$.

The idea of the attack on PD^2 pairs is now to construct an action of G on a tree, thus yielding a decomposition into pieces which are less complex in an appropriate sense, so that pieces of small enough complexity can be classified; and then to piece these back together to give G.

5.3. Proof that PD^2 groups and pairs are geometric

If G is a PD^2 group, then in particular it satisfies duality with $\mathbb{Z}$ coefficients, so has homology isomorphic to that of a uniquely determined closed surface M_G; a corresponding statement holds for pairs. If $\chi(M_G) \leq 0$, then $H_1(M_G; \mathbb{Z})$ is infinite, hence in particular there is a group epimorphism $\pi : G \to \mathbb{Z}$. If we set $H := \operatorname{Ker} \pi$, it follows that there is a splitting $G = H*_H$. We cannot use this splitting, since the group H is not f.g. Instead, we have the following.

Theorem 5.3 *[Eckmann and Müller 1980] Let G be a finitely presented group that admits an infinite cyclic factor group. Then G splits over an f.g. subgroup L as $G = G_0*_L$.*

Proof Take a presentation $G = \langle a_1, \ldots, a_k \,|\, r_1, \ldots, r_m \rangle$; we may suppose the epimorphism $\pi : G \to \mathbb{Z}$ satisfies $\pi(a_1) = 1$ and $\pi(a_i) = 0$ for $i > 1$. It follows that in the expression of each r_j in terms of the generators a_i, the sum of the exponents to which a_1 appears is 0, and hence that r_j can also be expressed as a word in the conjugates $b_{i,n} := a_1^{-n} a_i a_1^n$ (with $2 \leq i \leq k$ and $n \in \mathbb{Z}$). As replacing r_j by any conjugate still gives a presentation, we may restrict to $n \geq 0$. Since there are only finitely many relations, it suffices to use the $b_{i,n}$ with $0 \leq i \leq N$ for some fixed N.

Write $G(m)$ for the subgroup of G generated by the elements $b_{i,n}$ with $0 \leq n \leq m$. Then conjugation in G by a_1 defines an isomorphism γ from $G(n-1)$ to a subgroup of $G(n)$. Let $H := G(n)*_{G(n-1)}$ be the extension defined by this isomorphism. There is a natural map $H \to G$, and it is easy to see that this is an isomorphism. □

The next step in the argument is a relative form, due to Müller, of Stallings' splitting theorem. Assume the hypothesis of Swarup's Theorem 4.5; for Müller's splitting theorem we suppose H a further f.g. subgroup of G, and choose a transversal X to H in G. The restriction map may be written

$$res : H^1(G; \mathbb{Z}G) \to H^1(H; \mathbb{Z}G) \cong \bigoplus_{x \in X} H^1(H; \mathbb{Z}H)x.$$

The minimal number of non-zero components of $res(c)$ for all $0 \neq c \in K(G; \mathbf{S})$ is called the *weight* $n(H)$ of H with respect to $(G; \mathbf{S})$. Note that $n(H) = 0$ if and only if $K(G; \mathbf{S} \cup \{H\}) \neq 0$, so that Theorem 4.5 applies with H added to the list $\mathbf{S}$ of subgroups S_i. For the case when $n(H) > 0$, we have

Theorem 5.4 *[Müller 1981] Assume the hypothesis of Theorem 4.5 and that H is a further subgroup of G with $n(H) > 0$. Then there is a G-tree T, adapted to $\mathbf{S}$, with finite edge groups, such that $G \backslash T$ has just one edge, and there is a H-stable subtree T_0 such that $H \backslash T_0$ has at most $n(H)$ edges.*

For the cases when $(G; \mathbf{S})$ is a PD^2 pair and G splits over an f.g. subgroup H, there are restrictions on the weight of H. By Lemma 5.1, each group in $\mathbf{S}$ is infinite cyclic and G is free. It was shown in [Bieri and Eckmann 1978] that we can suppose that G has a free set of generators $g_1, \ldots, g_n$ such that, for $1 \leq i \leq m$, g_i generates S_i where $\mathbf{S} = \{S_0, \ldots, S_m\}$. Set $\mathbf{S}' = \{S_1, \ldots, S_m\}$. We wish to apply Theorem 5.4 to $(G, \mathbf{S}')$ and the subgroup $H = S_0$. Applying Poincaré duality, we identify

$$H^1(S_i; \mathbb{Z}G) \cong H_0(S_i; \mathbb{Z}G) \cong \mathbb{Z} \otimes_{\mathbb{Z}S_i} \mathbb{Z}G,$$

$$\left(H^1(G;\mathbb{Z}G) \to \oplus H^1(S_i;\mathbb{Z}G)\right) \cong \left(H_1(G,\mathbf{S};\mathbb{Z}G) \to \oplus H_0(S_i;\mathbb{Z}G)\right),$$

and since $H_1(G;\mathbb{Z}G) = 0$, $H_0(G:\mathbb{Z}G) \cong \mathbb{Z}$ the latter map is injective with cokernel $\mathbb{Z}$. Hence the image of $K(G;\mathbf{S})$ in $\mathbb{Z}\otimes_{\mathbb{Z}S_0}\mathbb{Z}G \cong \oplus_{x\in X}\mathbb{Z}x$ is the kernel of the augmentation map, so $n(S_0) = 2$, as we can take just two non-zero components, with coefficients $+1$ and -1. Now Müller's splitting theorem has the

Corollary 5.2 *If G is torsion free and $n(H) = 1$ then for the splitting of G given by T we have one of*

(i) $G = G_1 * G_2$, $H = H_1 * H_2$, $H_i \subseteq G_i$;
(ii) $G = G_1 * \mathbb{Z}(t)$, $H = H_1 * tH_2t^{-1}$, $H_1, H_2 \subseteq G_1$;
(iii) $G = H = \mathbb{Z}(t)$, all S_i trivial.

If $n(H) = 2$ and H is infinite cyclic, we have one of

(iv) $G = G_1 * G_2$, $H = \langle g_1g_2\rangle$, $1 \neq g_i \in G_i$;
(v) $G = G_1 * \mathbb{Z}(t)$, $H = \langle tg_1t^{-1}g_2\rangle$, $1 \neq g_1, g_2 \in G_1$;
(vi) $G = \mathbb{Z}(t)$, $H = \langle t^2\rangle$, all S_i trivial.

Sketch of proof of Theorem 5.2. In the case when $S \neq \emptyset$, make an induction on the rank of G. If this rank is 1, we find that $\bigoplus_{i=0}^m(\mathbb{Z}\otimes_{\mathbb{Z}S_i}\mathbb{Z}G)$ is a torus group, corresponding to an annulus. Thus either $m = 1$ and $S_0 = S_1 = G$ or $m = 0$ and S_0 has index 2 in G, corresponding to a Möbius strip.

If G has rank at least 2 we take $T = S_0$ and show, as above, that T has weight 2. There are now two cases.

Case 1: $G = G_1 * G_2$; S_0 is generated by g_1g_2 with $g_i \in G_i$; $S_1, \ldots, S_k$ are (conjugate to) subgroups of G_1 and $S_{k+1}, \ldots, S_m$ are subgroups of G_2. We may suppose the rank of G_2 no less than that of G_1. Write H for the infinite cyclic group generated by g_2 and G_1' for $G_1 * H$. Then we have a splitting $G = G_1' *_H G_2$ with $S_0, \ldots, S_k \subset G_1'$ and $S_{k+1}, \ldots, S_m \subset G_2$. Thus $(G_1'; S_0, \ldots, S_k)$ and $(G_2; S_{k+1}, \ldots, S_m)$ are PD^2 pairs, and we may proceed by induction. An extra argument is needed if $H = G_2$. Also if G_1 and G_2 both have rank 1 we cannot use induction. But here we must have $k = 1$, $m = 2$, corresponding to a 'pair of pants'.

Case 2: $G = G_1 * P$, where P is cyclic, generated by p; S_0 is generated by $q = ph_1p^{-1}h_2$ with h_1, h_2 non-trivial elements of G_1, and $S_1, \ldots, S_m$ are (conjugate to) subgroups of G_1.

We can write G as an HNN extension $(G_1 * A)*_B$, where A, B are infinite cyclic, generated by a, ag_2^{-1}, where $S_0 = A$, $S_1, \ldots, S_m$ are in $G * A$, and $p^{-1}(ag_2^{-1})p = g_1$. Then if H_i is the subgroup generated by g_i, $(G_1 * A; A, A_1, B, S_1, \ldots, S_m)$ and hence $(G_1; S_1, \ldots, S_m, H_1, H_2)$ is a PD^2 pair, and again we can use induction.

It remains to treat the case $S = \emptyset$. We first consider the case when $H_1(G;\mathbb{Q}) \neq 0$. Then there is a surjection $G \to \mathbb{Z}$, hence a splitting of G; by Theorem 5.3, G splits over an f.g. subgroup L. We now make an induction on the rank of L. If this rank is 1, L is a PD^1 group, so splitting gives a PD^2 pair with non-empty boundary, which is a surface pair by what we have already proved, so glueing together again shows G is a surface group.

Otherwise, L has infinitely many ends. Consider the case when $G = G_1 *_L G_2$. In the exact sequence

$$0 \to H^1(G_1;\mathbb{Z}G) \oplus H^1(G_2;\mathbb{Z}G) \xrightarrow{(res_1,-res_2)} H^1(L;\mathbb{Z}G) \xrightarrow{\delta} H^2(G;\mathbb{Z}G)$$

since the restriction of δ to $H^1(L;\mathbb{Z}L)$ cannot be injective, this subgroup must intersect the image of $(res_1, -res_2)$ non-trivially. Hence L has weight 1 with respect to either $(G_1, \emptyset)$ or $(G_2, \emptyset)$: say the former. Thus either

(1) $G_1 = H_1 * H_2$, $L = L_1 * L_2$ with $L_i \subseteq H_i$ and either $G = H_1 *_{L_1} (H_2 *_{L_2} G_2)$ or the same with suffixes 1 and 2 interchanged gives a splitting of G over a subgroup of rank less than that of L. Or

(2) $G_1 = H * T$, T generated by t, $L = L_1 * tL_2t^{-1}$, L_1, L_2 non-trivial subgroups of H. Then $G = (H *_{L_1} G_2) *_{L_2}$ splits over L_2, again of lesser rank.

The case when $G = G_1 *_L$ is handled similarly.

It remains to consider the case when $\mathbf{S} = \emptyset$ and $H_1(G;\mathbb{Q}) = 0$: here we seek a contradiction. The argument, due to [Eckmann and Linnell 1983], is completely different from the other cases. First, passing to a subgroup of index 2 if necessary, we may suppose G orientable. An argument involving traces shows that, for any non-zero finitely generated projective $\mathbb{Z}G$-module M, $\dim_{\mathbb{Q}}(\mathbb{Q} \otimes_{\mathbb{Z}G} M) \neq 0$.

Now G has c.d. 2, so there is an exact sequence

$$0 \to P \to (\mathbb{Z}G)^d \to \mathbb{Z}G \to \mathbb{Z} \xrightarrow{\epsilon} 0.$$

Applying duality shows that there is a dual sequence

$$0 \to \mathbb{Z} \to (\mathbb{Z}G)^d \to P^* \to \mathbb{Z} \xrightarrow{\gamma} 0, \qquad (1)$$

where the dual module $P^* = \mathrm{Hom}_{\mathbb{Z}G}(P, \mathbb{Z}G)$ is again finitely generated projective. By Schanuel's lemma, there is an isomorphism $P^* \oplus \mathfrak{G} \cong \mathbb{Z}G \oplus L$, where $\mathfrak{G} := \mathrm{Ker}\,\epsilon$ and $L := \mathrm{Ker}\,\gamma$. There is a surjection $\mathbb{Z}G^d \to L$, and hence $\mathbb{Z}G^{d+1} \to P^* \oplus \mathfrak{G} \to P^*$. The kernel K is finitely generated projective and non-zero, so $\dim_{\mathbb{Q}}(\mathbb{Q} \otimes_{\mathbb{Z}G} K) \geq 1$, and hence $\dim_{\mathbb{Q}}(\mathbb{Q} \otimes_{\mathbb{Z}G} P^*) \leq d$.

The Euler characteristic $\chi(G)$ can be obtained by applying $\mathbb{Z} \otimes_{\mathbb{Z}G}$ to (1) and taking the alternating sum of the ranks; thus $\chi(G) = \dim_{\mathbb{Q}}(\mathbb{Q} \otimes_{\mathbb{Z}G} P^*) - d + 1 \leq 1$. On the other hand since G is an orientable Poincaré duality group, $\chi(G) = 2 - \beta_1(G)$. Hence $\beta_1(G) \geq 1$, a contradiction. □

We will discuss PD^2_k groups fully in §10.1. Natural examples are groups G with a surface subgroup H, $|G : H| = N < \infty$. Then G is PD^2k for any ring k in which N is invertible. In the converse direction,

Theorem 5.5 *[Eckmann and Müller 1982] Let G be a virtual PD^2 group with $\beta_1(G) > 0$. Then G is a finite-by-2-orbifold group.*

It was shown in [Dicks and Dunwoody 1989] that if char k=0, the torsion-free PD^2_k groups are the infinite surface groups.

6. Splittings over infinite groups

6.1. Relative ends

In this section we abandon the historic order in favour of a more coherent account.

Suppose G splits over a subgroup H: let T be the corresponding G-tree, e an edge with stabiliser H and initial vertex v. If we remove the interior of e, T splits into two components: T_0 (containing v) and T_1. Let $B := \{b \in G \mid bv \in T_0\}$.

Since both v and T_0 are preserved by H, $HBH = B$.

Both T_0 and T_1 are infinite, for otherwise there would be a vertex of valence 1 and hence, since G is transitive on edges, only one edge in T. Hence neither B nor B^* is a finite union of cosets Hx.

For any $g \in G$, $B + Bg$ is a finite union of cosets Hx_i. To see this, observe that if $y \in B + Bg$, one of yv and $yg^{-1}v$ is in T_0, the other in T_1, so they are separated by e. Hence v and $g^{-1}v$ are separated by $y^{-1}e$. But there are only finitely many edges in the tree separating these two vertices (they are those on the geodesic joining them), so $y^{-1}e$ is one of a finite list of edges $x_i^{-1}e$. As H is the stabiliser of e, $y \in \bigcup Hx_i$.

A theory concerning the existence of such subsets B is a relative form of the theory of ends, giving an invariant $e(G,H)$; there is also an alternative theory with a different invariant $\tilde{e}(G,H)$. We can give either definition in geometric or algebraic terms. First we recast the usual theory of ends.

(Geom) Let X be a locally finite simplicial complex. Consider the family $\mathcal{A}$ of finite subcomplexes of X, ordered by inclusion; for each $A \in \mathcal{A}$ write CA for the (finite) set of infinite connected components of $X - A$. If $A \subseteq B \in \mathcal{A}$ there is a natural surjection $C(B) \to C(A)$. Then $lim_{\leftarrow A \in \mathcal{A}} C(A)$ is a compact totally disconnected space, which we denote $\mathcal{E}X$, and call the space of ends of X. The number of ends $e(X) := \#\mathcal{E}X = \sup_{A \in \mathcal{A}} \#CA$.

If G acts freely on X with compact quotient (hence is f.g.), e.g. if X is the Cayley graph $\Gamma(G;S)$, then $\mathcal{E}X$ depends only on G and can be denoted $\mathcal{E}G$.

(Alg) Let k be a field (usually $\mathbb{F}_2$); write QG for the subgroup of $\overline{kG}$ consisting of almost invariant sets — i.e. those whose image in $\overline{kG}/kG$ is G-invariant — and define $e(G) := \dim (QG/kG)$. This is independent of k. We have a short exact sequence $0 \to kG \to \overline{kG} \to k_eG \to 0$ and QG is the set of elements of $\overline{kG}$ whose image in k_eG is G-invariant, so $e(G) := \dim H^0(G : k_eG)$.

Corresponding definitions relative to a subgroup H of G are:

(Geom) If G acts freely on X with compact quotient, then $\mathcal{E}(H\backslash X)$ depends only on G and H and can be denoted $\mathcal{E}(H\backslash G)$; write $e(G,H)$ for its cardinality.

(Alg) G acts on the set $H\backslash G$ of cosets, so the groups $k(H\backslash G)$ of finite and $\overline{k(H\backslash G)}$ of infinite linear combinations are kG-modules; write $k_e(H\backslash G)$ for the quotient. Now set $e(G,H) := \dim{}_k H^0(G; k_e(H\backslash G))$. This does not depend on the field k.

As H may not be fixed, it is more convenient to work with subsets of G than with subsets of $H\backslash G$. We say a subset $X = HX$ is H-almost invariant (H-a.i. for short) if $H\backslash X$ is an almost G-invariant subset of $H\backslash G$ (i.e. for any $g \in G$,

$H\backslash X =^a Hg\backslash X$). We can write $Q(H\backslash G)$ for the group of H-a.i. subsets, or more generally for the subgroup of elements of $\overline{k(H\backslash G)}$ whose images in $k_e(H\backslash G)$ are G-invariant, so $e(G,H) = \dim_k Q(H\backslash G)/k(H\backslash G)$.

The first version of the definition of relative ends was given by [Houghton 1974]. The above (algebraic form) follows the account of [Scott 1977]. Scott showed that $e(G,H) = 0$ if and only if $|G : H|$ is finite; if G_1 is a subgroup of finite index in G that contains H, then $e(G,H) = e(G_1,H)$; and if H_1 is a subgroup of finite index n in H, then $e(G,H) \leq e(G,H_1) \leq ne(G,H)$. Easy examples show that $e(G,H)$ can take any finite value as well as ∞.

The exact sequence $0 \to k(H\backslash G) \to \overline{k(H\backslash G)} \to k_e(H\backslash G) \to 0$ gives rise to a cohomology sequence. Since $k(H\backslash G) \cong kG \otimes_{kH} k$ and $\overline{k(H\backslash G)} \cong \mathrm{Hom}_{kH}(kG,k)$. Shapiro's lemma gives $H_q(G;k(H\backslash G)) \cong H_q(H;k)$ and $H^q(G;\overline{k(H\backslash G)}) \cong H^q(H;k)$.

There is also a rather different relative notion.

(Geom). Let X be a locally finite one-ended simplicial complex on which a discrete group H acts properly. Consider the family $\mathcal{A}_H$ of subcomplexes A of X with $H\backslash A$ finite, ordered by inclusion; for each $A \in \mathcal{A}_H$ write $C_H A$ for the (finite) set of connected components Y of $X - A$ with $H\backslash Y$ infinite. If $A \subseteq B \in \mathcal{A}_H$ there is a natural surjection $C_H(B) \to C_H(A)$. Then $lim_{\leftarrow A\in\mathcal{A}_H} C_H(A)$ is a compact totally disconnected space, which we denote $\mathcal{E}_H(X)$, and call the space of coends of X relative to H. The number of coends is $e_H(X) := \#\mathcal{E}_H(X) = \sup_{A\in\mathcal{A}_H} \#C_H(A)$.

If G acts freely on X with compact quotient (hence is f.g.), and H is a subgroup of G, then $\mathcal{E}_H(X)$ depends only on G and H and can be denoted $\mathcal{E}_H(G)$. We set $\tilde{e}(G,H) := \#\mathcal{E}_H(G)$. This version is due to [Bowditch 2002].

(Alg) Let H be a subgroup of G. A subset B of G is called H-finite if it is contained in a finite union of right cosets Hx. Write $F_H G$ for the set of H-finite subsets of G or more generally, for any k, $\mathrm{Ind}_H^G \overline{kH}$, regarded as a subgroup of $\overline{kG}$. Now set $\tilde{e}(G,H) := \dim (\overline{kG}/\mathrm{Ind}_H^G \overline{kH})^G$.

Say that B is *almost invariant rel.* H if, for all $g \in G$, $B + Bg$ is H-finite — denote the group of these by $Q_H G$ — and is *proper* if neither B nor B^* is H-finite. Then $\tilde{e}(G,H) = \dim Q_H G/F_H G$. Note that if B is almost invariant rel. H, the saturation $X = HB$ has $HX = X$ and is H-a.i.

This notion was introduced by [Kropholler and Roller 1989a], who give the elementary properties: $\tilde{e}(G,H) = 0$ if and only if $|G : H|$ is finite; if G_1 is a subgroup of finite index in G that contains H, then $\tilde{e}(G,H) = \tilde{e}(G_1,H)$; and if H_1 is a subgroup of finite index in H, then $e(G,H) = e(G,H_1)$. Moreover, $e(G,H) \leq \tilde{e}(G,H)$; if $\tilde{e}(G,H) < \infty$, there is a subgroup H_1 of finite index in H such that $e(G,H_1) = \tilde{e}(G,H_1) = \tilde{e}(G,H)$; and if $H \lhd G$ then $\tilde{e}(G,H) = e(G,H) = e(G/H)$.

We showed at the start of this section that if G splits over H, we can construct a proper H-a.i. subset B of G, and hence $e(G,H) \geq 2$. Can one prove a converse, that if $e(G,H) \geq 2$, there is a splitting? The hypothesis gives us an H-a.i. subset X, and we can seek to apply Dunwoody's method to construct a G-tree, taking the G-translates of X and X^* as edges. It is necessary for the gX to be nested, so we seek to modify X to achieve this.

Some restrictions are certainly necessary. In Theorem 3.3 one does not obtain a splitting over the trivial group, but over a finite group. It is thus natural to look for a theorem asserting the existence of a splitting over a subgroup H_1 containing H and such that $|H_1 : H| < \infty$. Secondly, easy examples show that G itself need not split. Consider, for example, a triangle group $\Delta_{2,3,6}$: this has no splitting, but has a subgroup of finite index isomorphic to $\mathbb{Z} \times \mathbb{Z}$, which splits over $\mathbb{Z}$. One way round this is to look for a splitting of a subgroup of finite index in G. Scott also gives the example where A and C are infinite simple groups, so A, C and $G = A * C$ have no subgroups of finite index, we have $e(G, C) = \infty$, but G does not split over C.

A first converse was proved by [Houghton 1974]. Suppose G is f.g., H is a subgroup of G of infinite index in its normaliser. Then $e(G, H) = 2$ if and only if there are subgroups $H \subset H_1 \lhd G_1 \subseteq G$ with $|G : G_1| < \infty$, $|H_1 : H| < \infty$, and G_1/H_1 infinite cyclic or dihedral. A more general result was obtained by Scott. First Scott proves a lemma which was later sharpened by [Kropholler and Roller 1989a] as follows.

Lemma 6.1 *If $S, T \subset G$ are f.g. groups, A is an almost invariant subset of G rel. S and B an almost invariant subset rel. T, then there is a finite $F \subset G$ such that if $g \in G - SFT$, gA and B are nested.*

Theorem 6.1 *[Scott 1977] Suppose G and H f.g. and $e(G, H) \geq 2$. Suppose also that H is closed in the profinite topology of G. Then G has a subgroup G_2 of finite index that contains H and splits over H.*

Sketch of proof Since $e(G, H) \geq 2$, there is a proper almost invariant subset E of G/H. Let $h_1, \ldots, h_r$ generate H. Write X for the (finite) union of the symmetric differences $h_i E + E$. This does not include the coset H/H. Since H is closed, there is a subgroup G_1 of finite index in G, containing H, with G_1/H disjoint from X. Thus $E_1 := E \cap G_1/H$ is H-invariant.

By Lemma 6.1, we can avoid the elements g such that E and gE are not nested by a further passage to a subgroup G_2 of finite index, and so suppose that for all $g \in G_2$, E and gE are nested. It now follows from Dunwoody's reworking of Stallings' arguments that G_2 splits over C. □

The B at the beginning of this section satisfied the stronger condition $HBH = B$. We will see that even if $e(G, H) \geq 2$ there is usually a further obstruction to finding an H-a.i. set B satisfying this additional condition. If B does, one may hope to show that the set of translates by G of B and B^* defines a tree on which G acts; the stabiliser of B itself then contains H. More explicitly, [Kropholler and Roller 1989b] made a conjecture which has motivated much subsequent work.

Conjecture 6.1 *Let H be a subgroup of G; suppose there exists a proper almost invariant set $X \subset G$ such that $HXH = X$. Then G splits over a subgroup commensurable with a subgroup of H.*

The first important result on this conjecture was obtained by [Dunwoody and Roller 1993].

Theorem 6.2 *Let G be f.g., H a polycyclic-by-finite subgroup, and $B = BH$ a subset which is proper almost invariant rel. H. Then there is a G-tree T such that G has no fixed point, acts transitively on the edges, and some G_e contains H as a subgroup of finite index.*

The plan is to find a B such that the sets $\{gB \,|\, g \in G\}$ are nested: thus write $\sigma(B) := \{g \in G \,|\, B$ and gB are not nested$\}$. They use Lemma 6.1, and a lemma stating that if $\sigma(B)$ is contained in a subgroup C and $g \in \sigma(B)$ then each $\sigma(B^? \cap gB^?) \subseteq C$. Then they find a subgroup S with $C \subset \mathrm{Comm}_G(S)$, and alter B till it is nested with gB for all $g \notin \mathrm{Comm}_G(S)$. The key step is to form a sequence B_i with $\sigma(B_i) \subseteq C$, $B_{i+1} = B_i^? \cap c_i B_i^?$ for some $c_i \in \sigma(B_i \cap C)$ and some choices of ?, and $\mu(B_i \cap C) > \mu(B_{i+1} \cap C)$, where μ is as in [Bergman 1968], and use Bergman's argument to show the sequence terminates.

6.2. The JSJ theorems

Much of the work on splitting theorems was inspired by analogies with 3-dimensional topology. The key to the modern understanding of 3-manifolds is the study of embedded tori in 3-manifolds. In this section we consider compact orientable 3-manifolds M.

A manifold is said to be *irreducible* if every embedded 2-sphere in it bounds a 3-disc. Two closed surfaces F_1, F_2 in M are *parallel* if they are the boundary components of an embedded copy of $F_1 \times I \subset M$. A surface $F \subset M$ is *incompressible* if the induced map $\pi_1(F) \to \pi_1(M)$ is injective; if F has several components, we require this for each component. A *Seifert fibre space* is a manifold M foliated by copies of S^1: in the orientable case, there is a fixed-point free action of S^1 on M. A manifold M is *atoroidal* if every torus embedded in M is parallel to a component of the boundary ∂M.

The following, due independently to [Jaco and Shalen 1979] and [Johannson 1979], is known as 'the JSJ theorem'. We first give the statement for closed manifolds, which is rather clearer.

Theorem 6.3 *Let M be a compact irreducible orientable 3-manifold with $\partial M = \emptyset$. There is a collection of embedded 2-dimensional tori, with no two parallel, such that each closed complementary component C is either a Seifert fibre space or atoroidal. A minimal set of such tori is unique up to isotopy.*

For the case of manifolds with boundary, we need to contemplate annuli (homeomorphs of $S^1 \times I$) as well as tori. Perhaps the neatest formulation is that due to [Neumann and Swarup 1997]. An embedding f of a surface S in M is *proper* if $f^{-1}(\partial M) = \partial S$. An annulus or torus S properly embedded in M is *essential* if it is incompressible and not boundary-parallel (more formally, a map $f : (X, Y) \to (M, N)$ with $N \subseteq \partial M$ is essential if, for each component of X, f_* is injective on $\pi_1(X)$ and f is not homotopic (as map of pairs) to a map into N); it is *canonical* if, in addition, any other properly embedded essential annulus or torus can be isotoped to be disjoint from S. Now assume M has incompressible boundary. Then a maximal collection of mutually disjoint canonical surfaces with no two parallel is unique up to isotopy, and detailed analysis of the pieces obtained by cutting along such a collection can be given. Moreover, in the case when each component of ∂M is a torus, there are no canonical annuli.

However, the following version of the result will serve as a better model for our analogy (the relation between the two is described by Neumann and Swarup). Consider manifolds M with boundary partitioned $\partial M = \partial_+ M \cup \partial_- M$, where $\partial_+ M$ and $\partial_- M$ are compact surfaces with $\partial\partial_+ M = \partial\partial_- M = \partial_+ M \cap \partial_- M$, and study pairs $(M, \partial_+ M)$. An I-pair is a bundle over a closed surface with fibre $(I, \partial I)$; an S^1-pair admits a fixed-point-free action of S^1 with quotient $(F, \partial_+ F)$ (F a compact surface); we call $(M, \partial_+ M)$ a Seifert pair if each connected component is either an I-pair or an S^1-pair. A Seifert pair is degenerate if, for some component $(M, \partial_+ M)$, either $\pi_1(M) = 1$ or $\pi_1(M) \cong \mathbb{Z}$ and $\partial_+ M = \emptyset$.

Theorem 6.4 *Let M be a compact orientable irreducible 3-manifold with incompressible boundary, $(M, \partial_+ M)$ be a manifold pair. There exists an essential Seifert pair $(C, \partial_+ C) \subset (M, \partial_+ M)$ such that for any non-degenerate connected Seifert pair $(N, \partial_+ N)$, any essential map $(N, \partial_+ N) \to (M, \partial_+ M)$ is homotopic (as map of pairs) to a map into $(C, \partial_+ C) \subset (M, \partial_+ M)$. The pair $(C, \partial_+ C)$ is unique up to isotopy.*

We have a natural decomposition of $(M, \partial_+ M)$ into $(C, \partial_+ C)$ and a complementary piece $(C^*, \partial_+ C^*)$. Each component of $C \cap C^*$ is an essential torus or an essential annulus in $(M, \partial_+ M)$; and there are no essential annuli or tori in $(C^*, \partial_+ C^*)$.

The decomposition can be indexed by a bipartite graph: we associate a black vertex to each component of C, a white vertex to each component of C^*, and to each component of $C \cap C^*$, an edge joining the corresponding vertices.

Some components of $(C^*, \partial_+ C^*)$ may be products of a torus or annulus with I. We may call such components inessential. When $\partial M = \emptyset$, an inessential component gives two parallel tori: only one contributes to Theorem 6.3.

Note for later reference that a manifold M is called *acylindrical* if ∂M is incompressible and M has no essential annuli. Whenever two loops in ∂M are homotopic to each other in M but not in ∂M, there is an essential map of an annulus (alias cylinder) to M taking the ends to the two curves; there is then also an essential annulus in M.

The next result can be deduced from the JSJ theory: an independent account, which gives the history of several earlier versions, in given by [Scott 1984].

Theorem 6.5 *(Torus theorem) Let M be a compact irreducible orientable 3-manifold such that $G = \pi_1(M)$ contains a torus subgroup, then either there is an embedded torus in M and G splits over a torus subgroup, or G has an infinite cyclic normal subgroup.*

The first proofs of these results used traditional techniques of 3-dimensional topology. Later accounts have made increasing use of the methods surveyed in this article. A crucial step was taken by [Scott 1984]; a further account is given in [Kropholler 1993]. In the next section we will discuss the generalisation in pure group theory, where we replace closed (or compact) n-manifolds M by PD^n groups G or pairs $(G; \mathbf{S})$, and codimension one submanifolds of M by PD^{n-1} subgroups H over which G splits; in §9.2 much wider generalisations.

6.3. Splittings of PD^n groups

Let G be a PD^n group. We would like conditions on (G, H) under which G does split over H. Then we seek a decomposition of a PD^3 group as a graph of groups whose edge groups are torus groups, which is canonical in some sense; and extensions to PD^n groups for higher values of n. The next developments occur in a sequence of papers by [Kropholler and Roller 1988a, 1988b, 1989a]: we state their results in the improved terminology of the third paper.

Suppose that G is a PD^n group and H a PD^{n-1} subgroup; in the arguments below I assume for simplicity that all PD groups occurring are orientable. We have isomorphisms

$$H^q(G; k(G/H)) \cong H_{n-q}(G; k(G/H)) \cong H_{n-q}(H; k) \cong H^{q-1}(H; k),$$

using in turn duality for G, Shapiro's lemma, and duality for H. Since, also by Shapiro's lemma, $H^q(G; \overline{k(G/H)}) \cong H^q(H; k)$, the first terms of the exact cohomology sequence of G corresponding to the coefficient sequence $0 \to k(G/H) \to \overline{k(G/H)} \to k_e(G/H) \to 0$ reduce to

$$0 \to H^{-1}(H; k) \to H^0(H; k) \to H^0(G; k_e(G/H)) \to H^0(H; k) \to H^1(H; k),$$

and hence to

$$0 \to 0 \to k \to H^0(G; k_e(G/H)) \to k \to H^1(H; k).$$

Thus $e(G, H)$ is equal to 1 or 2, and for splitting we require $e(G, H) = 2$.

If $e(G, H) = 2$ there is essentially only one almost invariant subset rel. H, as

$$H^1(G; \overline{kH} \otimes_{kH} kG) \cong H_{n-1}(G; \overline{kH} \otimes_{kH} kG) \cong H_{n-1}(H; \overline{kH}) \cong H^0(H; \overline{kH}) \cong k,$$

using in succession duality for G, Shapiro's lemma, duality for H and the lemma again. If $k = \mathbb{F}_2$, there is a unique non-zero element $\gamma \in H^1(G; \overline{kH} \otimes_{kH} kG)$, and we write $\sigma(G, H)$ for the restriction of γ to $H^1(H; \overline{kH} \otimes_{kH} kG)$. We have $\sigma(G, H) = 0$ if and only if there is a proper almost invariant subset B of G rel. H with $B = BH$.

This relates the obstruction to that encountered by [Scott 1977] and to [Dunwoody 1979].

In some sense, $\sigma(G,H)$ represents the self-intersection of the $(n-1)$ dimensional manifold mapped into the n dimensional one: in the geometric case this is an $(n-2)$ dimensional submanifold. Here it is shown that $H^1(H;\overline{kH}\otimes_{kH}kG)$ vanishes unless there is an element $g\in G$ such that c.d. $(H\cap H^g)=n-2$.

Suppose G is a PD^n group and H a PD^{n-j} subgroup. Then

$$H^q(G;\mathrm{Ind}_H^G\overline{kH})\cong H_{n-q}(G;\mathrm{Ind}_H^G\overline{kH})\cong H_{n-q}(H;\overline{kH})\cong H^{q-j}(H;\overline{kH})\cong H^{q-j}(1;k),$$

giving k for $q=j$ and 0 for $q\neq j$. It now follows from the cohomology exact sequence of $0\to F_HG\to\overline{kG}\to\overline{kG}/F_HG\to 0$ that $\tilde{e}(G,H)=2$ if $j=1$ and 1 if $j\geq 2$.

Write $H\sim H'$ if some conjugate of H is commensurable with H'.

Lemma 6.2 *If $H\subseteq G$ are f.g. groups with $\tilde{e}(G,H)$ finite, then either $|\mathrm{Comm}_G(H):H|<\infty$ or H has a subgroup H_0 of finite index such that $|N_G(H_0):H_0|=\infty$.*

The proof is not difficult. If H has infinite index in $\mathrm{Comm}_G(H)$, we wish to show that $N_G(H)$ has finite index in it. Choose a proper H-invariant subset B of G and consider $K:=\{g\in\mathrm{Comm}_G(H)\,|\,gB=^a B\}$. Since $\tilde{e}(G,H)$ is finite, this has finite index in $\mathrm{Comm}_G(H)$, and we can now juggle with almost invariant sets and replacing H,K by suitable commensurable groups.

Proposition 6.6 *Let G be a PD^n group, H a PD^{n-1} subgroup. The set of subgroups $K\sim H$ of G with $e(G,K)=2$ has a unique maximal element $H^\dagger$, so $N_G(H^\dagger)=\mathrm{Comm}_G(H^\dagger)$. The quotient $CH:=N_G(H^\dagger)/H^\dagger$ is isomorphic to 1, $\mathbb{Z}_2$, $\mathbb{Z}$ or to $\mathbb{Z}_2*\mathbb{Z}_2$.*

Sketch of proof If $|\mathrm{Comm}_G(H):H|<\infty$, we can suppose $\mathrm{Comm}_G(H)=H$; there is a proper almost invariant subset B rel. H, and $H^\dagger$ consists of those $h\in H$ such that hB is almost equal to either B (rather than B^*).

Otherwise, it follows that $|N_G(H):H|=\infty$ and then it can be shown that $|G:N_G(H)|<\infty$. In particular H has only finitely many conjugates under $\mathrm{Comm}_G(H)$; we can replace H by their intersection. The quotient $\mathrm{Comm}_G(H)/H$ is now 2-ended, and we can take $H^\dagger/H$ as its maximal finite normal subgroup. □

The main result of [Kropholler and Roller 1988a] is

Theorem 6.7 *Let G be a PD^n group, H a PD^{n-1} subgroup. If G splits over H then $H=H^\dagger$. G splits over $H^\dagger$ if and only if $\sigma(G,H)=0$. The splitting is then uniquely determined by $H^\dagger$. Moreover,*

(i) If $CH=1$, $H^\dagger$ is self normalising.

(ii) If $|CH|=2$, $G=N_G(H^\dagger)*_{H^\dagger}B$ for some B.

(iii) If CH is infinite, $H^\dagger\lhd G$.

Sketch of proof In case (iii) we need to show that if $\sigma(G,H)=0$ then $H \lhd G$, for then as the quotient is 2-ended G is either $H*_H$ or $H' *_H H''$ with H of index 2 in H', H''. If T is a transversal to H in its normaliser K, $\sigma(G,H)$ is the image of

$$H^1(G; \operatorname{Ind}_H^G \overline{kH}) \to H^1(K; \operatorname{Ind}_H^G \overline{kH}) \to H^1(H; \operatorname{Ind}_H^G \overline{kH}).$$

We can expand $H^1(K; \operatorname{Ind}_H^G \overline{kH}) = \oplus_{t \in T} H^1(K; \operatorname{Ind}_{H^t}^G \overline{kH}t)$. The obstruction $\sigma(G,H)$ maps diagonally to the direct sum, so must have zero image in each component $H^1(H; \operatorname{Ind}_{H^t}^G \overline{kH}t)$. But this implies that t commensurises H.

If CH is finite, the obstruction vanishes automatically. There is a proper almost invariant set B rel. H with $B = BH$. It can be verified that the set of translates of B and B^* satisfies Dunwoody's tree axioms (T1)-(T5), and the action on this tree gives the desired splitting. □

In this result there is no longer a need to pass from G to a subgroup of finite index, except to find a subgroup on which the obstruction $\sigma(G,H)$ becomes zero.

In the second paper, these results were relativised. If $(G, \mathbf{S})$ is a PD^n pair and H a PD^{n-1} subgroup, then $H^1(G, \mathbf{S}; \operatorname{Ind}_H^G \overline{kH}) \cong k$. Write $\sigma(G, \mathbf{S}; H, K)$ for the restriction of the non-zero element (when $k = \mathbb{F}_2$) to $H^1(K; \operatorname{Ind}_H^G \overline{kH})$, where K is also a PD^{n-1} subgroup; if $K = H$ we write just $\sigma(G, \mathbf{S}; H)$.

Theorem 6.8 *[Kropholler and Roller 1988b] Let $(G; \mathbf{S})$ be a PD^n pair and H a PD^{n-1} subgroup of G with $H \not\sim S_i$ for each i. Then G admits a splitting, adapted to $\mathbf{S}$, over a subgroup commensurable with H, if and only if $\sigma(G, \mathbf{S}; H) = 0$.*

Theorem 6.9 *[Kropholler and Roller 1988b] Let G be a PD^n group and $\mathbf{S}$ a finite family of mutually non-conjugate PD^{n-1} subgroups $S_i = S_i^\dagger$. Then G can be expressed as the fundamental group of a graph of groups with edge groups S_i if and only if, for all $i, j \in J$, $\sigma(G : S_i, S_j) = 0$.*

The splittings of 3-manifolds were over embedded tori: this corresponds to torus groups. More generally, [Kropholler and Roller 1989a] introduce the class $\mathcal{C}_n$ of polycyclic-by-finite groups G of Hirsch length n. Observe that $\mathcal{C}_0$ consists of finite, or equivalently 0-ended groups; $\mathcal{C}_1$ consists of 2-ended groups. A group G in $\mathcal{C}_2$ has a torus subgroup of finite index. A torsion free group in $\mathcal{C}_n$ is a PD^n group.

The introduction of $\mathcal{C}_n$ is also motivated by

Lemma 6.3 *If a PD^n group G splits over a soluble-by-finite subgroup H, then $H \in \mathcal{C}_{n-1}$.*

Now suppose G a group, $H \in \mathcal{C}_{n-1}$ a subgroup of infinite index in G and finite index in $\operatorname{Comm}_G(H)$. A $\mathcal{C}_{n-2}$ subgroup C of H is said to be a *singularity controller* if, for some $g \in G$, $C \sim H \cap H^g$. There are various equivalent conditions; also a uniqueness result. A sample result is

Proposition 6.10 *Let $G \notin \mathcal{C}_n$ be a PD^n group, and $S, T \in \mathcal{C}_{n-1}$ subgroups.*

(i) If $\sigma(G; S, T) \neq 0$, then T has a unique commensurability class of singularity controllers.

(ii) If $\sigma(G, S) \neq 0$ then S has a normal subgroup N such that S/N is infinite cyclic or dihedral.

The methods of Kropholler and Roller are applied more specifically to PD^3 groups in [Kropholler 1990a], which gives an algebraic proof of the torus theorem. The theory of singularity controllers is used to deal with the obstruction $\sigma(G, H)$. In this paper a number of techniques are imported from pure group theory. Say that G satisfies *Max-c* if the set of centralisers of subgroups of G satisfies the maximal (ascending chain) condition. An important technical result states that if $(G, \mathbf{S})$ is a PD^n pair and G satisfies Max-c, then for every $\mathcal{C}_{n-1}$ subgroup H of G, every $\mathcal{C}_{n-2}$ subgroup K of H has a subgroup K_1 of finite index such that $N_G(K_1) = \mathrm{Comm}_G(K)$. This in turn is used to give tight control either on G or on singularity controllers.

The key result of [Kropholler 1990a], whose proof is again based on the methods of Stallings and Dunwoody, is a rather technical generalisation — whose proof involves an induction on length of chains of subgroups — of

Theorem 6.11 *Let G be a group, H a subgroup, $0 \neq \xi \in H^1(G; F_H G)$. If $\mathrm{Res}_H \xi = 0$ and $\tilde{e}(G, H \cap H^g) = 1$ for $g \notin H$, then G splits over a subgroup of H.*

A subgroup H of G is called *malnormal* if, for all $g \in (G - H)$, $H \cap g^{-1}Hg = \{1\}$. This implies $H^1(H; F_H G) = 0$, and hence leads to

Corollary 6.1 *If H is a proper malnormal subgroup of G and G and H are one-ended, then G splits over a subgroup of H if and only if $\tilde{e}(G, H) \geq 2$.*

In the rest of this section we state the results of [Kropholler 1990a], in slightly simplified form. The proofs also involve the methods of [Müller 1981].

Theorem 6.12 *Let $(G, \mathbf{S})$ be an orientable PD^3 pair such that G satisfies Max-c. If H is a torus subgroup of G which is not conjugate to a subgroup of any S_i then either G splits over a torus subgroup or G has an infinite cyclic normal subgroup.*

This is closely analogous to the strong torus theorem for 3-manifolds, and [Kropholler 1993] can be used to prove it since, by [Kropholler 1990b], 3-manifold groups satisfy Max-c. It is unknown whether all PD^3 groups satisfy Max-c: [Mess 1990] showed that PD^n groups for large n need not.

Iterating such splittings leads to

Theorem 6.13 *Let G be an orientable PD^3-group with Max-c which is not abelian-by-finite. Then there is a unique reduced G-tree Y such that $G\backslash Y$ is finite, each edge group is a torus group, for each vertex v either G_v has an infinite cyclic normal subgroup or the vertex pair is atoroidal; and every torus subgroup of G fixes a vertex of Y.*

Here a pair $(G, \mathbf{S})$ is called *atoroidal* if every torus subgroup of G is conjugate to a subgroup of some $S_i \in \mathbf{S}$. This theorem gives all one could wish, except for precision on vertex pairs of Seifert type and the undesirable hypothesis Max-c.

Corresponding results are also obtained in higher dimensions. Here we define generalised atoroidal by replacing torus subgroups by subgroups in $\mathcal{C}_{n-1}$.

Theorem 6.14 *Let $(G, \mathbf{S})$ be an orientable PD^n pair such that G satisfies Max-c. If $H \in \mathcal{C}_{n-1}$ is a subgroup of G which is not conjugate to a subgroup of any S_i then either $G \in \mathcal{C}_n$, G splits over a subgroup $K \in \mathcal{C}_{n-1}$, or G has a normal subgroup $K \in \mathcal{C}_{n-2}$.*

Theorem 6.15 *Let $(G, \mathbf{S})$ be a PD^n-group such that G satisfies Max-c but is not in $\mathcal{C}_n$. Then there is a unique reduced G-tree Y, adapted to $\mathbf{S}$, such that $G \backslash Y$ is finite, each edge group is in $\mathcal{C}_{n-1}$ (hence is a PD^{n-1} group), for each vertex v, either G_v has a normal subgroup in $\mathcal{C}_{n-2}$ or the vertex pair is generalised atoroidal; and every subgroup in $\mathcal{C}_{n-1}$ of G fixes a vertex of Y.*

7. Geometry of groups

7.1. Quasi-isometry and hyperbolic groups

During the 1980s, Gromov introduced the dramatic new idea of regarding an infinite group, with the word metric, as a geometric object in its own right: perhaps the key references are [Gromov 1981, 1987]. This led to a new and very active area of study. We will make no attempt to survey these developments here, but need to recall some key definitions since the two areas have interacted increasingly since about 1990. There are many accounts, and several introductions, e.g. [Ghys et al. 1990], [Bridson and Haefliger 1999]. Unattributed results below may be found in these references.

A map $f : X \to Y$ between metric spaces (X, d_X), (Y, d_Y) is a *quasi-isometry* if, for some $C > 0$,

(i) for each $y \in Y$ there exists $x \in X$ with $d_Y(f(x), y) < C$ and

(ii) for any $x, x' \in X$, we have

$$C^{-1} d_X(x, x') - C < d_Y(f(x), f(x')) < C d_X(x, x') + C.$$

Quasi-isometry of metric spaces is an equivalence relation, and *coarse geometry* is the study of properties invariant under this relation.

If G is a group with a finite set S of generators, we make the Cayley graph $\Gamma(G; S)$ into a metric space by giving all the edges length 1. The group G acts properly and isometrically on $\Gamma(G, S)$, and the quotient is a finite union of circles, hence is compact (so the action is uniform). The induced metric on the subset G is called the word metric. It is unique up to quasi-isometry, so the coarse geometry of G is well-defined.

If F is a finite normal subgroup of G, the projection $G \to G/F$ is a quasi-isometry. If G' is a subgroup of finite index in G, the inclusion $G' \to G$ is a quasi-isometry.

There has been much work on the classification of groups up to quasi-isometry, some of which we will summarise in §9.4.

A path joining two points x, y in a metric space (X, d) is said to be a *geodesic* if it has length $d(x, y)$; we say that (X, d) is *geodesic* if any two points in X can be joined by a geodesic. A geodesic metric space (X, d) is *hyperbolic* if there exists $0 < \delta \in \mathbb{R}$ such that, for every geodesic triangle in X, each side is contained in the δ-neighbourhood of the union of the other two sides. A geodesic metric space (X, d) is said to satisfy CAT(k) (where $k \in \mathbb{R}$) if every geodesic triangle in X is 'thinner' than a geodesic triangle with the same side lengths in a surface with constant curvature k. If $k < 0$, this implies hyperbolicity.

Now let G be a discrete group acting properly, isometrically and uniformly on a metric space (X, d). Then if $x \in X$, the map $g \mapsto g.x$ is a quasi-isometry from G (with the word metric) to X. We saw in §3.1 that for X a CW-complex the number of ends of G in this situation is equal to the number of ends of X. Indeed, the number of ends of a geodesic metric space is invariant under quasi-isometry. In view of Theorem 3.3, this gives a first link between quasi-isometry of groups and group splittings.

An f.g. group G is said to be *(word-)hyperbolic* if, for some (and hence, for every) choice of finite generating set S, the Cayley graph $\Gamma(G; S)$ is hyperbolic. This implies that G is f.p. Numerous conditions equivalent to hyperbolicity can be formulated; see e.g. [Ghys et al. 1990]. In a certain sense, 'most' groups are hyperbolic.

A metric space (X, d) is *proper* if closed metric balls $\{y \in X \mid d(x, y) \leq C\}$ are compact. If S is finite, the Cayley graph $\Gamma(G; S)$ is proper; more generally, if the metric space (X, d) admits a proper, isometric and uniform action of a discrete group G, (X, d) is proper.

Hyperbolicity of geodesic metric spaces is invariant under quasi-isometry. Thus if G acts properly, isometrically and uniformly on a proper geodesic metric space (X, d), the group G is hyperbolic if and only if the space (X, d) is.

A subset Y of a geodesic metric space X is *quasiconvex* if there is a constant $c > 0$ such that for all $x, x' \in Y$, each geodesic joining x to x' is contained in the c-neighbourhood of Y. A subgroup H of an f.g. hyperbolic group G is said to be quasiconvex in G if it is so in the Cayley graph (the condition is independent of choice of finite set of generators).

The following may be found in [Haefliger and Bridson 1999, pp 460–464]. A subgroup of a hyperbolic group is quasiconvex if and only if it is quasi-isometrically embedded; a quasiconvex subgroup of a hyperbolic group is hyperbolic; intersections of quasiconvex subgroups and centralisers of f.g. subgroups are quasiconvex.

Conjecture 6.1 was reformulated by [Sageev 1997] as follows.

Conjecture 7.1 *Let $H \subset G$ be f.g. groups with $e(G, H) > 1$. Then either G splits over a subgroup K with $|K : K \cap H| < \infty$ or there is a subgroup H' of H with $e(H, H') > 1$.*

He then proved it in the case when G is hyperbolic and H a quasiconvex subgroup.

Metric concepts are used in the main result on developability of complexes of groups obtained at the end of [Bridson and Haefliger 1999]. Let X be a complex of groups G_σ (with associated monomorphisms and conjugating elements). Recall that the local complex over the star of a cell σ has a development $\tilde{X}(st(\sigma))$.

Theorem 7.1 *[Bridson and Haefliger 1999] Let $(X, \{G_\sigma\})$ be a complex of groups; suppose X has a metric such that each cell is isometric to a euclidean polyhedron and each of the local developments $\tilde{X}(st(\sigma))$ satisfies CAT(0); then the given complex is developable.*

7.2. The boundary of a hyperbolic group

The boundary ∂X of a geodesic metric space (X, d) may be defined as the space of equivalence classes of geodesic rays $\rho : [0, \infty) \to X$, where the rays ρ_1, ρ_2 are equivalent if the distances $d(\rho_1(t), \rho_2(t))$ are uniformly bounded. We can topologise using the compact-open topology on the set of rays. Note that if (X, d) is a tree (with edges of length 1) this agrees with our previous definition.

For a hyperbolic space, one can proceed using sequences. In any metric space we can define

$$\delta_x(y, z) := \frac{1}{2}(d(x, y) + d(x, z) - d(y, z)).$$

Fix a base point $x_0 \in X$, write δ_0 for δ_{x_0}, and say that $\{x_n\}$ converges to infinity in X if $\delta_0(x_i, x_j) \to \infty$ as $i, j \to \infty$; and that two sequences $\{x_n\}$, $\{y_n\}$, each converging to infinity, are equivalent if $\delta_0(x_i, y_j) \to \infty$ as $i, j \to \infty$. The set of equivalence classes does not depend on x_0, so also defines a 'boundary'; again a topology can be defined abstractly. For a proper hyperbolic metric space X, there is a natural homeomorphism to the boundary of the previous paragraph.

There is a trichotomy for isometries h of hyperbolic spaces (X, d), analogous to that for isometries of the hyperbolic plane $\mathbb{H}^2$:

- h is elliptic: if the orbit of any $x \in X$ under $\{h^n \mid n \in \mathbb{Z}\}$ is bounded;
- h is parabolic: if it has just one fixed point h_+ in ∂X; for any $x \in X$, $h^n(x) \to h_+$ and $h^{-n}(x) \to h_+$ as $n \to \infty$;
- h is hyperbolic (or loxodromic): if it has just two fixed points h_+, h_- in ∂X; for any $x \in X$, $h^n(x) \to h_+$ and $h^{-n}(x) \to h_-$ as $n \to \infty$.

A quasi-isometry $f : X \to Y$ of proper geodesic spaces induces a homeomorphism $\partial X \to \partial Y$. If G is a hyperbolic group, its Cayley graph $\Gamma(G; S)$ is a proper hyperbolic space, and its boundary is independent of the choice of S: we may thus define it to be ∂G.

There are many analogies between ∂G and the action of G on it and actions of Kleinian groups on hyperbolic space: we refer to [Kapovich and Benakli 2002] for an excellent survey. A crude classification is given by

Proposition 7.2 *If G is a hyperbolic group with 0 ends (i.e. G finite), $\partial G = \emptyset$; if G has 2 ends, ∂G consists of 2 points. Otherwise, G has a free subgroup of rank 2, and ∂G is an infinite perfect compact metric space.*

A compact Hausdorff space X is said to be perfect if it has no isolated points. We say that G is elementary if it has 0 or 2 ends.

7.3. Convergence groups

Starting from a quite different viewpoint, that of abstracting geometric properties of discrete quasiconformal groups acting on spheres, [Gehring and Martin 1987] defined a *convergence group* to be a group G of self-homeomorphisms of S^n such that from each infinite subset of G one can choose a sequence $\{f_n\}$ such that either

(i) there is a self-homeomorphism f of S^n such that $f_n \to f$ and $f_n^{-1} \to f^{-1}$ uniformly, or

(ii) there exist points $\alpha, \omega \in S^n$ such that $f_n(x) \to \omega$ uniformly on compact subsets of $S^n - \{\alpha\}$ and $f_n^{-1}(x) \to \alpha$ uniformly on compact subsets of $S^n - \{\omega\}$.

Following what has become current usage, we use the term to mean what Gehring and Martin called 'discrete convergence group', where (ii) holds in all cases.

Gehring and Martin obtained a number of consequences of this definition, gave a preliminary classification of convergence groups, raised the question whether convergence groups are conjugate (in the group of all self-homeomorphisms of S^n) to Fuchsian groups, and conjectured that this was the case when $n = 1$.

If M is a compact Hausdorff space, and $\Phi(M) \subset M^3$ denotes the set of distinct triples of elements of M, then a discrete group G of homeomorphisms of M is said to be a convergence group on M if the induced action on $\Phi(M)$ is proper. For $M = S^n$, this is equivalent to the above definition. We speak of a uniform convergence group if the action on $\Phi(M)$ is uniform.

The so-called Seifert conjecture in 3-manifold theory states that that if M is a closed irreducible orientable 3-manifold such that $\pi_1(M)$ has an infinite cyclic normal subgroup, then M has a Seifert fibration. It was shown in [Scott 1983] that if M, N are two such manifolds with $\pi_1(M) \cong \pi_1(N)$ and N is a Seifert fibre space, then M and N are homeomorphic. Thus the problem depends only on the fundamental group.

Using this, it was shown by Mess that if convergence groups on S^1 are conjugate to Fuchsian groups, then the Seifert conjecture holds. Unfortunately, he did not publish an account of this work. More recent accounts of these ideas may be found in [Maillot 2001, P99] and stronger results are obtained in [Bowditch P99].

Considerable progress on the convergence group conjecture was made by [Tukia 1988], who proved it in many cases. The completion of the proof was announced at the International Congress in 1990, and independent accounts were published by [Gabai 1992] and [Casson and Jungreis 1994].

Theorem 7.3 *Every convergence group on S^1 is conjugate (in the group of all self-homeomorphisms of S^1) to a Fuchsian group.*

This theorem has numerous applications to group theory. Many of these depend on reformulation of the notion of convergence group in somewhat more algebraic terms.

Let G be a group of homeomorphisms of S^1. Write $\mathcal{L}$ for the space of unordered pairs of points of S^1: elements of $\mathcal{L}$ will be called axes, and the two elements of the pair the end points of the corresponding axis. We say that $S, T \in \mathcal{L}$ cross if the end points of S separate those of T. The proof of the theorem eventually depends on combinatorial arguments involving crossings. Similar arguments are used in some of the applications.

Consider a G-invariant discrete subset $\mathcal{A}$ of $\mathcal{L}$. Say that $(G, \mathcal{A})$ is a convergence pair if, for each $T \in \mathcal{A}$, the stabiliser of T in G is a convergence group with non-empty limit set. Say that $\mathcal{A}$ is endpoint-disjoint if, for distinct $S, T \in \mathcal{A}$, $S \cap T = \emptyset$. A crossing sequence in $\mathcal{A}$ from S to T is a sequence $S = T_0, T_1, \ldots, T_n = T$ of elements of $\mathcal{A}$ such that, for $1 \leq i \leq n$, T_{i-1} crosses T_i. We say $\mathcal{A}$ is cross-connected if any two elements of $\mathcal{A}$ are joined by a crossing sequence in $\mathcal{A}$. Then the following is the main theorem of [Swenson 2000].

Theorem 7.4 *A group G of homeomorphisms of S^1 is a convergence group if and only if there is a discrete, G-invariant $\mathcal{A} \subset \mathcal{L}$ such that $(G, \mathcal{A})$ is a cross-connected, endpoint-disjoint convergence pair.*

7.4. The action of a hyperbolic group on its boundary

A hyperbolic group G acts on ∂G as a uniform convergence group. Conversely,

Theorem 7.5 *[Bowditch 1998b] Let X be a perfect compact metric space, G a group of homeomorphisms. The following are equivalent:*

G acts on X as a uniform convergence group,

G is a non-elementary hyperbolic group and X is G-homeomorphic to ∂G.

This is a highly non-trivial result. Part of the proof consists in establishing a type of quasi-conformal structure on ∂G.

The conditions studied in the early part of this survey, when restricted to hyperbolic groups, are reflected in the geometry of ∂G as follows.

Theorem 7.6 *Let G be non-elementary hyperbolic. Then*

(i) G splits over a finite group if and only if ∂G is disconnected.

(ii) The following are equivalent: $\dim \partial G = 0$*; ∂G is homeomorphic to the Cantor set; G is virtually free.*

The next case to consider is when ∂G is homeomorphic to S^1, which was analysed in the preceding section; in general if $\dim \partial G \leq 1$ we have

Theorem 7.7 *[Kapovich and Kleiner 2000] Let G be a hyperbolic group with $\dim \partial G = 1$, which does not split over a 0- or 2-ended group. Then ∂G is homeomorphic to either S^1, a Sierpinski gasket or a Menger curve.*

If $\partial G = S^1$ then by Theorem 7.3, G is a Fuchsian group. The case when we have Menger's universal curve is the generic one: it is hard to say more. The Sierpinski gasket is obtained from a 2-disc by removing the interiors of a sequence of disjoint discs, so has a natural boundary which is a union of circles S_i. In this case Kapovich and Kleiner further show

Theorem 7.8 *The circles S_i fall into finitely many orbits under G. The stabiliser of S_i is a virtually Fuchsian group H_i, quasi-convex in G, which acts on S_i as a uniform convergence group.*

Let $\mathbf{H}$ consist of one representative from each conjugacy class of subgroups H_i. Then the double $D(G,\mathbf{H})$ is hyperbolic, G is a quasiconvex subgroup of it, $\partial D(G,\mathbf{H})$ is homeomorphic to S^2, and $D(G,\mathbf{H})$ is a PD^3 group (over $\mathbb{Q}$; if torsion-free, over $\mathbb{Z}$).

It follows from Lemma 5.2 that $(G,\mathbf{H})$ is a PD^3 pair over $\mathbb{Q}$; if G is torsion-free, over $\mathbb{Z}$.

In general, for G hyperbolic, the topology of ∂G can be investigated as follows. The *Rips complex* is defined to be the simplicial complex whose vertices are the elements of G and such that $g_1,\ldots,g_n$ span a simplex if $d(x_i,x_j) \leq d$ for all $1 \leq i,j \leq n$. If d is large enough, $P(G) = P_d(G)$ is contractible. The stabilisers of the action of G on $P(G)$ are finite, so if G is torsion free, we have a $K(G,1)$ which is a finite CW complex.

According to [Bestvina and Mess 1991] there is a natural topology on $\overline{P(G)} := P(G) \cup \partial G$ under which it is compact, metrisable, finite dimensional and an absolute retract; moreover, ∂G is a Z-set, i.e. for any open $U \subseteq \overline{P(G)}$, the inclusion $U - \partial G \to U$ is a homotopy equivalence. It follows that, for any k, there are isomorphisms $H^i(G;kG) \to \check{H}^{i-1}(\partial G;k)$ of kG-modules. We have $\dim_k \partial G = \max\{n \mid H^n(G;kG) \neq 0\}$, and so if $\mathrm{c.d.}_k G < \infty$, we have $\dim_k \partial G = \mathrm{c.d.}_k G - 1$.

In particular, G is a PD^n group if and only if G is torsion-free and ∂G has the integral Čech cohomology of an $(n-1)$-sphere. It then follows that ∂G is also a homology manifold.

Finally, if G is 1-ended and, for some i and k, $H^i(G;kG)$ is finitely generated and non-zero, then ∂G is locally connected.

8. $\mathbb{R}$-trees

8.1. Definitions of Λ-trees and elementary remarks

In [Lyndon 1963] real valued length functions were considered, and it was conjectured that that a group admitting a real valued length function satisfying (LM1) is isomorphic to a free product of subgroups of the additive group $\mathbb{R}$. A formal definition of $\mathbb{R}$-trees was first given in [Tits 1977]. Tits showed that $\mathbb{R}$-trees behave in some ways like hyperbolic spaces, and gave an analogy between the space of ends of an $\mathbb{R}$-tree and the sphere at infinity of hyperbolic space. Examples of interest arise in various ways: we will discuss the Λ tree of a valued field below. Given a measured foliation of codimension 1 on a manifold, the space of leaves has the structure of a $\mathbb{R}$-tree. [Paulin 1997] described an $\mathbb{R}$-tree structure on the free product $\mathbb{R} * \mathbb{R}$.

To avoid later repetition we now define Λ-trees for any ordered abelian group Λ; we also give some elementary consequences, mainly following the treatment of [Alperin and Bass 1987].

Define a 'metric' on Λ by $d(x,y) = |x-y| := \max(x-y, y-x)$. We define the (closed) *interval* $[x,y]$ in Λ (where $x \leq y$) to be the subset $\{z \in \Lambda \,|\, x \leq z \leq y\}$; its end points are x and y. A Λ-metric space satisfies the usual axioms for a metric space except that distances take values in Λ. We will call a subset of a Λ-metric space isometric to a closed interval in Λ an *arc*: the parametrisation is unique up to translation and reflection in Λ.

A Λ-metric space X is said to be a Λ-*tree* if

(RT1) Any two points in X are the end points of a unique arc.

(RT2) If two arcs in X have a common endpoint, then their intersection is an arc (possibly a single point).

(RT3) If the intersection of two arcs in X is a common endpoint, then their union is an arc.

Given three points x, y, z of a Λ-tree X, it follows from (RT2) that, for some $w \in X$, $[x,y] \cap [x,z] = [x,w]$. If the points x, y, z are taken in a different order, we obtain the same 'mid-point' w. We have $d(x,w) = \frac{1}{2}(d(x,y) + d(x,z) - d(y,z)) = \delta_x(y,z)$; w is the unique closest point to x on the arc yz. The intersection of two arcs is an arc; hence the intersection of two closed subtrees is a closed subtree. Generalising the mid-point we have a convexity property: if X_1 and X_2 are disjoint closed subtrees, there are unique points $x_i \in X_i$ such that any arc joining X_1 to X_2 contains the arc x_1x_2.

A Λ-metric space (X,d) with a base point x_0 is isometric to a Λ-tree if and only if the following conditions (ST1)–(ST3) hold; if only (ST1) and (ST2) hold, X is isometric to a subspace of a Λ-tree, and the subtree spanned by X is uniquely determined.

(ST1) for all $x, y \in X$, $\delta_{x_0}(x,y) \in \Lambda$;

(ST2) for all $x, y, z \in X$, $\delta_{x_0}(x,z) \geq \min(\delta_{x_0}(x,y), \delta_{x_0}(y,z))$

(ST3) for all $x \in X$ with $d(x_0, x) = d$ there exist a subset A_x of X and an isometry $\alpha_x : ([0,d], 0, d) \to (A_x, x_0, x)$.

A function $L : G \to \Lambda$ is a *length function* if it satisfies (L0), (L1) and (L2); we often require also

($L\Lambda^+$) $\forall s, t \in G$, $\delta_L(s,t) \in \Lambda$.

If G acts by isometries on a Λ-tree X, $x_0 \in X$, and we define L by $L(s) := d(x_0, gx_0)$ then L satisfies (L0) trivially; (L1) since G acts by isometries, and (L2) by (ST2); (ST1) is equivalent to $L\Lambda^+$. Conversely [Chiswell 1976b] and [Imrich 1979] observed that the argument of Proposition 2.6 leading from a length function on a group G to a tree T on which G acts applies also when the length function is allowed arbitrary real values; this yields a metric space, on which G acts by isometries, which is path-connected and contractible. It was later shown by [Alperin and Moss 1985] that Chiswell's spaces are $\mathbb{R}$-trees in this sense. More generally, for any Λ-length function L on G there exist a Λ-tree T, with base point x_0, and an action of G by isometries of T, such that for all $s \in G$, $L(s) = d(sx_0, x_0)$. The subtree of T spanned by $G.x_0$ is unique up to isomorphism.

If $h : \Lambda \to \Lambda'$ is a homomorphism of ordered abelian groups, and X is a Λ-tree, then if h is injective we can regard X as a Λ'-metric space, and conditions (ST1) and (ST2) hold, so there is a unique Λ'-tree X' spanned by X. If h is not injective, and we set $x \sim x'$: if $d(x,x') \in \operatorname{Ker} h$ then $\sim$ is an equivalence relation, and the function $h(\delta_{x_0}(x,y))$ is defined on $X/\sim$: again we obtain a Λ'-tree X'. A form of this construction is given in [Morgan and Shalen 1984].

In particular, for any Λ there is a homomorphism $h : \Lambda \to \mathbb{R}$, unique up to multiplication by a positive constant: its kernel is the subgroup of 'infinitesimal' elements, i.e. those $z \in \Lambda$ such that for some $x \in \Lambda$ we have $-x < nz < x$ for all $n \in \mathbb{Z}$. Thus any Λ-tree X determines an essentially unique $\mathbb{R}$-tree X_R. Moreover, X_R is a complete metric space and the union of segments in X_R with end points in X is dense in X_R. If $h(\Lambda)$ is dense in $\mathbb{R}$, X is dense in X_R. If $\Lambda = \mathbb{Z}$ then X_R is a geometric tree with 0-skeleton X and edges of length 1.

8.2. Structure of G-Λ-trees

Say that a subtree of a Λ-tree X is *linear* if it is isometric to a (not necessarily closed) subset of Λ. A *ray* from $x \in X$ is a linear subtree which is maximal subject to having x as an end point. If the ray is an arc, its other end point is called a closed end of X. If not, we define its *end* ϵ to be the equivalence class of rays, where two are equivalent if their intersection is a ray; the ray can then be denoted $[x, \epsilon)$. A linear subtree which is the union of two rays with a common endpoint is a *line*.

An automorphism g of a Λ-tree X is called an *inversion* if g has no fixed point, but stabilises an arc xy, so interchanges x and y. As there is no fixed point, $d(x,y) \notin 2\Lambda$.

We can always avoid inversions by going to the associated Λ'-tree with $\Lambda' = \frac{1}{2}\Lambda$ (corresponding to subdivision in the case of trees).

There is a generalisation of the classification in Lemma 2.1 to automorphisms g of a Λ-tree X. Either g is an inversion or it has a fixed point (in these cases we call g elliptic); otherwise we call g hyperbolic, and it has an invariant line.

If g is not an inversion, $\lambda_X(g) := \min_{x\in X} d(x, gx)$ exists. Define the *axis* $A_g := \{p \in X \mid d(p, gp) = \lambda_X(g)\}$. Then A_g is a closed, g-invariant subtree; in particular, it is non-empty. If g is elliptic, A_g is the fixed set of g.

If g is elliptic, we define the *translation length function* $\lambda_X(g)$ to be 0. If g is hyperbolic, A_g is a line and $g|A_g$ is a translation by some positive element of Λ, which we define to be $\lambda_X(g)$. For any $x \in X$ there is a unique shortest arc xy with $y \in A_g$, and $d(x, gx) = \lambda_X(g) + 2d(x, y)$.

Now let G act by isometries on a Λ-tree X. A non-trivial result states that if g, h are not inversions, $d(A_g, A_h) = \frac{1}{2}\max(\lambda_X(gh) - \lambda_X(g) - \lambda_X(h), 0)$. Define the core of the action to be $C := \bigcap_{g\in G} A_g$. As before, we say that a G-Λ-tree X is minimal if there is no proper G-invariant subtree.

There is a generalisation, due to [Alperin and Bass 1987], of the classification in Lemma 2.2 of actions on trees; partial results were also obtained by [Culler and Morgan 1987]. We state it in our own terminology.

Proposition 8.1 *Let G act isometrically without inversions on a Λ-tree X. Then one of the following holds:*

Elliptic *G has a fixed point. Every element of G is elliptic, $C = X^G \neq \emptyset$.*

Parabolic *G has no fixed point in X but has one in ∂X. Thus there is a ray ρ such that for every $g \in G$, $g\rho \cap \rho$ is a ray. There is a homomorphism $\omega : G \to \Lambda$ such that for each $g \in G$, $\lambda_X(g) = |\omega(g)|$. We have 3 subcases:*

> **Linear** *$C \neq \emptyset$. Then C is a linear subtree on which G acts by translations, and defines two points of ∂X fixed by G. We have $\omega \neq 0$.*
>
> *Otherwise $C = \emptyset$, there is a unique $\epsilon \in \partial X$ fixed under G and, for all $g \in G$ and $x \in A_g$, $[x, \epsilon) \subseteq A_g$.*
>
> **Strictly parabolic** *$\omega \neq 0$.*
>
> **Weakly elliptic** *$\omega = 0$.*

Dihedral *G has no fixed point in $X \cup \partial X$, but there is an invariant pair of points in ∂X, these are joined by a unique line, and G has a subgroup of index 2 of Linear type.*

Cut type *$X = X_1 \cup X_2$ is the disjoint union of two G-invariant Λ-trees, each of parabolic type. For $x_1 \in X_1$ and $x_2 \in X_2$, the arc $[x_1x_2]$ is the disjoint union of the rays $[x_1\epsilon_1)$ and $(\epsilon_2x_2]$.*

Hyperbolic *There exist two hyperbolic elements of G such that the intersection of their axes is compact. Sufficiently high powers of these elements generate a free subgroup of G.*

Equivalent characterisations of the hyperbolic and dihedral types are that the restriction of λ_X to the commutator subgroup $[G, G]$ is non-zero; that for some hyperbolic elements $g, h \in G$, $\lambda_X(g^{-1}h^{-1}gh) \neq 0$; that there exist $g, h \in G$ with $\lambda_X(g) + \lambda_X(h) - \lambda_X(gh) < 0$ (or equivalently, $d(A_g, A_h) > 0$). Alperin and Bass call all the others abelian actions.

We have $\lambda_X \equiv 0$ if and only if the type is elliptic or weakly elliptic.

Culler and Morgan call the elliptic, linear, dihedral and hyperbolic actions semisimple; in these cases there is a unique minimal invariant tree T and $\lambda_X = \lambda_T$: T is a point in the elliptic case, a line in the linear and dihedral cases, in the hyperbolic case, if $A_g \cap A_h = \emptyset$ and xy is the shortest arc joining A_g to A_h it is the subtree spanned by the orbit $G.x$. Any translation length function corresponds to a unique semisimple action on a minimal G-tree.

The distinctively new possibility is the cut type. In this case, if $\lambda_X(g) \neq 0$ and g moves x_1 along the ray towards ϵ_1 and x_2, then for any $n \in \mathbb{N}$ we still have $g^n x_1 \in X_1$, and hence $n\lambda_X(g) < d(x_1, x_2)$. Thus Λ is non-archimedean; this cannot occur if $\Lambda = \mathbb{Z}$ or $\mathbb{R}$.

Both [Alperin and Bass 1987] and [Culler and Morgan 1987] proceed to detailed studies of translation length functions. They give lists of axioms in an attempt to characterise those functions which arise from actions on $\mathbb{R}$-trees. In [Parry 1991] it was shown that the following set of axioms on $\lambda : G \to \Lambda$, where Λ is an ordered abelian group, suffices for this purpose:

(TL0) For all g, $\lambda(g) \geq 0$.

(TL1) If $\lambda(g) > 0$ and $\lambda(h) > 0$ then $\max\{0, \lambda(gh) - \lambda(g) - \lambda(h)\} \in 2\Lambda$.

(TL2) $\lambda(ghg^{-1}) = \lambda(h)$.

(TL3) Either $\lambda(gh) = \lambda(gh^{-1})$ or $\max\{\lambda(gh), \lambda(gh^{-1})\} \leq \lambda(g) + \lambda(h)$.

(TL4) If $\lambda(g) > 0$ and $\lambda(h) > 0$ then either $\lambda(gh) = \lambda(gh^{-1}) > \lambda(g) + \lambda(h)$ or $\max\{\lambda(gh), \lambda(gh^{-1})\} = \lambda(g) + \lambda(h)$.

These are essentially the axioms of Culler and Morgan. Parry observed that these already imply that $\lambda(g) = \lambda(g^{-1})$ (take $g = 1$ in (TL3)) and $\lambda(1) = 0$ (take $h = 1$ in (TL4)). He also added Axiom (TL1), which is needed for general Λ. Chiswell pointed out that (TL0) was implicitly assumed. The idea is to use the axioms to define what is the set of branch points of a Λ-tree on which G acts.

8.3. The Λ-tree of a valued field

Serre's construction of a tree of lattices generalises easily. Let F be a field with a valuation v with value group Λ and valuation ring $\mathcal{O}$. Let V be a 2 dimensional vector space over F. An $\mathcal{O}$-lattice in V is a free $\mathcal{O}$-module of rank 2 spanning V. We say lattices L, L' are homothetic if, for some $\alpha \neq 0 \in F$, $\alpha L = L'$. Write X for the set of homothety classes of $\mathcal{O}$-lattices in V.

If L, L' are $\mathcal{O}$-lattices in V, we now define a distance $d(L, L') \in \Lambda$ which depends only on the homothety classes of L, L'. Choose α such that $L'' = \alpha L' \subseteq L$ and L/L'' is cyclic: this L'' is uniquely determined, so if we write $L/L'' \cong \mathcal{O}/\beta\mathcal{O}$ then $v(\beta)$ is uniquely determined by (the homothety classes of) L and L'. Define $d(\{L\}, \{L'\}) :=$

$v(\beta)$. This gives a Λ-metric on X. The classes of the lattices M with $L'' \subseteq M \subseteq L$ form an arc in X joining $[L]$ to $[L']$.

With this metric, X has the natural structure of a Λ-tree. The group $GL(V)$ acts on this tree by isometries. The stabilisers are as before: see §2.5; also, the stabiliser in $SL(V)$ of an arc of length α is isomorphic to the subgroup of $SL_2(\mathcal{O}_v)$ of matrices with $v(c) \geq \alpha$.

If $\gamma \in SL_(V)$ acts hyperbolically, it is semisimple, so we may take it as the diagonal matrix $diag(a, a^{-1})$ with $v(a) > 0$; then it takes $\mathcal{O}^2$ to a lattice equivalent to $a^2\mathcal{O} \oplus \mathcal{O}$, so has translation length $2v(a)$. Now $v(\text{trace}\,\gamma) = v(a + a^{-1}) = \min(v(a), v(a^{-1}) = -v(a)$, so $\lambda_X(\gamma) = -2v(\text{trace}\,\gamma)$; and in general $\lambda_X(\gamma) = -2\min(0, v(\text{trace}\,\gamma))$.

A generalisation of this construction is given by [Morgan 1986]. Let K be a field with valuation v having value group Λ, valuation ring $\mathcal{O}$, maximal ideal $\mathfrak{m}$ and residue field k which is formally real. Write q for the quadratic form $q(\mathbf{x}) = x_0x_1 + \frac{1}{2}\sum_2^n x_i^2$ on the $(n+1)$ dimensional vector space V over K. Consider the unimodular $\mathcal{O}$-lattices in V.

Such a lattice L has a basis $\{e_0, \ldots, e_n\}$ such that $e_0.e_1 = 1 = e_j.e_j$ for $j > 1$ and $e_i.e_j = 0$ otherwise; call it a standard basis. Given two unimodular $\mathcal{O}$-lattices L, L' in V, there exist a standard basis for L and $\alpha \in K$ such that $\{\alpha e_0, \alpha^{-1}e_1, e_2 \ldots, e_n\}$ is a basis for L'. Now define X to be the set of these lattices and define $d(L, L') := v(\alpha)$; then X has the structure of a Λ-tree.

More generally, if K is a field with a non-Archimedean valuation with value group Λ, and G a connected semi-simple algebraic group over K of K-rank 1, then the Bruhat-Tits building of G is a Λ-tree.

[Culler and Shalen 1983] introduced new ideas, leading to new applications of splittings. For G a group with a finite set S of n generators, the set of matrix entries $\rho(S)$ of homomorphisms $\rho : G \to SL_2(\mathbb{C})$ is a closed algebraic subset $R(G)$ of $\mathbb{C}^{4n}$. Each $g \in G$ defines a map $\tau_g : R(G) \to C$ by taking the trace of $\rho(g)$. They show that these functions are all determined by the finite subset obtained by letting g run through the products (in some order, but without repetition) of the elements in a subset of S. Taking these characters only defines a map $R(G) \to \mathbb{C}^N$ (for some N) whose image $X(G)$ is a closed algebraic subset.

Let C be an affine curve contained in $X(G)$. Then the corresponding homomorphisms assemble to $G \to SL_2(F)$, where F is the function field of C. Each point x of the completion $\overline{C}$ of C determines a discrete valuation of F, so Serre's theory can be applied to obtain an action of G on a tree. Culler and Shalen show that $g \in G$ lies in a vertex group if and only if the function on C given by evaluating the character at g does not have a pole at x. Thus if x is an ideal point of C we obtain a splitting of G.

The paper continues by taking G to be the fundamental group of a 3-manifold M, and showing that a splitting of G implies a splitting of M by incompressible surfaces, none boundary-parallel, with fundamental groups contained in edge groups of the splitting and fundamental groups of complementary components in vertex groups. Combining this with known results in 3-dimensional topology, they obtain a number of important applications: notably a new proof of the 'Smith conjecture' — that an action of a cyclic group on S^3 whose fixed point set is homeomorphic to a circle is equivalent to an action by rotations. For an account of the original proof see [Morgan et al. 1984].

This work was developed by [Morgan and Shalen 1984]. With the variety $X(G)$ in mind, they construct a compactification defined for arbitrary affine varieties X: roughly, choose a countable set $\mathcal{F}$ of functions f regular on X, map X into a projective space by sending $x \in X$ to $\theta_F(x) = \{\log(|f(x)|) + 2\}_{f \in F}$, and take the closure. An ideal point then has projective coordinates giving the relative growth rates of the functions f along a sequence converging to the point. Such points are shown to correspond to valuations v of the function field $k(X)$ belonging to a certain subset S_0 of the set of all valuations. If v is such a valuation, with value group Λ_v, we obtain an action of G on a Λ_v-tree. Moreover, the ideal points corresponding to discrete valuations are dense in the boundary. Also, the arc stabilisers of the action are virtually abelian.

Now consider the case when $G = \pi_1(S)$ is the fundamental group of a closed surface S: write M for the Teichmüller space of S. By Thurston's construction, this may be regarded as a space of measured foliations. Every point of M gives an isometric G action on an $\mathbb{R}$-tree as follows: given a measured foliation $\mathcal{F}$ of S, the action is the G-action on the leaf space of the lift of $\mathcal{F}$ to $\mathbb{H}^2$. This action is small. Since edge stabilisers have infinite index in G, they have c.d. 1 by Proposition 5.1, so are free; as they do not contain rank 2 free groups, they are trivial or infinite cyclic. It was conjectured that every minimal small action of G on an $\mathbb{R}$-tree arises in this way. This conjecture was proved by [Morgan and Otal 1993] and [Skora 1996]: see also [Otal 1996].

[Morgan and Shalen 1984] apply these ideas, and extensive further geometrical arguments, to give new proofs of two fundamental results of [Thurston 1986]; that the space of hyperbolic structures on an acylindrical 3-manifold is compact, and that the Thurston boundary of Teichmüller space consists of projective measured laminations. In [Morgan and Shalen 1991] they showed more generally that for G f.g., not virtually abelian, and not splitting over a virtually abelian group, the space of conjugacy classes of representations $G \to SL_2(\mathbb{C})$ is compact.

In the situation studied by Culler and Shalen, if $x_i \in C$ converges to x, the corresponding characters, taken in the projective space of the space of all (class) functions on G, converge to the translation length function of the corresponding splitting. In general, translation length functions may be regarded as elements of the function space $\mathbb{R}^G$, and determine points in the corresponding projective space $P(G)$. Since they are constant on conjugacy classes in G, they lie in the conjugacy-invariant subspace. Denote by $\Psi LF(G)$ the set of classes of pseudo-length functions and by $PLF(G)$ the subset of translation length functions of actions on $\mathbb{R}$-trees. As above, these are analogues of Teichmüller space.

Theorem 8.2 *[Culler and Morgan 1987] The subspaces $\Psi LF(G) \subseteq PLF(G)$ of $\mathbb{R}^G$ are compact.*

A survey of group actions on $\mathbb{R}$-trees and applications is given in [Morgan 1992]. An account of a wide variety of topics including group actions on $\mathbb{R}$-trees and many applications to 3-manifold topology is given in the book [Kapovich 2002].

The fact that surface groups act freely on $\mathbb{R}$-trees gives a counterexample to Lyndon's conjecture. An earlier, less explicit example was given by [Alperin and Moss 1985], and an infinitely generated example by [Promislow 1985]. This led to the speculation that groups acting freely on $\mathbb{R}$-trees might be free products of free abelian groups and surface groups.

8.4. Rips' theorem

This conjecture was — rather surprisingly — resolved in the affirmative by some highly original work of E. Rips, expounded in lectures in 1991.

Theorem 8.3 *Any f.g. group G admitting a free action on an $\mathbb{R}$-tree is a free product of surface groups and free abelian groups.*

There are no published accounts by Rips himself: the key references are [Gaboriau et al. 1994] and [Bestvina and Feighn 1995], who generalised the result as follows.

For any arc $[x, y]$ we have the stabiliser $G_{[x,y]}$. But arcs in an $\mathbb{R}$-tree do not correspond to edges of a tree, but rather to geodesics. When we pass from an arc to a subarc, the stabiliser will increase. One can construct examples (e.g. the action of $SL_2(K)$ where K is a field with a valuation with value group $\mathbb{R}$) where all these stabilisers are different: one cannot really expect a structure theory here. We call an arc $[x, y]$ *stable* if the stabiliser of any subarc is the same as that of $[x, y]$, and call the G-$\mathbb{R}$-tree *stable* if every arc contains a stable subarc. If arc stabilisers are 2-ended or, more generally, noetherian, this condition is automatic.

[Bestvina and Feighn 1995] introduce a 2-dimensional type of object, termed a band complex. There is a notion of a resolution of a tree by (the universal cover of) a band complex. A stable G-$\mathbb{R}$-tree is said to be pure if it admits a minimal resolving complex X such that Y is connected and $\pi_1(Y) \to \pi_1(X)$ is onto. A series of 6 geometric types of moves is introduced for replacing one band complex by another. Using these, there is an algorithm for putting a band complex into a sort of normal form. Their main theorem is the following.

Theorem 8.4 *Let G be an f.g. group with a faithful stable minimal action on an $\mathbb{R}$-tree T. Then either*

(i) G splits over a group H with a normal subgroup K fixing an arc of T and H/K cyclic, or

(ii) T is a line. G splits over a free abelian group H.

Moreover if **S** *is a set of f.g. subgroups of* G *each of which fixes a point of* T*, the splitting in (i) or (ii) can be supposed adapted to* **S**.

The original proofs assumed that G was f.p.; the extension to f.g. groups is due to [Sela 1997b].

Another account of this result, combining ideas from both [Gaboriau et al. 1994] and [Bestvina and Feighn 1995] to simplify the details, was given by [Paulin 1997]. This approach begins from the observation that the leaf space of a possibly singular foliation with a transverse measure has a natural $\mathbb{R}$-tree structure. Here a resolution of an $\mathbb{R}$-tree by a band complex is replaced by a resolution by a foliation.

8.5. Applications of Rips' theorem

Theorem 8.5 *[Paulin 1991, 1997] If* G *is f.g., hyperbolic, not 2-ended, and does not split over a 2-ended group,* $\mathrm{Out}\,(G)$ *is finite.*

Sketch of proof Suppose ϕ_n a sequence of automorphisms of G giving distinct elements of $\mathrm{Out}\,(G)$; let S be a generating set of G. Since G is not 2-ended, the function $f_i(g) := \max_{s \in S} d(g, \phi_i(s)g)$ attains its minimum λ_i on a finite subset F_i of G. Choose $x_i \in F_i$. It follows from distinctness of the ϕ_i that $\lambda_i \to \infty$.

Then the ultralimit of the based metric spaces $(G, \lambda_i^{-1} d_i, x_i)$ is an $\mathbb{R}$-tree, with an isometric action of G. There can be no global fixed point, and Paulin shows that the arc stabilisers are 2-ended, giving a contradiction. □

[Paulin 1997] also shows how to deduce that if G is hyperbolic and H is f.g. and 1-ended, then G has only finitely many conjugacy classes of subgroups isomorphic to H (for torsion free hyperbolic groups this is due to [Gromov 1987]).

A structure theorem for stable actions of groups on $\mathbb{R}$-trees, not just for their resolutions, is obtained by [Sela 1997a]. A lengthy statement appears on p 77 of [Rips and Sela 1997]. It refers to a large number of definitions from [Rips and Sela 1994] and [Sela 1997a], so we will not repeat it.

A number of important applications of Rips' work were made in [Sela 1995]. These concern the class of torsion free groups which are (word) hyperbolic. Sela calls an action essential if it is non-elliptic and each arc stabiliser is either trivial or a maximal cyclic subgroup.

It follows from Rips' theory that a torsion free hyperbolic group admits an essential action on an $\mathbb{R}$-tree if and only if it admits an essential action on a simplicial tree. Since any non-trivial free product or group split over $\mathbb{Z}$ has an infinite outer automorphism group, it follows from Theorem 8.5 that

Lemma 8.1 *A torsion free hyperbolic group has infinite outer automorphism group if and only if it admits an essential action on an* $\mathbb{R}$*-tree.*

Sela's main theorem is

Theorem 8.6 *[Sela 1995] Let G_1, G_2 be torsion free hyperbolic groups which admit no essential action on an $\mathbb{R}$-tree. Then it is decidable whether G_1 and G_2 are isomorphic, and if they are, it is possible to find effectively all conjugacy classes of isomorphisms between them.*

Corollary 8.1 *The following problems are decidable:*

(a) existence or not of a homotopy equivalence between closed aspherical manifolds with hyperbolic fundamental group,

(b) the homeomorphism problem for closed hyperbolic manifolds,

(c) the homeomorphism problem for closed, negatively curved manifolds of dimension ≥ 5,

(d) the problem of conjugacy of pseudo-Anosov automorphisms in the mapping class group of a closed surface, and

(e) conjugacy in $\mathrm{Out}\,(F_n)$ of irreducible automorphisms of the free group F_n.

General references on $\mathbb{R}$-trees are [Bestvina 2002], [Chiswell 2001] and [Paulin 1997].

8.6. Actions on real trees, dendrons and pretrees

The theory of the preceding sections concerns isometric actions on $\mathbb{R}$-trees. There are some situations where there is an action but no natural metric, and others — work in this direction has been particularly motivated by the study of the natural action of a hyperbolic group G on ∂G — where we have an action on an object which is treelike in some more general way. We now mention briefly some results in this direction.

Following [Bowditch 1998c] we define a *real tree* to be a Hausdorff topological space T which is uniquely arc connected and locally arc connected. Thus any two points $x, y \in T$ are connected by a unique interval $[x, y]$ (homeomorph of $[0, 1]$); and for any point $x \in T$ and any neighbourhood U of x there is a neighbourhood V of x such that $y \in V$ implies $[x, y] \subset U$. A *dendron* is a compact real tree.

A metric d on a real tree T is *monotone* if, for any $x, y, z \in T$ with $z \in [x, y]$ we have $d(x, z) \leq d(x, y)$; and is *convex* if, for all such x, y, z, we have $d(x, z) + d(z, y) = d(x, y)$. An equivalent definition of an $\mathbb{R}$-tree is a real tree with a continuous convex metric (it is known that such metrics always exist).

Theorem 8.7 *[Bowditch 1998c] Let G be an f.p. group acting without fixed points on a real tree T preserving a monotone metric. Then G also acts isometrically and without fixed points on an $\mathbb{R}$-tree Σ, so that each Σ arc stabiliser is contained in a T arc stabiliser, and if $\{G_i \mid i \in \mathbb{N}\}$ is an ascending chain of Σ arc stabilisers, there is an ascending chain $\{H_i \mid i \in \mathbb{N}\}$ of T arc stabilisers with $G_i \subseteq H_i$ for each i.*

Corollary 8.2 *(i) If G is an f.p. group acting stably without fixed points on a real tree T preserving a monotone metric and with finite arc stabilisers, then either G is virtually abelian or it splits over a finite or 2-ended subgroup.*

(ii) If G is an infinite f.p. group satisfying a.c.c. for finite subgroups, and G admits a discrete convergence action on a dendron, then G splits over a finite or 2-ended subgroup.

(iii) If G is a one-ended hyperbolic group with a global cut point in its boundary, then G splits over a two-ended subgroup.

Here corollaries (i) and (ii) follow using earlier results in this chapter. To obtain (iii), Bowditch proved that under these hypotheses ∂G admits a G-invariant quotient which is a non-trivial dendron, and G acts as a convergence group on it.

The main technique used in the proof of the theorem is the use of foliations on 2-complexes.

A related and somewhat stronger result was obtained by Levitt. An action of a group G on a real tree T is said to satisfy the non-nesting condition if, for any $g \in G$ and arc J of T, $g(J) \subseteq J$ implies $g(J) = J$. This condition was introduced by Bestvina, and substitutes for Rips' stability hypothesis.

Theorem 8.8 *[Levitt 1998] If G is an f.p. group with a fixed-point free action satisfying the non-nesting condition on a real tree T, then G admits a fixed-point free action on an $\mathbb{R}$-tree T_0. Any T_0-arc stabiliser is a T-arc stabiliser.*

Levitt describes his argument as follows. First, using a construction of Rips, we obtain a finite system $\mathcal{K} = \{\phi_i : A_i \to B_i\}$ of homeomorphisms between closed subtrees of a compact tree K with finitely many vertices. We prove that it suffices to construct a non-trivial non-atomic $\mathcal{K}$-invariant measure on K. We pass from $\mathcal{K}$ to a pseudogroup Ψ of homeomorphisms of S^1 by collapsing each component of the complement of an infinite minimal $\mathcal{K}$-invariant set (there is a technical difficulty that domains of $\mathcal{K}$ are compact and those of Ψ must be open). Finally the proof uses a theorem of Sacksteder that if orbits of Ψ are dense and Ψ satisfies a non nesting condition, there exists a Ψ-invariant probability measure.

A further generalisation in this direction was given by [Bowditch and Crisp 2001]. Here we focus on the betweenness relation for points in a tree and define a *pretree* to be a set T and a ternary relation (xyz) holding for some triples of points, satisfying

(PT1) If (xyz) then $x \neq z$.

(PT2) If (xyz) then (zyx).

(PT3) If (xyz) holds then (xzy) does not.

(PT4) If (xyz) and $w \neq y$ then either (xyw) or (wyz).

Any finite pretree can be embedded in a tree so that the betweenness relation is induced from that of the tree. For any pretree, define the arc $[x, y]$ by

$$[x, y] := \{x\} \cup \{z \mid (xzy)\} \cup \{y\}.$$

A *median* of $x, y, z \in T$ is an element of $[x, y] \cap [y, z] \cap [z, x]$. If such a point exists, it is unique. A *median pretree* is a pretree in which every triple of points has a median.

An action of a group G on a pretree T is *archimedean* if, for any $x \in T$ and $g \in G$, any interval in T contains only finitely many of the points $g^n(x)\, n \in \mathbb{N}$. For actions on real trees, this is equivalent to the non-nesting hypothesis. The action is 2-nontrivial if every orbit contains more than 2 points.

Theorem 8.9 *[Bowditch and Crisp 2001] If a countable group G admits a 2-nontrivial archimedean action on a median pretree T_0, then it admits a fixed point free non-nesting action on a real tree T_1. If the set of edge groups of T_0 satisfies a.c.c. we may suppose that every arc stabiliser of T_1 is an arc stabiliser of T_0.*

Corollary 8.3 *(i) If an f.p. group G admits a 2-nontrivial archimedean action on a median pretree T_0, then it admits a fixed point free stable action on a $\mathbb{R}$-tree T_2. If the set of edge groups of T_0 satisfies a.c.c. we may suppose that every arc stabiliser of T_2 is an arc stabiliser of T_0.*

The authors develop the theory of pretrees, recovering much of the familiar geometry from this set of axioms, and are then able to apply the main arguments of this chapter.

9. Further splitting theorems

The new tools described in the preceding three sections have led to a large number of new splitting theorems for groups. Much of this work was motivated by the topological JSJ Theorem 6.3, and all the splittings obtained are called 'the JSJ splitting' regardless of how close the analogy is. In Stallings theory we considered splittings over finite (or 0-ended) groups; in the next section we study splittings over $\mathbb{Z}$ and other two-ended groups; then we consider further cases. For the JSJ splitting of a closed 3-manifold, the manifold is cut by tori, whose fundamental groups are $\cong \mathbb{Z}\times\mathbb{Z} \in \mathcal{C}_2$, not $\mathcal{C}_1$. Thus the theorems in §9.1 are not close analogues of the geometric JSJ theorem.

A good theory of splittings will have at least three ingredients: an effective condition for existence of splittings (such a theorem is called, after the analogy with 3-manifold theory, a torus theorem); a bound on iterated splittings (an accessibility theorem); and a canonical decomposition of groups belonging to some fairly wide class (a JSJ theorem).

9.1. Splittings over two-ended subgroups

In this section we concentrate on those results which concern splittings with edge groups in $\mathcal{C}_1$. To see that care is necessary, consider the fundamental group G of a closed surface S. Any essential embedded curve on S defines a splitting of G over an infinite cyclic group, and the choice of splitting is no more unique than the choice of the curve (up to homotopy). It turns out that it is convenient to regard such a group G itself as a building block for splittings. This corresponds to a Seifert piece of a JSJ splitting.

The first theorem of this type was obtained by [Sela 1997a]: he assumed G hyperbolic and with one end. Call a graph of groups decomposition a Z-splitting if the edge groups are infinite cyclic. The idea is to use a group theoretic analogue of Dehn twists. If $G = G_1 *_{\mathbb{Z}} G_2$ and $1 \neq s \in \mathbb{Z}$, define an automorphism by $D_s(g_1) = g_1$ if $g_1 \in G_1$ and $D_s(g_2) = s^{-1}g_2s$ if $g_2 \in G_2$ (similarly for $G_1 *_{\mathbb{Z}}$). Provided neither G_1 nor G_2 has infinite centre, this is outer, and of infinite order in Out (G). We can now use D_t to produce a non-trivial G-action on an $\mathbb{R}$-tree. This is the basis for an inductive construction of such actions. The arguments of [Sela 1997a] make extensive use of the notion of acylindrical splitting introduced in [Sela 1997b] (which includes an accessibility theorem for actions having a uniform bound on diameters of fixed point sets) and Rips' classification of stable actions of groups on $\mathbb{R}$-trees.

The splitting obtained is not unique up to automorphisms of G, but uniqueness up to certain 'moves' can be established. It provides the basis for understanding the dynamics of automorphisms of G. It also serves as a key object in Sela's solution Theorem 8.6 of the isomorphism problem for hyperbolic groups.

In the same paper, a canonical decomposition is constructed for a discrete f.g. subgroup of a rank 1 Lie group. Suppose G a real, non-compact, almost simple Lie group of rank 1, $\Gamma \subset G$ a discrete torsion-free subgroup of G which is not virtually nilpotent, not a free product, and geometrically finite. Then Out (Γ) acts discretely and uniformly on the space of G-conjugacy classes of faithful discrete representations $\rho : \Gamma \to G$ such that the injectivity radius of $X/\rho(\Gamma)$ is bounded below (where X is the symmetric space of G). Moreover, Out (Γ) has a subgroup of finite index which is the direct product of an f.g. free abelian group and surface groups.

In [Rips and Sela 1997] it is observed that hyperbolicity is not necessary for the theory. It is shown that provided G is one-ended and torsion free, sequences of proper unfoldings must terminate.

Given two splittings of G with edge groups C_1 and C_2, the second is said to be elliptic (E) with respect to the first if C_2 fixes a vertex in the first tree; and is hyperbolic (H) if some element of C_2 acts hyperbolically on the first tree. Taking into account the comparison in the reverse order, we can classify the pair of splittings as (E-E), (E-H) etc. All cases can be illustrated with geometric splittings of surface groups. It was shown by [Sela 1997a] for Z-splittings of a group which is not a free product and more generally by [Fujiwara and Papasoglu P98] if G has no splitting over a subgroup of infinite index in either C_1 or C_2 that only the cases (E-E) and (H-H) are possible.

Consider a vertex pair $(H, \mathbf{S})$ of a Z-splitting of G which is a 2-orbifold pair (Sela's terminology is 'quadratically hanging'). A simple closed curve in the orbifold is weakly essential if no power is homotopic into the boundary. The splitting is (essential) maximal if, for every Z-splitting $G = A *_C B$ or $A*_C$ (with non-cyclic vertex groups), H either is elliptic or contains a conjugate of C; if a further technical condition is satisfied, it is CMQ.

Theorem 9.1 *[Rips and Sela 1997] Let G be an f.p. group with a single end. There exists a reduced unfolded Z-splitting J of G, with tree T_J, such that*

(i) Every CMQ subgroup is conjugate to a vertex group; every non-CMQ vertex group is elliptic in every Z-splitting of G.

(ii) Any elementary Z-splitting over C which is hyperbolic in another elementary Z-splitting is obtained by cutting a 2-orbifold corresponding to a CMQ subgroup of G along a weakly essential simple closed curve.

(iii) For any elementary Z-splitting Θ of G which is elliptic with respect to another elementary Z-splitting of G, there is a G-equivariant simplicial map from a subdivision of T_J to the tree T_Θ.

(iv) For any Z-splitting Λ of G there exist a Z-splitting Λ_1 obtained from J by splitting the CMQ subgroups along weakly essential simple closed curves on their corresponding 2-orbifolds and a simplicial G-map from a subdivision of T_{Λ_1} to T_Λ.

There is also a uniqueness clause. Using Theorem 4.13, this can be restated. It is shown in [Forester P01] that a G-tree satisfying (i)-(iv) is unique up to elementary deformation. Forester also constructs examples to show that this uniqueness statement cannot be improved. For this he used the class of generalised Baumslag-Solitar groups, which are the fundamental groups of finite graphs of groups with all edge and vertex groups isomorphic to $\mathbb{Z}$.

An algebraic annulus theorem was obtained (in 1995) by [Scott and Swarup 2000a], for the case when G is torsion free hyperbolic and the subgroup $J \cong \mathbb{Z}$.

There is no analogue of the Dehn twists used by Sela for groups split over $\mathbb{F}_2 * \mathbb{F}_2$, and [Miller et al. 1999] give an example of a hyperbolic group G with a natural splitting and with $\mathrm{Out}\,(G)$ trivial.

An approach to splittings of (non-elementary) hyperbolic groups G using the boundary ∂G was developed by Bowditch. There is a natural map from ∂G onto the space of ends of G. Thus if G has more than one end, ∂G is disconnected, so G splits over a finite group. The converse also holds [Ghys et al. 1990].

It is now natural to restrict to 1-ended hyperbolic groups, where ∂G is connected. It was shown by [Bestvina and Mess 1991] that ∂G is locally connected if there are no global cut points, and by [Swarup 1996] (using earlier results by a large number of authors) that ∂G has no global cut point.

If $G = A *_T B$ is split, where T is infinite cyclic with generator t, then $lim_{n\to\infty} t^n$ defines a point $t_+ \in \partial G$, and one can see that t_+ is a local cut point. [Bowditch 1998a] obtained a major result in this area: that G splits over a 2-ended subgroup if and only if ∂G has a local cut point. This is deduced from

Theorem 9.2 *Supose G 1-ended hyperbolic, not Fuchsian. Then there is a canonical splitting of G as a finite graph of groups with 2-ended edge groups and each vertex group of one of 3 types:*

(i) a 2-ended subgroup;

(ii) the vertex pair is a Fuchsian pair;

(iii) a non-elementary quasiconvex subgroup of G not of type (ii).

The types are mutually exclusive, and no two vertices of the same type are adjacent.

If $H \subset G$ is a 2-ended subgroup such that (G, H) has more than one end, then H can be conjugated into a vertex group of type (i) or (ii).

The proof proceeds by studying the topology of the boundary. The splitting obtained is determined uniquely (not merely up to isomorphism) by the action of G on ∂G.

The note [Papasoglu P01b] outlines an alternative proof for part of this result, and points out that for the groups of the theorem it establishes the quasi-isometry invariance of the existence of a splitting over a two-ended group.

Outside the class of hyperbolic groups, the results are less clean.

Theorem 9.3 *[Bowditch 2002] Suppose G f.g., one-ended; $\mathcal{H}$ a finite union of G-conjugacy classes of two-ended subgroups H_i with $\tilde{e}(G, H_i) \geq 2$ and $\mathrm{Comm}_G(H_i) = H_i$. Then there is a finite bipartite graph (Γ, A) with fundamental group G. Each edge group is 2-ended, and each element of $\mathcal{H}$ is conjugate to a black vertex group. For each black vertex v either $G_v \in \mathcal{H}$ or the vertex pair is a Fuchsian pair.*

Since G is assumed to split over each $H \in \mathcal{H}$, the key construction is to adapt these splittings so as to be compatible. If each H has only finitely many coends, the result is essentially unique: it may be that this holds for any self-commensurising 2-ended subgroup. A (more complicated) result is also given for 2-ended subgroups with large commensurisers. Note also that if G is f.p. the process of extending the splitting to accommodate more subgroups H_i terminates by Theorem 4.11, so a somewhat canonical splitting is eventually obtained.

A geometrical criterion for splitting is

Theorem 9.4 *[Papasoglu P01a] Let G be a one-ended f.p. group not commensurable to a surface group. Then G splits over a two-ended group if and only if the Cayley graph of G is separated by a quasi-line.*

9.2. General splitting theorems

A more general splitting theorem was obtained, by completely different methods, by [Dunwoody and Sageev 1999]. First, they choose a suitable class $\mathcal{C}$ of groups, and seek a splitting of G such that any $\mathcal{C}$-subgroup over which G splits is contained in a vertex group.

To be able to apply the accessibility theorem 4.11, it is prudent to assume that all groups in $\mathcal{C}$ are small. In this paper the stronger assumption is made that groups in $\mathcal{C}$ are noetherian.

A class $\mathcal{C}$ is said to be *closed* if whenever $G \in \mathcal{C}$ and G' is a group such that we have normal subgroups $N \lhd G$ and $N' \lhd G'$ such that $N \cong N'$ and G/N and G'/N' are either both finite or both have 2 ends, then $G' \in \mathcal{C}$. The classes $\mathcal{C}_n$ introduced in §6.3 are examples of closed classes of noetherian groups.

If $\mathcal{K}$ is a class of groups, write $\mathcal{ZK}$ for the class of groups G having a normal subgroup $H \in \mathcal{K}$ with 2-ended quotient. If $\mathcal{K}$ consists of noetherian groups, or is closed, then $\mathcal{ZK}$ has the same property. We have $\mathcal{ZC}_{n-1} = \mathcal{C}_n$.

If H and K are subgroups of G, say that H is *smaller* than K if $H \cap K$ has finite index in H and infinite index in K.

Theorem 9.5 *[Dunwoody and Sageev 1999] Suppose G is a finitely presented group and $\mathcal{K}$ is a closed class of noetherian groups. Suppose G does not split over any subgroup smaller than an element of $\mathcal{ZK}$. Then either G is an extension of a $\mathcal{K}$-group by a closed 2-orbifold group or there exists a decomposition of G as a bipartite graph of groups such that*

(i) A black vertex pair is either a $\mathcal{K}$-by-2-orbifold pair with G_v maximal (subject to this) or an inessential pair with $G_v \in \mathcal{ZK}$.

(ii) Every $\mathcal{ZK}$ group over which G splits is conjugate into a vertex group, and has a finite index subgroup conjugate into a black vertex group.

(iii) For every splitting $G = A *_C B$ or $G = A*_C$ over a $\mathcal{ZK}$ group, each white vertex group is conjugate into A or B.

(iv) For each $\mathcal{K}$-by-2-orbifold pair $(H, \mathbf{S})$ which is a vertex pair in a splitting of G, H is conjugate into a vertex group. If H contains a $\mathcal{ZK}$ subgroup which is not an adjacent edge group, H is conjugate into a black vertex group.

It is shown by [Forester P01] that the resulting splitting is unique up to elementary deformation.

The authors describe their methods as follows. Our approach does not make use of the theory of $\mathbb{R}$-trees, working instead entirely in the theory of simplicial trees. We use the theory of tracks, as described in §4.2. Work of [Delzant 1996] employs certain types of moves on such tracks to produce a quotient presentation 2-complex in which various algebraic features of the group in question become apparent. We proceed in the same spirit as Delzant to produce a 2-complex with a group action in which one can readily see the $\mathcal{K}$-by-2-orbifold pairs which carry the splittings over the noetherian groups in question. The idea in the 2-ended case, briefly, is this. Suppose one has a splitting of a one-ended, f.p. group G as an amalgam over a 2-ended group. The preimage of a G-map from the universal cover $\tilde{K}$ of a presentation 2-complex to the Bass-Serre tree produces a collection of 2-ended tracks in $\tilde{K}$. One now wants to 'zip' each track in an equivariant way to an embedded line in the quotient complex. More precisely we produce a quotient complex of $\tilde{K}$ which is still simply connected, admits a uniform action of G, and in which the images of all the tracks are lines. We will then want to zip a sequence of tracks for different splittings without harming the previously zipped tracks. This will then produce a simply connected complex with a G action so that the quotient under the action of G is built out of subcomplexes, some of which are 2-orbifolds attached to other subcomplexes along their boundaries. We then appeal to Theorem 4.11 to tell us that the process terminates, producing the desired complex.

[Fujiwara and Papasoglu P98] developed a rather different approach. The rough idea is to construct a splitting which contains in some sense all splittings of G over noetherian subgroups. Given two such splittings, over C_1 and C_2, of (H-H) type, a subgroup S of G is an enclosing group if

(i) There is a decomposition of G as graph of groups with noetherian edge groups which are elliptic with respect to the given splittings; S is a vertex group and contains conjugates of C_1 and C_2.

(ii) Any group S' satisfying (i) contains a conjugate of S.

(iii) S is an extension of a 2-orbifold group by a normal subgroup F of C_1.

Their key result states that if G is f.g. and the given splittings are minimal, an enclosing subgroup exists. The idea of the proof is to consider the induced action of G on the product of the two trees, and work on graphs contained in this 2-complex.

Now a decomposition may be refined successively to enclose more and more splittings. This needs an extension of the above argument, and an application of Theorem 4.11 to show that the procedure terminates. They obtain

Theorem 9.6 *[Fujiwara and Papasoglu P98] For any f.p. group G, there is a G-tree T with noetherian edge groups such that*

(i) For any G-tree T_1 corresponding to a splitting $G = A *_C B$ or $A*_C$ over a noetherian subgroup C, or more generally, for any G-tree T_1 with noetherian edge groups; there exist a G-tree T' obtained from T by elementary unfolding and by splitting some vertex groups along noetherian subgroups, and a G-map from a subdivision of T' to T_1.

(ii) Any enclosing group S for a finite family of mutually (H-H) minimal splittings of G is contained in a vertex group of T.

Moreover, the vertex groups in (ii) are of a special type. An addendum states that for any subset X of G there is a corresponding decomposition adapted to X in the sense that each element of X is elliptic on T. If $X = H$ is a subgroup, this implies that the action of H is either elliptic or weakly elliptic: thus if H is not the union of a sequence of proper subgroups, the action is elliptic.

It is again shown by [Forester P01] that the resulting splitting is unique up to elementary deformation.

While Theorem 9.5 gives a rather detailed description of the type of splitting obtained, it does not include a criterion for the existence of a non-trivial splitting.

Theorem 9.7 *[Dunwoody and Swenson 2000] Let G be f.g. and let $J \in \mathcal{C}_n$ be a subgroup of G such that $\tilde{e}(G, J) \geq 2$ but no subgroup $H \subset J$ with $|J : H| = \infty$ has $\tilde{e}(G, H) \geq 2$. Then one of the following holds.*

(i) $G \in \mathcal{C}_{n+1}$ and $|G : N_G(J)| < \infty$.

(ii) G splits over a subgroup commensurable with J.

(iii) G is a $\mathcal{C}_{n-1}$-by-Fuchsian group.

(iv) There is a decomposition of G as a finite star-shaped graph of groups with central vertex pair a $\mathcal{C}_{n-1}$-by-Fuchsian pair.

Indeed Dunwoody and Swenson say that the proof of the torus and annulus theorems in [Scott 1980] points towards the subsequent algebraic versions. Their outline of the proof is as follows.

Let G be an f.g. group, J a subgroup with $\tilde{e}(G, J) \geq 2$. Thus if Γ is the Cayley graph of G with respect to a finite generating set, then $J\backslash\Gamma$ is a locally finite graph with more than one end. We show that it is possible to attach finitely many G-orbits of discs to Γ so that the resulting 2-complex $\tilde{K}$ contains a separating track T, i.e. it is possible to cut the complex into two pieces, where the cut T is connected, 1 -dimensional, and is in general position with respect to $\tilde{K}^1$. Also T projects to a compact separating track in $J - \tilde{K}$. If G is finitely presented then $\tilde{K}$ could be taken to be the universal cover of a Cayley complex K for G.

We prove our main result by analysing how T behaves with respect to translation by elements of G. We say that $g(T)$ crosses T if both components of $\tilde{K} - T$ contain points of $g(T)$ arbitrarily far from T. If there is no element $g \in G$ for which $g(T)$ crosses T then we show that G splits over a subgroup commensurable with J. The harder case to deal with is when there exists $g \in G$ for which $g(T)$ crosses T. We assume that G is not in $\mathcal{C}_{n+1}$, allowing reduction to the case where $n = 1$.

Let H be the subgroup of G generated by those $g \in G$ for which $g(T)$ crosses T. There is a subgroup $N < J$ which is in $\mathcal{C}_{n-1}$, with $N \triangleleft H$. An action of H/N on S^1 is obtained by considering the action of H/N on the set of ends of the translates $h(T/N)$, $h \in H/N$. Since $G \notin \mathcal{C}_{n+1}$, we are able to show that this action is a convergence action by showing that the action satisfies the conditions of [Swenson 2000]. It now follows by Theorem 7.3 that the action is conjugate to the action of a Fuchsian group. If the action of this Fuchsian group is uniform, then we have (iii). If not we show that H is a vertex group in a graph of groups decomposition of G and we have (iv).

Combining Theorems 9.5 and 9.7 leads to the following.

Theorem 9.8 *[Dunwoody and Swenson 2000] Let $n \geq 1$ and let G be f.p., not virtually polycyclic, not $\mathcal{C}_{n-1}$-by-Fuchsian, and suppose G has no subgroup $H \in \mathcal{C}_{n-1}$ with $\tilde{e}(G,H) \geq 2$.*

Then G has a finite bipartite graph of groups decomposition such that each black vertex pair is either inessential with $G_v \in \mathcal{C}_n$, or a $\mathcal{C}_{n-1}$-by-Fuchsian pair. Moreover,

(a) For every splitting $G = A *_C B$ or $G = A*_C$ of G over a $\mathcal{C}_n$ group C, each white vertex group is conjugate into A or B.

(b) Every $\mathcal{C}_{n-1}$-by-Fuchsian subgroup over which G splits is conjugate into a black vertex group.

(c) Every subgroup $J \in \mathcal{C}_n$ of G with $\tilde{e}(G,J) \geq 2$ is conjugate into a vertex group and has a subgroup of finite index conjugate into a black vertex group.

More general results have been obtained by Scott and Swarup by developing another line of argument suggested by analogies with 3-dimensional geometry. Recall that given a group G and subgroup H, a subset X of G is said to be H-a.i. if $HX = X$ and $H\backslash X$ is an almost G-invariant subset of $H\backslash G$; and proper if both $H\backslash X$ and its complement are infinite. Proper H-a.i. sets exist if and only if $e(G,H) > 1$. Scott and Swarup picture a.i. sets as immersed submanifolds of a manifold.

If H and K are subgroups of G, X is H-a.i. and Y is K-a.i., X is said to cross Y if each of the four sets $X^? \cap Y^?$ maps to an infinite subset of $K\backslash G$. If X and Y are both proper, X crosses Y if and only if Y crosses X; hence X and Y are not nested. We will say that $X \cap Y$ is small if it is H-finite.

It follows from Lemma 6.1 that if G, H and K are all f.g., the elements $g \in G$ such that gX crosses Y form a finite collection of double cosets KgH. Counting these double cosets defines the intersection number $i(H\backslash X, K\backslash Y)$. The symmetry of intersection numbers was proved in [Scott 1998]. Intersection numbers are independent of a number of choices; if the almost invariant sets X and Y correspond to splittings of G, the intersection number depends only on the splitting. They are imagined as analogues of geometric intersection numbers, e.g. of curves in a surface, and thus relate to the crossings of axes used by Bowditch. The invariant $\sigma(G, H)$ introduced by Kropholler and Roller is the self-intersection number of the unique H-a.i. set in that situation.

It was shown by [Scott and Swarup 2000b] that if $i(H\backslash X, H\backslash X) = 0$ then G splits over a subgroup commensurable with H, the splitting being determined by X. Further, given a collection of splittings of G over subgroups H_i corresponding to H_i-a.i. sets X_i, there is a splitting of G refining all these if and only if all the intersection numbers $i(H_i\backslash X_i, H_j\backslash X_j)$ vanish; moreover, this splitting is uniquely determined. This is proved assuming the subgroups H_i f.g., but it is shown in [Scott and Swarup P02] that this hypothesis is unnecessary. These generalise Theorems 6.8 and 6.9 of [Kropholler and Roller 1988b].

The a.i. set X crosses Y strongly if both $\delta X \cap Y$ and $\delta X \cap Y^*$ map to infinite sets modulo K. This condition likewise is very robust. If X and Y correspond to splittings σ_1 and σ_2 of G, then σ_1 crosses σ_2 strongly if and only if σ_1 is hyperbolic with respect to σ_2 in the sense of Sela.

The main idea of the development in [Scott and Swarup P02] is a notion of regular neighbourhood of a collection of a.i. sets. This generalises the enclosing notion of [Fujiwara and Papasoglu P98]. Suppose given a finite collection of subgroups H_i $(i \in I)$ of G and H_i-a.i. sets X_i; write $E := \{gX_i, gX_i^* \mid g \in G, i \in I\}$. Say E is in good position if given $X, Y \in E$ with two of $X^? \cap Y^?$ small, one of these sets is empty. Thus we can define $X \leq Y$ if $X \cap Y^*$ is either empty or the only small set of the four; this is a partial order. Write V' for the equivalence classes of E under the relation generated by $X \sim X^*$, $X \sim Y$ if X and Y cross (these classes are called cross-connected components). The partial order induces the structure of a pretree on V', which is discrete in the sense that intervals $[x, y]$ are finite sets. The group G acts on V' with finite stabilisers. There is a canonical construction of a bipartite tree from a discrete pretree T': the vertex set V consists of (white vertices) vertices of T' and (black vertices) stars of T'. The corresponding graph of groups is denoted by $\Gamma(\{X_i\}; G)$, and called a regular neighbourhood of the given collection of a.i. sets and their translates. With some effort the construction can be extended to cases when the 'good position' condition fails. It sometimes extends also when I is infinite: the idea is to include more and more elements of I and use an accessibility theorem to show that the process terminates. An element of E is said to be isolated if it crosses no element of E.

A key property of these regular neighbourhoods is an enclosing property. Here a vertex v of a G-tree T is said to enclose an H-a.i. set X if, for each edge e with $t(e) = v$, and $T_e^{\pm}$ the components T with the interior of e removed with $v \in T_e^+$, $X \cap \{g \in G \,|\, gv \in T_e^-\}$ is small. This notion is again robust; if X corresponds to a splitting σ of G, and v encloses X, we say that v encloses σ, and this depends only on σ not on the particular X. Suppose Γ a regular neighbourhood, H a further f.g. subgroup of G and X an H-a.i. set. Then if X does not cross any element of E, it is enclosed by a black vertex of Γ. This leads to a characterisation of regular neighbourhoods (of a collection of H_λ-a.i. sets X_λ) by the following conditions:

(RN0) G acts minimally on T and the quotient graph of groups Γ is bipartite.

(RN1) Each X_λ is enclosed by some white vertex of T, and each white vertex encloses some element of E.

(RN2) A splitting σ of G which does not cross any element of E is enclosed by some black vertex of T.

(RN3) A white vertex v of T is said to be isolated if, for some $X \in E$, the elements of E enclosed by v are just those equivalent to X. This defines a bijection with the set of isolated elements of E. Also any non-isolated vertex must enclose some non-isolated element of E.

Regular neighbourhoods satisfying these conditions are unique up to isomorphism; we have just sketched the existence construction.

The authors outline the proof of their splitting theorem as follows. We want to form a regular neighbourhood of an infinite family of almost invariant subsets of G. The first step is to show that the cross-connected components are of two types, those which contain only strong crossings and those which contain only weak crossings. The structure of the strong crossing components is handled by the techniques of [Bowditch 2002] and of [Dunwoody and Swenson 2000]. If commensurisers are small — i.e. $|\mathrm{Comm}_G(H) : H| < \infty$ — the structure of weak crossing components is easy to describe using regular neighbourhoods. If the commensuriser of a subgroup H of G is large, we let $B(H)$ denote the Boolean algebra of all proper a.i. subsets of G over subgroups commensurable with H. We show that $B(H)$ is finitely generated over $\mathrm{Comm}_G(H)$. The proof depends on standard accessibility results and on techniques of [Dunwoody and Roller 1993] for a special case of the annulus theorem. To obtain canonical decompositions, the only remaining difficulty is to show that the pretree which we construct from the cross-connected components is discrete. This is clear in the case when all commensurisers are small, and is proved in general using again the fact that $B(H)$ is finitely generated over $\mathrm{Comm}_G(H)$.

A proper H-a.i. subset X of an f.g. group G is said to be *n-canonical* if, for any subgroup $K \in \mathcal{C}_k$ of G with $k < n$ and any K-a.i. set Y we have $i(X, Y) = 0$.

Theorem 9.9 *[Scott and Swarup P02] Let G be f.p., not split over any $H \in \mathcal{C}_m$ with $m < n$ ($n \geq 1$). Write $\mathcal{F}_n$ for the set of equivalence classes of all proper a.i. sets over subgroups $H \in \mathcal{C}_n$ of G. Then the regular neighbourhood $\Gamma(\mathcal{F}_n, G)$ exists. For each black vertex w, either*

(a) w is isolated and $G_w \in \mathcal{C}_n$,

(b) G_w is $\mathrm{Comm}_G(H)$ *for some $H \in \mathcal{C}_n$ with $e(G,H) \geq 2$, or*
(c) G_w is a $\mathcal{C}_{n-1}$-by-Fuchsian group.
If e is an edge incident to a vertex of type (a) or (c), $G_e \in \mathcal{C}_n$.

In addition we have the properties (RN0)-(RN3), so any splitting of G over a $\mathcal{C}_n$-subgroup is enclosed by a black vertex; any a.i. subset over an f.g. subgroup that crosses no element of $\mathcal{F}_n$ is enclosed by a white vertex. If, for each subgroup $H \in \mathcal{C}_n$ of G, $|\mathrm{Comm}_G(H) : H| < \infty$ then the vertex groups of type (b) are in $\mathcal{C}_n$. The n-canonical splittings of G over $\mathcal{C}_n$ subgroups are the edge splittings over those edges e such that $G_e \in \mathcal{C}_n$; and the whole construction is invariant under Aut (G). To obtain this uniqueness we have lost something: the vertex groups of type (b), and those at white vertices, need not be f.g. The case $n = 1$ gives a splitting theorem over 2-ended subgroups.

Scott and Swarup proceed to splittings over subgroups belonging to two consecutive classes $\mathcal{C}_n$, seeking a closer analogue to the topological JSJ splitting where the edge subgroups may be $\mathbb{Z}$ or $\mathbb{Z} \times \mathbb{Z}$.

Theorem 9.10 *[Scott and Swarup P02] Let G be f.p., not split over any $H \in \mathcal{C}_m$ with $m < n$. Write $\mathcal{F}_{n,n+1}$ for the set of equivalence classes of all proper a.i. sets over subgroups $H \in \mathcal{C}_n$ and n-canonical subgroups $H \in \mathcal{C}_{n+1}$ of G. Then the regular neighbourhood $\Gamma(\mathcal{F}_{n,n+1}, G)$ exists. For each black vertex w, either (a) w is isolated and $G_w \in (\mathcal{C}_n \cup \mathcal{C}_{n+1})$, (b) G_w is* $\mathrm{Comm}_G(H)$ *for some $H \in (\mathcal{C}_n \cup \mathcal{C}_{n+1})$ with $e(G,H) \geq 2$, or (c) G_w is a $(\mathcal{C}_{n-1} \cup \mathcal{C}_n)$-by-Fuchsian group. If e is an edge incident to a vertex of type (a) or (c), $G_e \in \mathcal{C}_n$.*

Again we have (RN0)-(RN3); for $k = n, n+1$ the k-canonical splittings of G over $\mathcal{C}_k$ subgroups are the edge splittings over those edges e such that $G_e \in \mathcal{C}_k$; and the whole construction is invariant under Aut (G).

Splittings over subgroups in a union of three or more of the classes $\mathcal{C}_n$ are also considered, but to obtain a result it seems to be necessary to restrict to virtually abelian groups. Define $\mathcal{F}_{1,\ldots,n}$ to be the class of H-a.i. subsets X of G such that for some i with $1 \leq i \leq n$ H is virtually abelian of rank i and for any virtually abelian subgroup K of G of rank $k < i$ and any K-a.i. set Y we have $i(X,Y) = 0$. Then the regular neighbourhood $\Gamma(\mathcal{F}_{1,\ldots,n}, G)$ exists and results corresponding to Theorem 9.10 hold.

9.3. Actions on CAT(0) cube complexes

Among the many ideas introduced and explored in [Gromov 1987] were certain types of cube complexes. For this purpose, a cube complex is formed by gluing unit Euclidean cubes along their faces by isometries. We further require that the gluing is such that each cube is embedded in the resulting space X.

We can compute the lengths of rectifiable paths in X. It was shown by Bridson (see e.g. [Bridson and Haefliger 1999]) that this defines a complete metric on X

and thus X acquires the structure of geodesic metric space. We require X to satisfy Gromov's CAT(0) condition. Since we have used unit cubes, this can be expressed combinatorially: e.g. there must be no bigons, two 2-cubes (squares) do not share adjacent edges on their boundaries, and — 'no triangle' condition — if three $(n+2)$-cubes share a common n-cube on their boundaries, and pairwise share common $(n+1)$-cubes, then they must all lie on the boundary of an $(n+3)$-cube.

Connected CAT(0) cube complexes X are necessarily simply-connected, and may be regarded as higher dimensional analogues of trees. A crucial geometric concept for cube complexes is that of a hyperplane. For the standard unit cube $I^n \subset \mathbb{R}^n$, we consider the geometric hyperplanes $x_i = \frac{1}{2}$. In general we can define a (oriented) hyperplane to be an equivalence class of (oriented) edges of X, where two parallel edges of a 2-cube are required to be equivalent. One can show that in a CAT(0) cube complex, a hyperplane determines a geometric hyperplane J (the union of the corresponding subsets of the cubes), and this has no self-intersections, and $X - J$ has exactly 2 components, say Y and Y^*. The set of hyperplanes determines a dual graph: each hyperplane defines a vertex, and two vertices are joined if and only if the corresponding hyperplanes intersect.

The study of actions of groups G on a CAT(0) cube complex X was developed by [Sageev 1995]. If J is a hyperplane in X, write G_J for the subgroup which stabilises J and preserves the two complementary components. For any vertex v of X we partition G into $V := \{g \in G \,|\, gv \in Y\}$ and $V^* := \{g \in G \,|\, gv \in Y^*\}$. We say the action is essential with respect to J if there is a vertex v such that both V and V^* contain infinitely many right cosets of G_J; and is essential if it is essential with respect to some hyperplane.

Theorem 9.11 *[Sageev 1995] Suppose G is f.g. Then there is an essential action of G on a CAT(0) cube complex if and only if G has a subgroup H with $e(G,H) > 1$.*

A key part of the proof is a construction, starting from a subgroup H of G and a proper H-a.i. set A of a cube complex on which G acts. Sageev also indicates that the construction generalises to the case when several such sets (for different subgroups H) are given.

An independent proof was given by [Gerasimov 1997], yielding a stronger result, with 'essential' replaced by 'fixed point free'.

In [Niblo and Roller 1998] a result is proved for arbitrary groups G; here the 'essential' hypothesis is replaced by the conditions that G has no fixed point and is transitive on hyperplanes.

A slightly different form of Scott's splitting obstruction (here the notion of 'crossing' is absent) is as follows. For G a group, H a subgroup and A a proper H-a.i. set, write $S_A(G,H)$ for the set of $g \in G$ such that all four sets $gA^? \cap A^?$ are non-empty. If H is f.g., this is a finite union of double cosets HgH. The following results are also close to those of [Dunwoody and Roller 1993] and [Scott and Swarup P02].

Theorem 9.12 *[Niblo 2002] Let G be f.p., H a subgroup, A an H-a.i. set. If $S_A(G,H)$ is non-empty and generates a proper subgroup G_1 of G, G splits over a subgroup of G_1.*

If moreover G_1 commensurises H, G splits over a subgroup commensurable with H.

Niblo points out that if $S_A(G,H) = \emptyset$ then the cube complex which Sageev constructs is in fact a tree, so that we obtain a splitting of G. Under the hypothesis of the theorem he shows that the index of G_1 is infinite, and the dual graph to the set of hyperplanes is disconnected. Using this disconnection he is able to retract the cube complex on a tree.

Cube complexes also lead [Niblo P02] to a new proof of Stallings' theorem. First we have

Theorem 9.13 *Let G be f.g. with $e(G) > 0$. Then there is a CAT(0) G-cube complex, with a single G-orbit of hyperplanes, which are compact, and G has an unbounded orbit.*

The 2-skeleton of this complex has precisely the properties need to apply the technique of [Dunwoody 1985a]. This leads to

Theorem 9.14 *Let G be an f.g. group which acts with an unbounded orbit on a CAT(0) cube complex with a single G-orbit of hyperplanes, which are compact. Then G splits over a subgroup commensurable with a hyperplane stabiliser.*

9.4. Splittings and coarse geometry

We have already noted that a quasi-isometry between groups preserves the number of ends, and hence the property of admitting a splitting over a finite group. More generally one can ask: given a splitting of G and a quasi-isometry $G \to G'$, is there a corresponding splitting of G'? There have been several recent results of this type. To put these in perspective we note a few facts about the quasi-isometry classification of groups which are not necessarily split.

Finite groups form a single quasi-isometry class, as do 2-ended groups, all of which act properly and uniformly on $\mathbb{R}$.

It follows from Theorem 7.3 that any group quasi-isometric to $\mathbb{H}^2$ is finite-by-Fuchsian, i.e. acts isometrically, properly and uniformly on $\mathbb{H}^2$. There are corresponding results for several other symmetric spaces of non-compact type, and quasi-isometry classifications of lattices in semisimple Lie groups, contained in numerous papers by many authors. See [Farb 1997] for a survey.

It follows from [Gromov 1981] that any group quasi-isometric to $\mathbb{R}^n$ acts isometrically, properly and uniformly on it, and similarly for nilpotent groups. It is shown in [Rieffel 2001] that a group quasi-isometric to $\mathbb{H}^2 \times \mathbb{R}$, or equivalently, to the universal cover $\widetilde{SL_2}(\mathbb{R})$, is an extension of a 2-ended group by a cocompact Fuchsian group; and this result was generalised by [Kleiner and Leeb 2001] to the product of a simply-connected nilpotent Lie group with a symmetric space of non-compact type.

This discussion includes all 2-orbifold groups and all groups in $\mathcal{C}_2$ or $\mathcal{C}_3$ except those corresponding to the Thurston geometry Sol. One may conjecture that any discrete group quasi-isometric to Sol is commensurable with a discrete uniform subgroup of Sol, and that if two such groups G_i $(i = 1, 2)$ are extensions of $\mathbb{Z}^2$ by $\mathbb{Z}$ acting by an automorphism with eigenvalues $\lambda_i, \lambda_i^{-1}$ with $\lambda_i > 1$, then G_1 and G_2 are commensurable if and only if λ_1, λ_2 are (integer) powers of a common number.

The solvable Baumslag-Solitar groups $BS(1,n) = \langle a, b \,|\, a^{-1}ba = b^n \rangle$ have also been studied: it was shown by [Farb and Mosher 1998] that $BS(1,m)$ and $BS(1,n)$ are quasi-isometric if and only if m, n are (integer) powers of a common number, and by [Farb and Mosher 1999] that any group quasi-isometric to $BS(1,m)$ is commensurable to it.

It was shown by [Papasoglu and Whyte 2002] if A and A' are quasi-isometric, then so are $A * B$ and $A' * B$, *except if* B and one of A, A' have order 2. If F is a finite subgroup of A, $A *_F$ is quasi-isometric to $A * \mathbb{Z}$. If F is a finite subgroup of both A and A', and not of index 2 in both, then $A * A'$ and $A *_F A'$ are quasi-isometric. Further, if (Γ, A) and (Δ, B) are graphs of groups with finite edge groups, and if each vertex group of either is quasi-isometric to a vertex group of the other, then $\pi_1(\Gamma, A)$ and $\pi_1(\Delta, B)$ are quasi-isometric. If the vertex groups of both are 1-ended, the converse holds.

Coarse PD^n spaces are defined as follows: the definition will give the reader some sense of what is involved, but to develop it properly involves extensive preliminaries. X is a bounded geometry metric simplicial complex with $C_*(X)$ uniformly acyclic, there exist a constant D, and chain mappings $C_*(X) \xrightarrow{P} C_c^{n-*}(X) \xrightarrow{\overline{P}} C_*(X)$ such that P and $\overline{P}$ have displacement $\leq D$, and there are D-Lipschitz chain homotopies of $P \circ \overline{P}$ and $\overline{P} \circ P$ to the respective identities.

We say that G is a coarse PD^n group if it acts discretely and uniformly on a coarse PD^n space; a dcoarse PD^n group if also it admits an n-dimensional $K(G,1)$ space. One may conjecture that a coarse PD^2 space is quasi-isometric to $\mathbb{R}^2$ or $\mathbb{H}^2$: if so, it would follow from results above that a coarse PD^2 group is a virtual surface group. If $n \neq 2$, a coarse PD^n group is dcoarse if and only if c.d.$G = n$.

An algebraic version of Scott's compact core theorem for 3-manifolds is

Theorem 9.15 *[Kapovich and Kleiner P99] Let X be a coarse PD^n space and G an $(n-1)$ dimensional duality group acting freely and simplicially on X. Then G has a finite set $H = \{H_i\}$ of subgroups such that (G, H) is a PD^n pair. There is a connected G-invariant subcomplex K of X such that K/G is compact; the stabilisers of the components of $X - K$ are conjugate to the H_i; and each component of $(X - K)/G$ is 1-ended.*

We turn to theorems asserting invariance of splitting under quasi-isometry.

Theorem 9.16 *[Mosher, Sageev and Whyte P00] If (Γ, G) is a graph of coarse PD^n groups whose tree has infinitely many ends, and H is quasi-isometric to $\pi_1(\Gamma, G)$, then H is the fundamental group of a graph of groups whose tree has infinitely many ends and with vertex and edge groups quasi-isometric to those of G.*

Let (Γ, G) be a graph of f.g. abelian groups such that for each vertex group G_v of dimension n, the incident edge groups G_e have dimension $< n$, and if any have dimension $n-1$, then the G_e span G_v. Then any f.g. group H quasi-isometric to $\pi_1(\Gamma, G)$ is the fundamental group of a graph of virtually abelian groups.

They obtain this as a special case of a more general but more technical result. A closely related theorem is

Theorem 9.17 *[Papasoglu P02] (i) Let (Γ, A) be a graph of groups with all edge and vertex groups dcoarse PD^n groups; set $G := \pi_1(\Gamma, A)$. Exclude the cases that Γ is a loop with both edge-to-vertex maps isomorphisms and that Γ has just one edge with each $|A_v : A_e| = 2$. Then any group H quasi-isometric to G splits over a group quasi-isometric to an edge group of (Γ, A).*

(ii) Let (Γ, A) be a graph of groups with all vertex groups dcoarse PD^n groups and all edge groups dominated by coarse PD^{n-1} spaces; set $G := \pi_1(\Gamma, A)$; assume G f.g. Then any group H quasi-isometric to G splits over a group quasi-isometric to an edge group of (Γ, A).

One consequence of (i) gives the results of Farb and Mosher about solvable Baumslag-Solitar groups. Another corollary is the version where 'dcoarse PD^n' is replaced by 'virtually $\mathbb{Z}^n$'. This is weaker than the preceding theorem, except for the improvement of $n-2$ to $n-1$ in (ii).

10. PD^2 and PD^3 complexes and pairs

10.1. PD^2 groups

Recent advances have completed the earlier results about PD^2_k groups. Let us say for short that a group H is k-finite if its order is finite and invertible in k. For completeness, we first discuss PD^0_k and PD^1_k. Recall that by Theorem 4.4 an f.g. group G with c.d.$_k G = 1$ has a free subgroup G_0 of finite index (it is not difficult to eliminate the f.g. hypothesis below).

A PD^0_k group G has c.d.$_k G = 0$, so G_0 exists and must be trivial. Hence G is k-finite. For a PD^0_k pair $(G, \mathbf{H})$, $\mathbf{H}$ must be empty.

A PD^1_k group G has c.d.$_k G = 1$. Thus G_0 exists, is also a duality group, so has rank 1. Hence G is 2-ended, and is an extension of a k-finite group by $\mathbb{Z}$ or (if $\frac{1}{2} \in k$) $\mathbb{Z}_2 * \mathbb{Z}_2$. If $(G, \mathbf{H})$ is a PD^1_k pair, then as above G is k-finite. The pair arises by the action of G on a tree, which (for duality) must be a star with 2 arms. Either $\#\mathbf{H} = 2$ and $G = H_1 = H_2$ or $\#\mathbf{H} = 1$ and $|G : H_1| = 2$.

The structure of PD^2_k groups is given by the following major result, which establishes a conjecture of [Dicks and Dunwoody 1989, Chap V].

Theorem 10.1 *Let G be a group and k a commutative ring. Then the following are equivalent:*

(i) There is a uniform action of G by isometries on either the Euclidean or the hyperbolic plane and the order of each point stabiliser is k-finite.

(ii) G admits a uniform action on $\mathbb{R}^2$ such that the point stabilisers are k-finite.

(iii) G is a finite-by-2-orbifold group, where the finite group and stabilisers are k-finite.

(iv) G is a PD^2_k group.

If $k = \mathbb{Q}$, a further equivalent condition is

(v) G is a virtual surface group.

Assume as usual for simplicity that G preserves orientation. Clearly (i) implies (ii). If (ii) holds, the quotient $G\backslash\mathbb{R}^2$ is a closed 2-orbifold and (iii) holds. Since the cone points have orders invertible in k we can regard the orbifold as a k-homology manifold, so duality holds over k. Since also the kernel K of the action is k-finite, G is a PD^2_k-group.

If (iii) holds and $K \lhd G$ is a finite normal subgroup with $H = G/K$ a 2-orbifold group, then H has a subgroup $H' = G'/K$ of finite index which is a surface group. The extension G' of K by H' need not split — there is an obstruction in $H^2(H'; C)$ where C is the centre of K: see e.g. [Brown 1982, 6.6]. We can find a subgroup H'' of finite index in H' such that the obstruction restricts to $0 \in H^2(H''; C)$ (e.g. pass first to a subgroup which acts trivially on C, and then to a further subgroup of index $|C|$). The induced extension splits, so G'' and hence G contains a copy of H'' as a subgroup of finite index, and (v) holds.

If $|G : G'|$ is invertible in k and G' is a PD^n_k group, so is G: thus if $k = \mathbb{Q}$, (v) implies (iv); also (v) implies (i) (with $k = \mathbb{Q}$) by the main theorem of [Kerckhoff 1983]. We can pass to general coefficient rings since if G is a PD^2_R group for R a commutative ring, then it is also a PD^2_k group for any field k such that there is a homomorphism $R \to k$; also, it is easy to see that duality fails in (i) and (ii) if the orders of point stabilisers are not invertible in k. Thus the main point is that (iv) implies the other conditions.

It is shown in [Dunwoody and Swenson 2000] that if G is f.p., is a PD^2_k group, and has an element of infinite order, it either has an infinite cyclic subgroup H with $\tilde{e}(G, H) \geq 2$, or has a quotient $\cong \mathbb{Z}$, so that Theorem 9.7 can be applied.

The full result is due to [Bowditch P99], who proves that (iv) implies (v) assuming k a field, and also discusses other equivalent conditions. Much of the effort in [Bowditch P99] goes into proving that G must contain an element of infinite order. He also argues directly using Theorem 7.3 in the form of Theorem 7.4, so avoids the f.p. hypothesis in Theorem 9.7.

Bowditch also obtains a stronger result.

Theorem 10.2 *[Bowditch P99] Suppose G is FP_2 over a field k and that $H^2(G; kG)$ has a 1-dimensional G-invariant subspace. Then G is a virtual surface group.*

The strange looking hypothesis is suggested by the results in [Farrell 1974]. Farrell shows, for example, that if G is f.p. and not a torsion group, any G-submodule of $H^2(G; \mathbb{F}_2 G)$ has dimension 0, 1 or ∞ over $\mathbb{F}_2$. The theorem resolves the first problem listed in [Hillman 1994].

There are two further characterisations, the first by quasi-isometry.

Theorem 10.3 *[Bowditch P99] Suppose G is f.g. and quasi-isometric to a complete path-metric space homeomorphic to $\mathbb{R}^2$. Then G is a virtual surface group.*

Theorem 10.4 *[Bowditch 2002] Let G be an f.g. one-ended group which contains elements g of infinite order, and such that for every such g, $\langle g \rangle$ has at least 2 coends. Then G is a Fuchsian group.*

Now let $(G, \mathbf{H})$ be a PD_k^2 pair. Again by Proposition 5.1, c.d.$_k G = 1$, so by Theorem 4.4 G has a free subgroup G_0 of finite index N, which we may suppose normal. By Lemma 5.2, the double $D(G, \mathbf{H})$ is a PD_k^2 group, hence by Theorem 10.1 admits a uniform action by isometries on either the Euclidean or hyperbolic plane E. There is a subgroup of index N in $D(G, \mathbf{H})$ which is a double cover $D(G_0, \mathbf{H_0})$ where, for each $H \in \mathbf{H}$, $\mathbf{H_0}$ contains $|H : H \cap G_0|$ copies of $H \cap G_0$. If E is Euclidean, G_0 is cyclic and acts properly uniformly on a strip $\mathbb{R} \times I$; G is 2-ended and preserves the strip, and $\#\mathbf{H}$ is 1 or 2 (as in the discussion of PD_k^1 pairs). If E is hyperbolic, G_0 is Fuchsian and acts properly on the convex hull X of its set of limit points. As G normalises G_0, it preserves X, and the quotient of $(G, \mathbf{H})$ by the (k-finite) kernel of the action is the orbifold fundamental group pair of X/G.

10.2. Topological PD^3 complexes and pairs

The recent advances in both 3 dimensional topology and group theory give grounds for hope of significant progress towards classification for orientable PD^3 spaces and pairs. The most natural approach is to attempt to mimic known theorems for compact 3 dimensional manifolds. We will thus seek an analogue to Thurston's programme of finding a natural decomposition into 'geometric' pieces. There are, however, numerous unanswered questions.

It would be simpler to confine attention to PD^3 spaces, but we will discuss the case of pairs also. We seek to reduce questions about PD^3 pairs in the homotopy theoretic sense to questions about PD^3 pairs in the group theoretic sense, and then to apply some of the splitting theorems above.

First observe that there is no hope of obtaining nice results for PD^n groups for larger n. We refer to [Davis 2000] for references for examples of the following for any $n \geq 4$:

(a) For $k = \mathbb{Z}[m^{-1}]$ a PD_k^n group not commensurable to a PD^n group.

(b) A hyperbolic PD^n group G with ∂G not 1-connected; another with ∂G not locally 1-connected (hence not an ANR).

(c) A PD^n group which is not f.p.

(d) An aspherical closed n-manifold with $\pi_1(M)$ not residually finite.

The first step in decomposing a 3-manifold is to cut along embedded 2-spheres till we obtain an irreducible manifold, which is either homeomorphic to $S^1 \times S^2$ or has vanishing π_2. For a PD^3 complex X we have

Theorem 10.5 *[Turaev 1981] If $\pi_1(X) = G_1 * G_2$ is a free product, X is homotopy equivalent to the connected sum of PD^3 complexes X_1 and X_2, with $\pi_1(X_i) \cong G_i$.*

Turaev obtains this theorem as a consequence of his proof that an oriented PD^3 complex is determined up to homotopy by its fundamental group π and the image of its fundamental class in $H_3(\pi;\mathbb{Z})$, together with a characterisation in [Turaev 1979] of which pairs (π, z) with $z \in H_3(\pi;\mathbb{Z})$ correspond to PD^3 complexes. Now while we can iterate the decomposition until $\pi_1(X)$ no longer splits as a free product, we would like to apply Theorem 3.3 to deduce that we can reduce to the case when $\pi_1(X)$ has at most one end. For

$$\pi_2(X) \cong \pi_2(\tilde{X}) \cong H_2(\tilde{X};\mathbb{Z}) \cong H^1(\tilde{X};\mathbb{Z}) = H^1(\pi;\mathbb{Z}\pi).$$

This is non-trivial if and only if π has at least 2 ends and hence, by Stallings' theorem, splits over a finite group. If we knew that π could not split over a non-trivial finite group (e.g. if π is torsion free), we could use Turaev's theorem to split until we reach PD^3 complexes X with either $\pi_2(X) = 0$ or $\pi_1(X) \cong \mathbb{Z}$.

If $G \cong \mathbb{Z}$, then $X \simeq S^1 \times S^2$. If $\pi_2(X) = 0$ and G is finite, $\tilde{X} \simeq S^3$ and [Cartan and Eilenberg 1956] G has periodic cohomology with period 4. Such groups have been classified: see e.g. [Milnor 1957], and [Thomas 1969] each choice of a generator $g \in H^4(G;\mathbb{Z})$ determines a PD^3 complex, unique up to homotopy equivalence. If $\pi_2(X) = 0$ and G is infinite, then $\tilde{X}$ is contractible, X is a $K(G,1)$ space, and G is a PD^3 group.

Conjecture 10.1 *If X is a PD^3 complex or (X,Y) is a PD^3 pair, $G = \pi_1(X)$ does not split over a non-trivial finite group.*

A new approach by Crisp improves Theorem 10.5 as follows.

Theorem 10.6 *[Crisp 2000] If X is an orientable PD^3 complex, $\pi_1(X) = G$, and $e(G) \geq 2$, then either G is a free product or G is virtually free.*

The key idea is to consider a finite cyclic subgroup C of an edge group and compare the calculation $H_s(C; H^1(G;\mathbb{Z}G)) \cong H_{s+3}(C;\mathbb{Z})$ using Poincaré duality with calculations using the decomposition afforded by the tree.

Thus to prove Conjecture 10.1 for PD^3 complexes it would suffice to consider the case when G is virtually free.

The situation for PD^3 pairs is much less satisfactory. An appendix to [Turaev 1979] sketches an extension of his characterisation theorem to the case of pairs, and it seems plausible that this could be used to extend Theorem 10.5 to the case of PD^3 pairs.

Conjecture 10.2 *If $(X, \partial X)$ is a PD^3 pair and $G = \pi_1(X) = G_1 * G_2$ is a free product, then $(X, \partial X)$ is homotopy equivalent to the boundary-connected sum of PD^3 pairs (X_1, Y_1) and (X_2, Y_2), with $\pi_1(X_i) \cong G_i$.*

Even if we assume Conjecture 10.2, and that G is torsion free, so we can reduce to that case when G has just one end, it does not follow that X is then a $K(G, 1)$. Comparing with the situation for 3-manifolds we see that what is required there is an application of the loop theorem to show that if, for some component S of ∂M, the map $\pi_1(S) \to \pi_1(M)$ is not injective, there is an embedded disc in M whose boundary is not nullhomotopic in S.

The best presently known analogue of the loop theorem for PD^3 pairs is the 'weak loop theorem' of [Thomas 1984]. It is desirable to obtain a much closer analogue of the result known for manifolds. However, this weak loop theorem already enables [Crisp 2000] to prove Theorem 10.6 for the case of PD^3 pairs. Thus if Conjectures 10.1 and 10.2 hold, we can split PD^3 pairs down to the case where G either has one end or is virtually free.

It follows from the geometric loop theorem that if (X, Y) is a PD^3 pair, and S a component of Y (which, by Theorem 5.2, we may take to be a closed surface), and $\pi_1(S) \to \pi_1(X)$ is not injective (i.e. S is compressible), then there exists an embedded loop L in S, defining a non-zero element of $\pi_1(S)$ but nullhomotopic in X. We may suppose L two-sided, otherwise the boundary of a tubular neighbourhood of L is two-sided and has the same property. Thus we have a homeomorphism of $S^1 \times [-1, 1]$ to a tubular neighbourhood of L. Attaching a handle $D^2 \times [-1, 1]$ along this neighbourhood yields another PD^3 pair (X', Y'). Since the boundary $L = S^1 \times 0$ is nullhomotopic in X, hence bounds a singular disc, we can extend the central disc $D^2 \times 0$ of the handle to a map of S^2 into X'.

Conjecture 10.3 *There is a splitting of (X', Y') (up to homotopy) as a connected sum $(X_1, Y_1)\#(X_2, Y_2)$ whose splitting sphere is homotopic to the one just described.*

If this holds, it follows that (X, Y) splits as the boundary-connected sum of (X_1, Y_1) and (X_2, Y_2). Thus if all Conjectures 10.1-10.3 hold, we may split any PD^3 pair by connected sum and boundary-connected sum till, for each piece (X, Y), X is a $K(G, 1)$ and for each component Y_i of Y, $H_i = \pi_1(Y_i)$ maps injectively to G. We can also fill in any boundary 2-spheres by 3-discs, and so suppose each boundary component also a $K(H_i, 1)$ (by a result of [Wall 1967] there is a unique way to remove these 3-discs again afterwards to recover the original pair). Thus $(G, \{H_i\})$ is a PD^3 pair in the group theoretic sense.

10.3. PD^3 groups and pairs

We now assume that $(G, \{H_i\})$ is an orientable PD^3 pair in the group theoretic sense. Each H_i is a PD^2 group, hence by Theorem 5.2 is a surface group. It is our belief that in this case there is always a compact 3-manifold M such that $(M, \partial M)$ is homotopy equivalent to $(K(G,1), \dot{\bigcup}_i K(H_i,1))$. We again break this into a series of lesser conjectures.

First recall Theorem 6.15, in the case $n = 3$:

Theorem 10.7 *Let $(G, \mathbf{S})$ be a PD^3-pair such that G satisfies Max-c but is not in $\mathcal{C}_3$. Then there is a unique reduced G-tree Y, adapted to $\mathbf{S}$, such that $G\backslash Y$ is finite, each edge group is in $\mathcal{C}_2$ (hence is a torus group), each vertex pair is either of Seifert type or atoroidal; and every torus subgroup of G fixes a vertex of Y.*

This theorem gives a good decomposition, but is subject to the undesirable hypothesis Max-c. For the case of PD^3 groups, we can apply the stronger result Theorem 9.8, taking $n = 2$. To accommodate the non-orientable case, call a pair (G, H) with G a torus group and $|G : H| = 2$ weakly inessential: this corresponds to the product of S^1 with a Möbius strip.

Theorem 10.8 *If G is an f.p. orientable PD^3 group, one of the following holds.*

(i) $G \in \mathcal{C}_3$.

(ii) G is a $\mathbb{Z}$-by-2-orbifold group.

(iii) G has a finite bipartite graph of groups decomposition such that each edge group is a torus group; for each black vertex, either the vertex group is a torus group and the vertex pair inessential, or the vertex pair is a maximal $\mathbb{Z}$-by-2-orbifold pair; and each white vertex pair is atoroidal or weakly inessential. Moreover,

(a) For every splitting $G = A *_C B$ or $G = A*_C$ of G over a torus group C, each white vertex group is conjugate into A or B.

(b) Every torus subgroup J of G, and every $\mathbb{Z}$-by-2-orbifold subgroup over which G splits, is conjugate into a black vertex group.

Proof Since G is a PD^3 group, we have c.d.$G = 3$; in particular, G is torsion free. If G is polycyclic-by-finite, (i) holds. Otherwise, we have already seen that for a subgroup $H \cong \mathbb{Z}$, $\tilde{e}(G, H) = 1$. Thus the hypotheses of Theorem 9.8 are satisfied. Case (i) of that theorem gives case (ii) above, and case (ii) corresponds to (iii).

In this final case, the structure of the black vertex pairs is already identified. It follows that each edge group is a torus group. It remains to show that the white vertex pairs are atoroidal. We saw (see remarks preceding Lemma 6.2) that for any torus subgroup $J \subset G$, $\tilde{e}(G, J) = 2$. Thus J has a subgroup J' of finite index conjugate into a black vertex group. If J is contained in a white vertex group, then J' is conjugate into two distinct vertex groups of the decomposition. It follows from the general discussion of subgroups of split groups that J' is contained in an edge group, with the edge incident to the given white vertex.

Denote the white vertex pair in question by $(G', \mathbf{S})$, and let H' be one of the groups in $\mathbf{S}$ which contains J'. We may suppose without loss of generality that J is a maximal torus subgroup of G', and that J' is a maximal subgroup which is conjugate into H'. Visualise this as a manifold pair $(M, \partial M)$; consider the covering space $\tilde{M}$ corresponding to the subgroup J'. The boundary components over H' correspond to double cosets of H' and J' in G'; to the double coset $J'gH'$ corresponds something with fundamental group $g^{-1}J'g \cap H'$. If $g \in J$, this is just the torus group J'. If the cosets $J'g, J'g'$ (with $g, g' \in J$) are distinct, so are $J'gH'$ and $J'g'H'$, for otherwise there exists $h \in H'$ with $J'gh = J'g'$ and so $h \in J$; by maximality of J', $h \in J'$, so $J'g = J'g'$. Thus in the cover $\tilde{M}$ — itself homotopy equivalent to a torus — there are at least $|J : J'|$ boundary components which are tori. Hence $H_3(\tilde{M}, \partial\tilde{M})$ has rank at least $|J : J'| - 1$. By duality, so has $H^0_c(\tilde{M})$. But this group vanishes if the covering is infinite and $\tilde{M}$ non-compact; if the cover is finite, the group has rank 1. Hence if $|J : J'| > 1$, the index is 2; the covering is finite, so J' has finite index in G'; as G' is torsion free, it is a torus group, so $G' = J$. Now $H' = J'$ is a torus group of index 2 in G', thus (G', H') is weakly inessential. Otherwise $J = J'$ is contained in a boundary group. Since this holds for any torus subgroup of G', the vertex pair is atoroidal. □

Swarup has indicated to me that the result can also be deduced from Theorem 9.9, using essentially the same argument to deal with commensurisers. Thus the black vertex pairs are given by the regular neighbourhood construction (this is closely related to the enclosing condition (b)), and the splitting is unique.

In cases (i) and (ii), $K(G, 1)$ is homotopy equivalent to a 3-manifold with geometric structure; of type $\mathbb{R}^3$, Nil or Sol for (i) and $\mathbb{H}^2 \times \mathbb{R}$ or $\widetilde{SL_2(\mathbb{R})}$ for (ii). Geometries of type S^3, $S^2 \times \mathbb{R}$ do not correspond to aspherical manifolds. We refer to [Scott 1983] for an account of these geometries. In case (iii) the black vertex pairs correspond to Seifert manifolds with boundary: the precise identification is essentially due to Theorem (7.2) on convergence groups. Several earlier partial results were obtained by [Hillman 1985, 1987, 1994].

Since the theorems of §9.2 have yet to be adapted to the relative case, we have at present no result for PD^3 pairs avoiding the hypothesis Max-c. Perhaps this version of the relative case will not be too difficult. We hope for more.

Conjecture 10.4 *The direct analogue of Theorem 6.4 holds also for group* PD^3 *pairs.*

One line of attack is to apply Theorem 10.8 to the double $D(G, \mathbf{S})$. Since the resulting splitting is unique, it is compatible with the involution σ interchanging the two copies of G. It would be necessary to investigate how σ restricts to the vertex pairs of the splitting.

It was conjectured by Kropholler that if $(G, \mathbf{S})$ is atoroidal, G is isomorphic to a discrete subgroup of $PSL_2(\mathbb{C})$ and $\mathbf{S}$ to the collection of peripheral subgroups. To obtain a more precise formulation (I am indebted to Misha Kapovich for help with the following), let us define an atoroidal PD^3 pair $(G; \mathbf{S})$ to be *strictly atoroidal* if, moreover, it is acylindrical, i.e.

(i) for distinct elements S_1, S_2 of $\mathbf{S}$, and any $g_1, g_2 \in G$, $g_1^{-1}S_1g_1 \cap g_2^{-1}S_2g_2$ consists only of the identity element, and

(ii) if $S \in \mathbf{S}$, and $g^{-1}Sg \cap S \neq \{1\}$, then $g^{-1}Sg = S$.

Conjecture 10.5 *If $(G, \mathbf{S})$ is a strictly atoroidal PD^3 pair, then G is hyperbolic relative to $\mathbf{S}$.*

Misha Kapovich and Bruce Kleiner have recently shown that if G is a PD^3 group which acts discretely, isometrically and cocompactly on a CAT(0) space, then G is either a hyperbolic group, a Seifert manifold group, or splits over a virtually abelian subgroup.

Conjecture 10.6 *If $(G, \mathbf{S})$ is a strictly atoroidal PD^3 pair such that G is hyperbolic relative to $\mathbf{S}$, then G is isomorphic to a discrete subgroup of $PSL_2(\mathbb{C})$ and $\mathbf{S}$ to the collection of peripheral subgroups.*

Cannon and co-workers have written several papers developing an approach to this problem in the case of PD^3 groups; I confine myself to citing [Cannon 1994] and [Cannon and Swenson 1998], and the outline in [Kapovich and Benakli 2002]. We saw in §7 that a hyperbolic group is a PD^3 group if and only if ∂G has the Čech homology of a 2-sphere; it is then known that ∂G is homeomorphic to S^2. If there is a homeomorphism preserving the quasi-conformal structure, or equivalently if G is itself quasi-isometric to hyperbolic space $\mathbb{H}^3$, a theorem of Sullivan applies to give the desired result.

If $\mathbf{S} \neq \emptyset$, then $\mathrm{c.d.}\, G = 2$. Hence $\dim \partial G = 1$. It follows from duality (c.f. Corollary 5.1) that G cannot split over a finite group, so ∂G is connected; it follows from the acylindricity hypothesis that ∂G has no local cut points. Thus Theorem 7.7 is applicable. In this case ∂G is either S^1 or the Sierpinski gasket, so we have the situation of Theorem 7.8. To see this, we form the double $D(G, \mathbf{S})$. Either G is a surface group or $D(G, \mathbf{S})$ is hyperbolic, with G a quasiconvex subgroup. Hence ∂G embeds in $\partial D(G, \mathbf{S})$, which is homeomorphic to S^2, so is planar, and thus ∂G cannot be a Menger curve.

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