

Uncountably many wild knots whose cyclic branched covering are S^3

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ABSTRACT

There is a disk in S^3 whose interior is PL embedded and whose boundary has a tame Cantor set of locally wild points, such that the n -fold cyclic coverings of S^3 branched over the boundary of the disk are all S^3 . An uncountable set of inequivalent wild knots with these properties is exhibited.

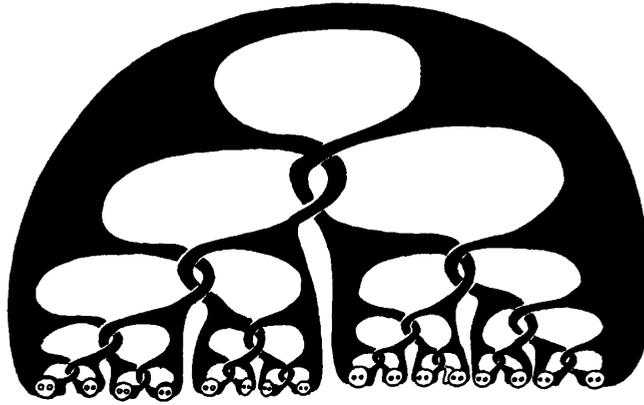
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1. Introduction

This paper deals with wild disks like the disk Δ of Figure 1. The interior of Δ is a PL embedded open disk in S^3 . Its boundary is a simple closed curve K in S^3 . The set T of points at which Δ is not locally tame is a tamely embedded Cantor set. We will prove the following theorem.

Theorem 1. *K is a prime, non trivial knot, but the n -fold cyclic covering of S^3 branched over it is S^3 , for every $n \geq 2$. The preimage of K is a simple closed curve in S^3 bounding a disk, and the set of points at which it is not locally tame is a wildly embedded Cantor set covering T . Moreover, uncountably many inequivalent knots having these properties can be exhibited.*

Figure 1: The disk Δ

By considering the preimage of Δ in any of the n -fold cyclic coverings of K we obtain disks satisfying the:

Corollary 2. *There exist a disk in S^3 whose boundary is a knot K which is wild at a wildly embedded Cantor set, such that the n -fold cyclic covering of S^3 branched over K is S^3 , for every $n \geq 2$. Moreover, there are uncountably many inequivalent knots having these properties.*

However, exhibiting such knots might be difficult. These two results are in sharp contradistinction with the PL case in which there exist exactly one such knot, namely, the trivial knot. *One wonders if every homotopy 3-sphere is always an (non PL) n -fold covering of S^3 , branched over a wild knot, with wild subset T , (where T might be a **knotted** Cantor set, as in Corollary 2), for some n .*

These results were, in part, announced in [11].

Acknowledgement 3. *I am grateful to Professor E. Outerelo for his help with Theorem 4.*

2. Preliminaries

A closed set X in a 3-manifold M is *tame* if there is a homeomorphism of M onto itself sending X onto a subpolyhedron of some rectilinear triangulation of M . If X is not tame then it is *wild*. The set X is *locally tame* at a point x of X if there exist a neighbourhood U of x in M and a homeomorphism of U into M that takes $U \cap X$ onto a tame set. Otherwise we say that X is *locally wild* at x . Bing [5] showed that a closed set is tame in S^3 if it is locally tame at each of its points. The set of points of X at which it is locally tame is open in X and is called the *tame subset* of X , while

the subset at which it is locally wild is closed, and is called the *wild subset* of X . A *knot* in M is an embedding of the 1-dimensional sphere S^1 in M . A *wild knot* has a non empty wild subset. A Cantor set A embedded in S^3 will be called an *Antoine's Cantor set* if $S^3 - A$ is not simply connected and A is not separated by any 2-sphere.

Following Fox [8], we say that a space X is locally connected in a space Y if there is a basis of Y such that $V \cap X$ is connected for every basic open set V . Freudenthal [9] (see [8]) has shown that every T_1 , connected, locally connected, locally compact, regular space X with a numerable basis, is contained in a compact space Y with the same properties in such a way that X is dense, open and locally connected in Y , and the *end space* $E(X) := Y - X$ is totally disconnected. Moreover, this compactification Y of X (*Freudenthal compactification*) is determined by these properties. As an important example we have:

Theorem 4. *Let T be a compact, totally disconnected subset of a compact, connected 3-manifold M , then M is the Freudenthal compactification of $M - T$.*

Proof. The only property which is not completely obvious to check is that $M - T$ is locally connected in M , but this is a consequence of the Jordan Separation Theorem (see [10], for instance). ■

We need to recall some concepts from decomposition theory (see [7]). A *decomposition* G of a topological space S is a partition of S . The elements g of the partition are subsets of S . If g contains more than one point, then g is *non-degenerate*. The *decomposition space* S/G is a topological space with underlying point set G and with the quotient space topology induced by the *decomposition map* $\pi : S \rightarrow S/G$ sending $s \in S$ to the unique element of G containing s . We say that G is *upper semi-continuous* if (1) π is closed and (2) each $g \in G$ is compact. An upper semi-continuous decomposition G is *monotone* when the elements g of G , besides being compact, are connected, that is, if the elements g of G are *continua*.

Let M_1, M_2, M_3, \dots be a sequence of compact, 3-manifolds PL embedded in S^3 such that each component of M_i is a handlebody and, for each positive integer j , $M_{j+1} \subset \text{Int}M_j$. We say that the sequence $\{M_i\}_{i=1}^{\infty}$ is a *defining sequence* of the upper semi-continuous monotone decomposition G of S^3 whose non degenerate elements are the connected components of $H = \bigcap_{i=1}^{\infty} M_i$. In the decomposition space S^3/G , the image of H is a compact, totally disconnected subset (see [1], [2]). Two such decompositions F and G of S^3 , with defining sequences $\{M_i\}_{i=1}^{\infty}$ and $\{N_i\}_{i=1}^{\infty}$ are called *equivalent* if and only if there exists a homeomorphism from $(S^3/F, (\bigcap_{i=1}^{\infty} M_i)/F)$ onto $(S^3/G, (\bigcap_{i=1}^{\infty} N_i)/G)$. In [2] is stated the following Theorem.

Theorem 5. *Suppose F and G are equivalent monotone decompositions of S^3 , with defining sequences $\{M_i\}_{i=1}^{\infty}$ and $\{N_i\}_{i=1}^{\infty}$. Assume also that the elements of F and G are cellular. Then, there exists a sequence $\{f_i\}_{i=1}^{\infty}$ of homeomorphisms from S^3 onto S^3 such that, for each i , $f_{i+1} \mid (S^3 - \text{Int}M_i) = f_i \mid (S^3 - \text{Int}M_i)$, and $\{f_i(M_i)\}_{i=1}^{\infty}$ is a defining sequence for G .*

Let K be a wild knot whose wild subset is a totally disconnected subset T of S^3 . There exist an n -fold cyclic covering $q : K_n \rightarrow S^3 - T$ branched over the tame subset $K - T$ of K . If $j : S^3 - T \rightarrow S^3$ denotes the inclusion map, then the composition $j \circ q$ is a spread (in Fox sense [8]) satisfying the conditions listed in [8] granting the existence and unicity of the Fox completion $\widehat{q} : \widehat{K}_n \rightarrow S^3$ of the spread $j \circ q$. The branched covering $\widehat{q} : \widehat{K}_n \rightarrow S^3$ is an n -fold cyclic covering branched over the knot K . The map \widehat{q} is induced by a cyclic action by homeomorphisms in the topological space \widehat{K}_n . The fixed point set of this action is the preimage $\widehat{q}^{-1}(K)$. Moreover $\widehat{q}^{-1}(T)$ is a totally disconnected subset of \widehat{K}_n . If \widehat{K}_n turns out to be a 3-manifold then $\widehat{q}^{-1}(K)$ is a wild knot with wild subset $\widehat{q}^{-1}(T)$. The space \widehat{K}_n is the Freudenthal compactification of K_n . This follows from the following direct generalization of the Compactification Theorem of Fox ([8], page 249):

Theorem 6. *Let $f : X \rightarrow B$ be a branched covering. Assume X and B are T_1 , connected, locally connected, locally compact, with base numerable and regular, but not compact. Let \widehat{B} be the Freudenthal compactification of B , and let j be the inclusion $j : B \subset \widehat{B}$. Let $g : Y \rightarrow \widehat{B}$ be the branched covering which is the Fox completion of $j \circ f : X \rightarrow \widehat{B}$. Then, Y is the Freudenthal compactification of X if \widehat{B} has a basis such that, for each basic open set W , the number of components of $f^{-1}(W)$ is finite.*

3. Description of Δ

In Figure 2 we see a 3-ball with two holes B . Consider B embedded in $S^3 = \mathbf{R}^3 + \infty$. Call B_{-1} the closure of the connected component of $S^3 - B$ containing ∞ . The outer boundary $B \cap B_{-1}$ is called S^+ . It bounds a ball \widehat{B} containing B . The other two boundary components of B are denoted S_0^-, S_1^- . They bound balls $\widehat{B}_0, \widehat{B}_1$, respectively. In B is embedded a surface F . The boundary of F intersects each component of the boundary of B in two disjoint arcs. The boundary of F minus the interior of these six arcs is denoted by R .

Suppose the orientation preserving similarity of \mathbf{R}^3 mapping S^+ onto S_i^- , takes $F \cap S^+$ onto $F \cap S_i^-$. Use this similarity σ_i to define $B_i = \sigma_i(B)$, and the surface $F_i = \sigma_i(F)$ inside B_i , $i = 0, 1$. Continue this process to infinity.

In this way, for every word e in the alphabet $\{0, 1\}$, we have a punctured ball B_e contained in a ball \widehat{B}_e , with an outer boundary component S_e^+ and two inside boundaries S_{e0}^-, S_{e1}^- , and an embedded surface F_e . The union of the balls \widehat{B}_e , such that the length of the word e is j , is denoted by \widehat{B}_j . The intersection $\bigcap_{j=0}^\infty \widehat{B}_j$ is a Cantor set T tamely embedded in S^3 . The union $F_{-1} \cup_e F_e \cup T$, where F_{-1} is the disk in B_{-1} shown in Figure 3, is the disk Δ . Clearly, the interior of Δ is PL embedded in S^3 . The boundary of Δ is a wild knot K with wild subset T .

For subsequent applications (section 7) the union of the punctured balls B_e such that the length of the word e is j is denoted by B_j . The intersection $B_j \cap \Delta$, denoted by Δ_j , is called the j -level of Δ . It consists of 2^j twisted bands like the one depicted

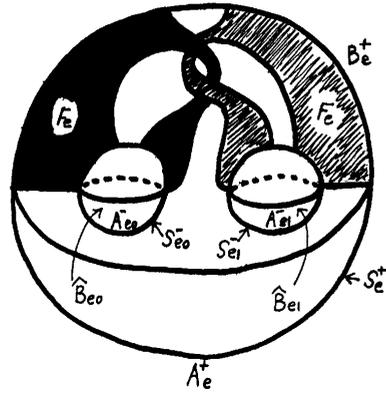


Figure 2: The ball B_e

in Figure 2.

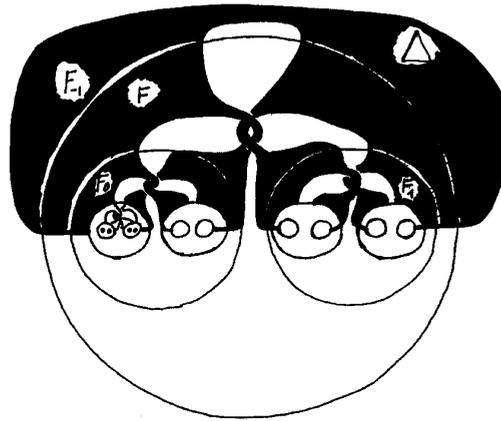


Figure 3

4. The cyclic coverings of B

First, we construct an n -fold cyclic covering of B branched over R as follows. After some deformation B , F , and R are shown in Figure 4. Cutting B open along F we obtain the *standard sheet* H shown in Figure 5. The n -fold cyclic covering of B branched over R is obtained by pasting n copies of the standard sheet H cyclically

around R . In figure 6 we see the case $n = 3$. Denote this n -fold cyclic covering by $p : X \rightarrow B_3$ branched over R . The manifold X is a handlebody of genus $n - 1$ (bounded by \tilde{S}^+) minus the interior of two disjoint handlebodies \hat{X}_0, \hat{X}_1 of genus $n - 1$, bounded by $\tilde{S}_0^-, \tilde{S}_1^-$. The manifold X is embedded in S^3 as shown in Figure 6. The covering p is induced by $2\pi/n$ rotation about the axis E . The closure of $S^3 - X$ is denoted by X_{-1} .

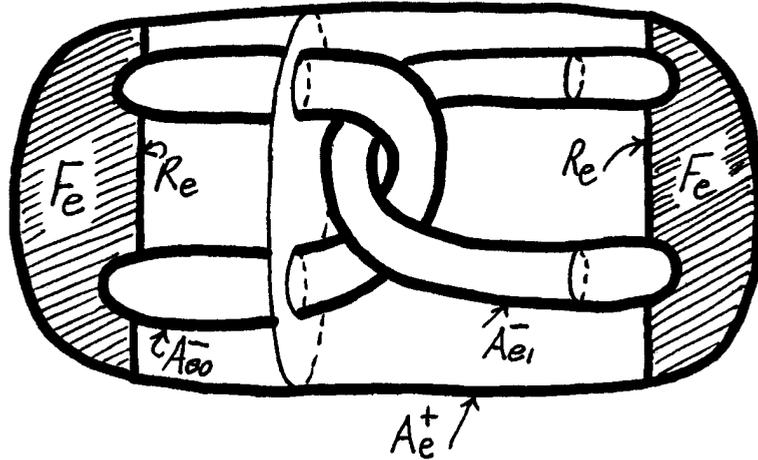


Figure 4

5. Proof of Theorem 1

For each word e in the alphabet $\{0, 1\}$ take a copy $p_e : X_e \rightarrow B_e$ of $p : X \rightarrow B$. Place X_{ei} inside \hat{X}_{ei} by identifying the boundary component \tilde{S}_{ei}^+ of X_{ei} with the boundary component \tilde{S}_{ei}^- of X_e in such a way that the arcs A_{ei}^+ in \tilde{S}_{ei}^+ are identified with the arcs A_{ei}^- in \tilde{S}_{ei}^- . Continue in this way up to infinity. Denote by M_j the union of the manifolds \hat{X}_e such that the length of the word e is j . Thus $M_0 = X$, $M_1 = \hat{X}_0 \cup \hat{X}_1$, $M_2 = \hat{X}_{00} \cup \hat{X}_{01} \cup \hat{X}_{10} \cup \hat{X}_{11}$. We obtain a manifold K_n such that $K_n = X_{-1} \cup_e X_e = S^3 - \cap_{j=1}^\infty M_j$. The coverings $p_e : X_e \rightarrow B_e$ match together to produce the n -fold cyclic covering $q_n : K_n \rightarrow S^3 - T$ branched over the tame subset $K - T$ of K and induced by the $2\pi/n$ rotation about the axis E .

We assume X_{ei} placed inside \hat{X}_{ei} in such a careful way that each connected component of the intersection $H_n = \cap_{j=0}^\infty M_j$ is a n -pod with the n -valent point lying in the axis E and such that its intersection with planes orthogonal to E is either empty or consists of just one point for each leg of the n -pod. The sequence $\{M_i\}_{i=0}^\infty$

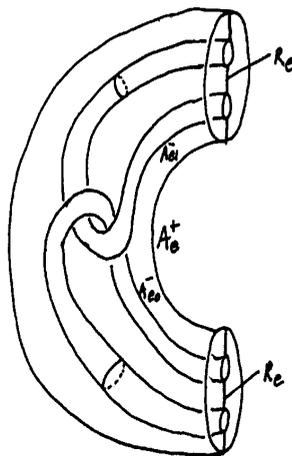


Figure 5: The standard sheet H

is a defining sequence of a monotone, upper semi-continuous decomposition G_n of S^3 whose non degenerate elements are the connected component of the intersection H_n . We have:

Lemma 7. *The decomposition space of G_n is S^3 for each $n \geq 2$, and the image of $H_n = \bigcap_{j=0}^{\infty} M_j$ in the decomposition space is a (wildly embedded) Antoine's Cantor set.*

We will defer the proof of this Lemma to the next section.

Corollary 8. *The Freudenthal compactification of K_n is S^3 for each $n \geq 2$, and the end space of K_n is a (wildly embedded) Antoine's Cantor set.*

Proof. If we denote by $f_n : S^3 \rightarrow S^3$ the decomposition map of G_n , then $f_n(H_n)$ is a Cantor set \tilde{T}_n in S^3 . Then $f_n | K_n : K_n \rightarrow f_n(K_n)$ is a homeomorphism and $S^3 - f_n(K_n) = \tilde{T}_n$ which is totally disconnected. By Theorem 4, S^3 is the Freudenthal compactification of K_n . ■

Since by Theorem 4 the Freudenthal compactification of $S^3 - T$ is S^3 , and by Corollary 8, the Freudenthal compactification of K_n is S^3 , then the n -fold cyclic covering $q_n : K_n \rightarrow S^3 - T$ branched over the tame subset $K - T$ of K and induced by the $2\pi/n$ rotation about the axis E , extends to an n -fold cyclic covering $\hat{q}_n : (\hat{K}_n = S^3) \rightarrow S^3$ branched over the knot K . The map \hat{q}_n is induced by a cyclic action by homeomorphisms in the topological space $(\hat{K}_n = S^3)$. This action is the projection, under f_n , of the $2\pi/n$ rotation about the axis E , because this rotation

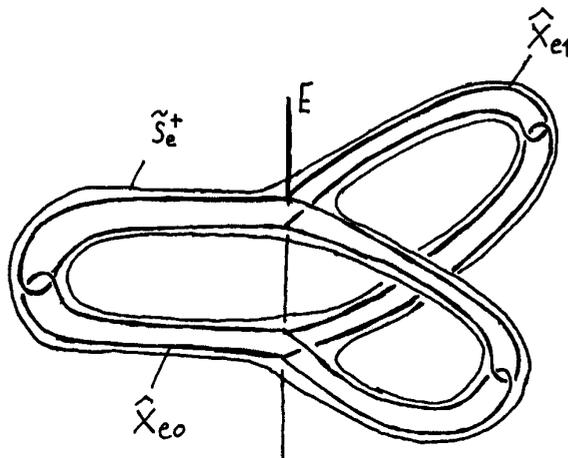


Figure 6: The case $n = 3$

preserves the elements of the decomposition G_n . The fixed point set of this action is then the set $f_n(E)$ which coincides with $\hat{q}_n^{-1}(K)$. Moreover $\hat{q}_n^{-1}(T) = \tilde{T}_n$. Consider the connected components of the preimage of the interior of the disk Δ . The closure of any of these connected components is a disk whose boundary is the wild knot $\hat{q}_n^{-1}(K)$. The wild subset of $\hat{q}_n^{-1}(K)$ is \tilde{T}_n , which, by Lemma 7, is an Antoine's Cantor set. Therefore the knot $\hat{q}_n^{-1}(K)$ is not trivial. Hence, the knot K is not trivial either.

Suppose Σ is a 2-sphere in S^3 cutting K transversally in two points. Then Σ cannot separate T , because it lifts to a sphere Σ_n in \hat{K}_n , and Σ_n cannot separate the Antoine's Cantor set \tilde{T}_n . It follows that the 3-ball C bounded by Σ and not containing T , lifts to a 3-ball C_n in \hat{K}_n . The induced n -fold branched covering $C_n \rightarrow C$, branched over a tame arc is then trivial. This shows that the knot K is prime. To complete the proof of Theorem 1 we have yet to prove Lemma 7 (section 6) and show that there are uncountably many inequivalent knots like K (section 7).

6. Proof of Lemma 7

The proof of Lemma 7, for $n \geq 2$, is *mutatis mutandis* the same as the one given in [4], Theorem 3.1. The case $n = 2$ is a variation of [3], section 3. However, after having worked out the details of the proof, I discovered the following fact. Consider the upper semi-continuous decomposition G_n^1 of S^3 described by Bing in [4], pages 85-86, and having a defining sequence whose first two stages are depicted in Figure 7 (which is the mirror image of Figure 2 in [4]). Call H_n^1 the union of the non degenerate

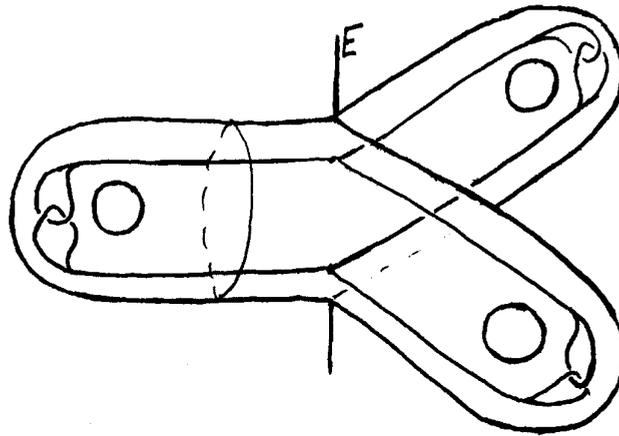


Figure 7

elements of G_n^1 . Then we have:

Theorem 9. *The open, 3-manifolds $S^3 - H_n$ and $S^3 - H_n^1$ are homeomorphic, for $n \geq 2$. Therefore their Freudenthal compactifications are homeomorphic. Thus, the decompositions G_n and G_n^1 of S^3 are equivalent.*

Remark 10. Then this common Freudenthal compactification, which is the decomposition space of G_n^1 , (or of G_n) is S^3 , by ([4], Theorem 3.1). As we said above, an analogous argument shows the same, directly, for the decomposition space of G_n . The proof of Theorem 9 shows that the involution defined in [12], coincides with the involution defined in ([4], Theorem 3.1). Moreover, the proof of Theorem 9 shows also that the cyclic actions, for $n \geq 2$, defined in this paper, and giving rise to the n -fold cyclic coverings of S^3 branched over K , coincide with the cyclic actions defined in ([4], Theorem 3.1).

Proof. The proof consists in constructing the branching set of the cyclic action in $S^3 - H_n^1$ and showing that this branching set is precisely our original $K - T$. See Figures 7, 8 and 9 and compare with sections 4 and 5. ■

The fact that $S^3 - \tilde{T}_n$ is not simply connected is contained in [4], page 89. (By computing $\pi_1(S^3 - \tilde{T}_n)$ directly, one can give an alternative proof (see[6]).) Moreover \tilde{T}_n is not separated by any 2-sphere, because \hat{X}_{e_0} and \hat{X}_{e_1} (Figure 6) are not separated by any 2-sphere. Thus \tilde{T}_n is an Antoine’s Cantor set.

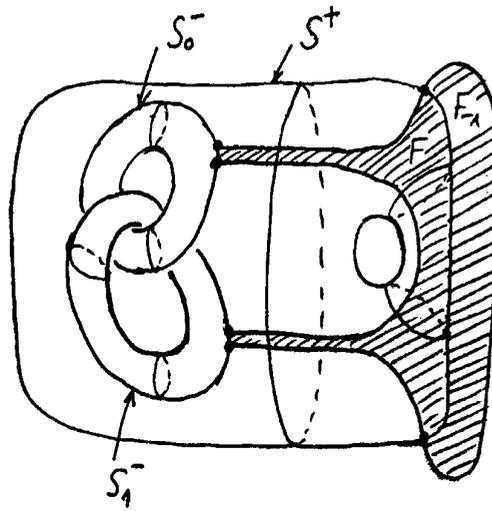


Figure 8

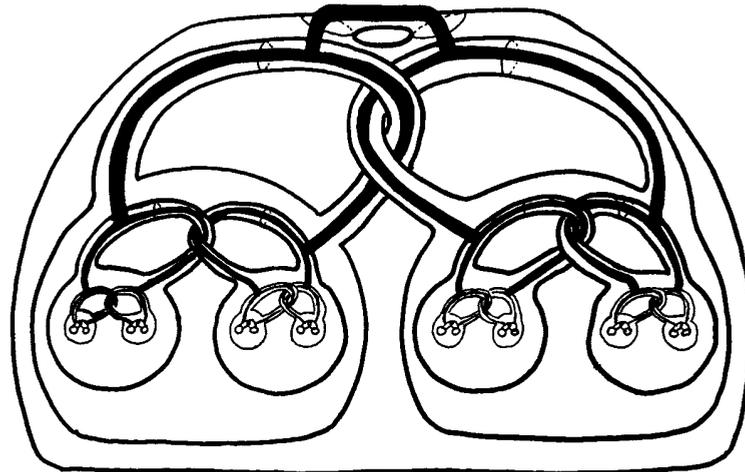


Figure 9

7. Uncountably many inequivalent knots

The disk Δ is by no means unique. An uncountable set of examples might be obtained by different procedures. Here is one (other examples, in section 9).

Consider the mirror image B^* of B . Construct Δ as above but substituting any number of B_e^* for B_e . In this way we obtain a set \mathbf{S} containing uncountably many disks like Δ , all having the same properties as Δ . It is clear that this construction amounts to substitute an arbitrary number of the left handed twists in the disk Δ of Figure 1, by right handed twists. None of the arguments in this paper suffers from this substitution, and therefore for this uncountable set of disks the theorems in this paper remain true. Only remains to see that in this set \mathbf{S} there are uncountably many different topological types. We now prove this.

Let Δ be a member of \mathbf{S} . Recall the construction of Δ in section 3. Here we are using B or B^* . The j -level of Δ will consists of 2^j twisted bands: $l_j(\Delta)$ twisted bands like the (-1) -band of Figure 2, and $r_j(\Delta)$ twisted bands like the mirror image of Figure 2, a $(+1)$ -band.

Theorem 11. *Let Δ, Δ^1 be two members of the set \mathbf{S} , such that $l_0(\Delta) = l_0(\Delta^1) = 1$, $r_0(\Delta) = r_0(\Delta^1) = 0$. Suppose that for some level $j \geq 1$, $l_j(\Delta) = r_j(\Delta^1) = 2^j$ and $r_j(\Delta) = l_j(\Delta^1) = 0$. Then the knots K, K^1 , bounding Δ, Δ^1 , are inequivalent, i.e. there is no homeomorphism f from S^3 onto S^3 taking K onto K^1 .*

Proof. On the contrary: assume such f exists. It induces a homeomorphism g from $S^3 - T$ onto $S^3 - T^1$ taking $K - T$ onto $K^1 - T^1$. Then g lifts to a homeomorphism \tilde{g} from the 2-fold covering K_2 of $S^3 - T$ branched over $K - T$ onto the 2-fold covering K_2^1 of $S^3 - T^1$ branched over $K^1 - T^1$. The manifold K_2 is $S^3 - \cap_{j=0}^\infty M_j$ where $\{M_j\}_{j=0}^\infty$ is a defining sequence such that each component of M_j is a solid torus. Call $\{N_j\}_{j=0}^\infty$ the corresponding defining sequence for K_2^1 . The decomposition spaces of the decompositions F and G , defined respectively by $\{M_j\}_{j=0}^\infty$ and $\{N_j\}_{j=0}^\infty$, are the Freudenthal compactifications of K_2 and K_2^1 . Since the homeomorphism \tilde{g} extends to these Freudenthal compactifications sending ends to ends, it follows that F and G are equivalent decompositions. On the other hand, it is obvious that the elements of F and G are cellular (see [7]). Then Theorem 5 grants the existence of a sequence $\{f_i\}_{i=0}^\infty$ of homeomorphisms from S^3 onto S^3 such that, for each $i \geq 0$, $f_{i+1} \mid (S^3 - IntM_i) = f_i \mid (S^3 - IntM_i)$, and $\{f_i(M_i)\}_{i=0}^\infty$ is a defining sequence for G .

Claim 12. *The sequence $\{f_i\}_{i=0}^\infty$ can be chosen to satisfy the additional condition $f_i(M_i) = N_i$ for each integer $i \geq 0$.*

Assume the claim is true. Then, since $S^3 - IntM_1$ and $S^3 - IntN_1$ are both, by the hypothesis $l_0(\Delta) = l_0(\Delta^1) = 1$, $r_0(\Delta) = r_0(\Delta^1) = 0$, the exterior of the link L of Figure 11, it follows that f_1 is an orientation-preserving homeomorphism from S^3 onto S^3 because L is not amphicheiral. Thus, all the homeomorphisms f_i are orientation-preserving. Let Y be a component of M_{j-1} . Then Y contains exactly two components Y_0, Y_1 of M_j . Let Z be $f_{j-1}(Y)$. This is a component of N_{j-1} , which contains exactly two components Z_0, Z_1 of N_j . The homeomorphism f_j , that agrees with f_{j-1} outside M_{j-1} must take $S^3 - Int(Y_0 \cup Y_1)$ onto $S^3 - Int(Z_0 \cup Z_1)$. However this is impossible because $S^3 - Int(Y_0 \cup Y_1)$ is, by the hypothesis $l_j(\Delta) = 2^j$, the

exterior of the link L , while $S^3 - Int(Z_0 \cup Z_1)$ is, by the hypothesis $r_j(\Delta^1) = 2^j$, the exterior of the link L^* where L^* denotes the mirror image of L . We have arrived to a contradiction and therefore the homeomorphism f does not exist. ■

Proof of Claim. We have two defining sequences $\{N_j\}_{j=0}^\infty$ and $\{f_i(M_i)\}_{i=0}^\infty$ for G . Call H the union of the non degenerate elements of G . Denote $\{f_i(M_i)\}_{i=0}^\infty$ by $\{M_i\}_{i=0}^\infty$. We want to prove that there exist a sequence $\{g_i\}_{i=0}^\infty$ of homeomorphisms from S^3 onto S^3 such that, for each $i \geq 0$, $g_{i+1} \mid (S^3 - IntM_i) = g_i \mid (S^3 - IntM_i)$, and $g_i(M_i) = N_i$. The proof follows the same lines as [13], Theorem 2 (see also Theorem 3).

We use the same argument of [13], 1197-1198 to assume that $(BdM_0) \cap (BdN_0) = \emptyset$. Since M_0 and N_0 both have exactly one component, it follows that M_0 and N_0 will be nested, since both contain H .

We now show that they are concentric. Suppose then that M_0 contains in its interior N_0 . Inside M_0 there are two components V_1 and V_2 of M_1 . As before, we may assume that they do not intersect BdN_0 . Since they cannot be in $M_0 - N_0$ (otherwise they do not intersect H), both they must be contained inside N_0 . Since $M_0 - Int(V_1 \cup V_2)$ is the exterior of the standard link of Figure 12 (or its mirror image), and *ad hoc* argument implies that BdN_0 is parallel to BdM_0 . We now complete the construction of g_0 by shoving M_0 onto N_0 .

Assume, by induction, that g_{i-1} exists. We may assume then that $M_{i-1} = N_{i-1}$. We need to construct a homeomorphism F of S^3 which is identity in $S^3 - IntM_{i-1}$ and such that $F(M_i) = N_i$. Working as above, inside each component of $M_{i-1} = N_{i-1}$, we may assume that the components of M_i and N_i have disjoint boundaries. Since all of them contain points of H in their interiors, and the number of components in M_i and in N_i is the same, they must be nested. And they must be concentric by the same argument used above. The claim is then proved. ■

Let $h : \mathbf{N} \rightarrow \{+1, -1\}$ be a function. Let $\Delta(h)$ be a members of the set \mathbf{S} , such that $l_0(\Delta) = l_0(\Delta^1) = 1$, $r_0(\Delta) = r_0(\Delta^1) = 0$ and such that all the bands in level $j \geq 1$, are $h(j)$ -bands.

Corollary 13. *The subset $\{\Delta(h) \in \mathbf{S} : \forall h : \mathbf{N} \rightarrow \{+1, -1\}\}$ is uncountable. Two arbitrary members of the subset have inequivalent boundaries. These boundaries satisfy the statement of Theorem 1.*

This completes the proof of Theorem 1.

8. Pasting together solid horned spheres

Consider the regular neighbourhood N_e of F_e in B_e (rel. R_e) shown in Figure 10. The union of the sets N_e , for all e , together with a regular neighbourhood N_{-1} of F_{-1} in B_{-1} (rel. R_{-1}), and together with T is a 3-ball C . The boundary of C is a 2-sphere in S^3 which is the celebrated Alexander horned sphere A (see [3], for instance; see also [5]). The equator K separates A in two copies of Δ , Δ^+ and Δ^- . The equator

K contains the Cantor set T which is the wild part of A . The set $(S^3 - C) \cup A$ is the solid horned sphere H of Bing [3], page 355, or *Alexander crumpled cube*. Since the manifold $\widehat{K}_n = S^3$ can be constructed pasting together n copies of H cyclically around the equator K , we have the following Corollary.

Corollary 14. *The result of pasting cyclically together, around the equator K , n copies of the Alexander crumpled cube H is S^3 , for $n \geq 2$.*

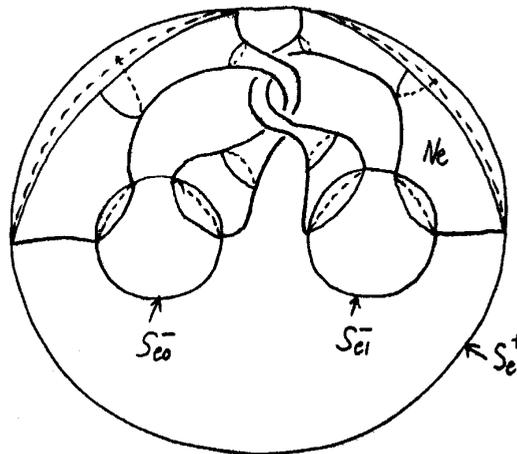


Figure 10

Remark 15. Bing showed in [3] that the double of H is S^3 . There has been a great deal of research on this topic.

9. Generalizations

An uncountable set of disks Δ all satisfying Theorem 1 has been constructed in section 7. There are different methods of constructing examples. Here we suggest some.

If instead of substituting left handed twists, as we have done in section 7, we put an arbitrary number of full twists (right of left handed), then none of the arguments in this paper suffers from this substitution, and therefore for this uncountable set of disks the theorems in this paper remain true. Only remains to see that in this set \mathbf{S} there are uncountably many different topological types.

If besides the above modifications we put an arbitrary number of half twists in an arbitrary number of the bands showed in Figure 1, we again have an uncountable set of disks for which the theorems in this paper still remain true. The addition of the twists

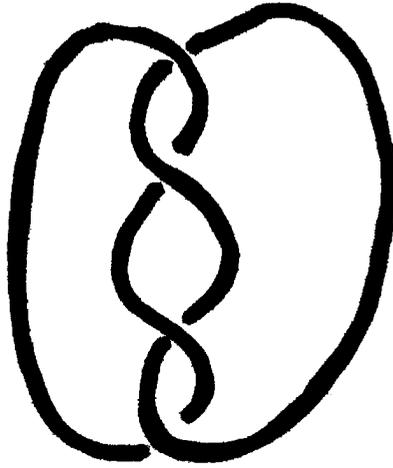


Figure 11

amounts to paste together the manifolds X_e using different sewing homeomorphisms. However the resulting open 3-manifold still embeds in S^3 and therefore the above arguments apply with trivial changes.

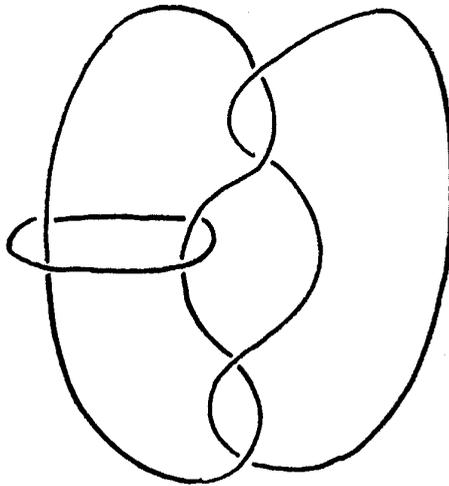


Figure 12

If instead of the strongly invertible link L of Figure 12, which is the basis for the constructions in this paper, we use other strongly invertible links of three or more trivial components we can realize *mutatis mutandis* the same construction as before. The real difficult point is to show that the Freudenthal compactifications of the open 3-manifolds, thus obtained, is S^3 (or just a closed 3-manifold). This is a relevant difficult question, that might be tackled using Bing methods, only if the open 3-manifold embeds in S^3 .

10. Some more results

Assume N is an arbitrary knot in S^3 . Consider the connected sum $N\#K$ of N and the wild knot K . Then the n -fold cyclic covering $(N\#K)_n$ of S^3 branched over $N\#K$ is homeomorphic to the n -fold cyclic covering N_n of S^3 branched over N . In fact, this covering is the connected sum $N_n\#K_n$, but K_n is homeomorphic to S^3 . We then have:

Theorem 16. *If M is a closed, oriented 3-manifold which is an n -fold cyclic branched covering of S^3 , then it is an n -fold cyclic covering of S^3 branched over a wild knot, in uncountably many different ways.*

Take the sphere S^3 as the boundary of the 4-ball D^4 . Push the interior of Δ slightly inside D^4 and call the resulting properly embedded disc $\widehat{\Delta}$. Then, we have:

Theorem 17. *(1) $(D^4, \widehat{\Delta})$ is not the standard pair of disks. (2) There are uncountably many such pairs of disks. (3) The n -fold cyclic coverings of D^4 branched over $\widehat{\Delta}$ is D^4 . (4) The double of $(D^4, \widehat{\Delta})$ is the standard pair (S^4, S^2) of spheres.*

Proof.

(1) $(D^4, \widehat{\Delta})$ is not the standard pair of disks because the boundary is not standard.

(2) A homeomorphism between two pairs would give a homeomorphism between their boundaries.

(3) The proof in [4], Theorem 3.1, can be modified to cover this case. The n -fold cyclic covering of D^4 branched over $\widehat{\Delta}$ is presented as a decomposition space. This decomposition of D^4 is similar to G_n , only that the handlebodies in the defining sequence of G_n are 4-dimensional. They enjoy the same mobility that allows Bing to obtain his result. We left the details to the reader.

(4) $(D^4, \widehat{\Delta})$ can be defined precisely as follows. Take a collar $S^3 \times [0, 1]$ in D^4 such that BdD^4 is identified with $S^3 \times \{0\}$. Then $\widehat{\Delta}$ is $Bd\Delta \times [0, 1] \cup \Delta \times \{1\}$. In the double of $(D^4, \widehat{\Delta})$ the 2-sphere $2\widehat{\Delta}$ is then the boundary of the double of $\Delta \times [0, 1]$ (double constructed along the part $\Delta \times \{0\}$ of the boundary of $\Delta \times [0, 1]$). Then $2\widehat{\Delta}$ is the boundary of a regular neighbourhood N of Δ in D^4 . Inside N take a locally finite set of mutually disjoint 4-balls containing in their interiors the bands of Δ . Inside these balls, and simultaneously, undo the bands by an isotopy which is the identity outside

the 4-balls. Now it is clear that the boundary of the so modified N is the standard 2-sphere. ■

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