Hypercyclic operators with an infinite dimensional closed subspace of periodic points

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Abstract

Let X be an infinite dimensional separable Banach space. There exists a hypercyclic operator on X which is equal to the identity operator on an infinite dimensional closed subspace of X.

1. Introduction and main results

In this note, we will be concerned with hypercyclic and chaotic linear operators on an infinite dimensional real or complex separable Banach space X. If T is a bounded operator on X, a vector x of X is said to be *hypercyclic* for T if the orbit of x under T, that is $Orb(x, T) = \{T^n x, n \geq 0\}$, is dense in X. If T has a hypercyclic vector, T is called a *hypercyclic operator*. There is a link between hypercyclicity and dynamics for linear operators on a Banach space. Recall that a continuous mapping ϕ of a metric space E is *chaotic* in the sense of Devaney ([8], p. 50) if it satisfies the following three conditions: it is topologically transitive (i.e. for every pair of non empty open subsets U, V of E, there exists an integer n such that $\phi^n(U) \cap V$ is non-void), it has a dense set of periodic points (a vector x of X is called *periodic* for ϕ when there exists a positive integer n such that $\phi^n x = x$), and it has sensitive dependence on initial conditions (there exists a positive real number δ such that for every x in X and every neighborhood V of X, there exists a vector y in V and a nonnegative integer

n such that $\phi^n x$ and $\phi^n y$ are more than the distance δ apart). It is shown in [3] that if ϕ is topologically transitive and has dense periodic points, then ϕ has sensitive dependence on initial conditions. In the special case where ϕ is a bounded operator on a separable Banach space X, it is shown in [9] that ϕ is topologically transitive if and only if it is hypercyclic. Thus ϕ is chaotic if and only if it is hypercyclic and has a dense set of periodic points.

A result of S. Ansari ([2]) and L. Bernal-González ([5]) states that every infinite dimensional separable Banach space supports a hypercyclic operator. It is interesting to notice here that this result cannot be extended to non-separable Banach spaces by replacing the concept of hypercyclicity by the concept of topological transitivity: a result of T. Bermudez and N.J. Kalton ([4]) implies that some non-separable spaces such as ℓ_∞ or $\mathcal{B}(\ell_2)$ do not support any topologically transitive operator. Another question which follows naturally from S. Ansari and L. Bernal-González result is to know whether every infinite dimensional separable Banach space supports a chaotic operator. The answer is no: J. Bonet, F. Martínez-Giménez and A. Peris proved in [6] that when X is a hereditarily indecomposable Banach space with hereditarily indecomposable dual, there exists no chaotic operator on X (see also [10], Section 5.3). This raises the following question regarding periodic points:

Question 1.1 [6] Does every infinite dimensional separable Banach space support a hypercyclic operator with an infinite dimensional closed subspace of periodic points?

The purpose of this note is to answer this question in the affirmative. We will prove the following stronger theorem:

Theorem 1.2. *Let X be an infinite dimensional separable Banach space. There exists a hypercyclic operator on X which is equal to the identity operator on an infinite dimensional closed subspace of X.*

This highlights the fact that a hypercyclic operator can be very far from having no non trivial invariant subspace. This theorem is also related to a result of [6], where it is proved that when X and its dual are hereditarily indecomposable and T is hypercyclic, there exists a positive integer N such that every periodic vector x satisfies $T^N x = x$. Since T is hypercyclic, T^N is hypercyclic too ([1]) and this implies that if T has an infinite dimensional closed subspace V of periodic points, then T^N is hypercyclic and equal to the identity operator on this same subspace V.

The main step in the proof of Theorem 1.2 is Theorem 1.3, in which we construct hypercyclic perturbations of the identity operator by some kind of shift operator with infinite dimensional kernel on one of the sequence spaces ℓ_p, $1 \le p < \infty$, or c_0. Theorem 1.3 is related to a result of H. Salas, who proved in [11] that when $(\omega_i)_{i \ge 0}$ is any bounded sequence of positive scalars and S is the backward shift defined on X by $Se_0 = 0$ and for every $i \ge 0$, $Se_{i+1} = \omega_i e_i$, then the operator $I + S$ is hypercyclic.

Theorem 1.3. *Let X be one of the sequence spaces ℓ_p, $1 \le p < \infty$, or c_0, with canonical basis $(e_i)_{i \ge 0}$. Let $(\omega_i)_{i \ge 0}$ be any bounded sequence of positive scalars and*

consider the operator T defined on X by the following relations: for every $i \geq 0$,

$$Te_{2i} = 0 \quad and \quad Te_{2i+1} = \omega_i e_i.$$

Then the operator $I+T$, where I denotes the identity operator on X, is hypercyclic on X, and it is equal to the identity operator on the closed space spanned by the vectors e_{2i}, $i \geq 0$.

Using the notion of quasi-extension of an operator on the space ℓ_1 to an operator on a general separable Banach space X introduced by S. Ansari in [2], we then deduce Theorem 1.2 from Theorem 1.3.

Remark 1.4. All the hypercyclic operators constructed in Theorem 1.3 satisfy the Hypercyclicity Criterion (see for instance [7]): since $(\omega_i)_{i \geq 0}$ is a sequence of positive scalars, this is a straightforward consequence of Corollary 2.8 of [7].

2. Proofs of Theorems 1.2 and 1.3

2.1. Proof of Theorem 1.3

We will prove Theorem 1.3 in the Hilbert space case when $p = 2$, the construction being the same in the general case.

Notation. $(e_i)_{i \geq 0}$ denotes the canonical basis of ℓ_2. A vector x of ℓ_2 is said to have finite support when it is a linear combination of finitely many basis vectors e_i, and if $x = \sum_{i \geq 0} x_i e_i$, $\operatorname{supp}(x) = \{i \geq 0; x_i \neq 0\}$ denotes the support of x.

Let $(z_k)_{k \geq 1}$ be a dense sequence of vectors of ℓ_2 with finite support such that for every $k \geq 1$, $\max \operatorname{supp}(z_k) \leq k$. Proceeding as in the proof of Theorem 3.3 in [11], we will construct inductively a fast increasing sequence $(n_k)_{k \geq 1}$ of integers and a sequence $(y_k)_{k \geq 1}$ of finitely supported vectors of ℓ_2 such that:

    (1) for every $k \geq 2$, $\max \operatorname{supp}(y_{k-1}) < \min \operatorname{supp}(y_k)$
    (2) for every $k \geq 1$, $\|y_k\| \leq 2^{-(k+1)}(1 + \|T\|)^{-(n_k+1)}$
    (3) for every $k \geq 1$, $\left\| (I+T)^{n_k} (\sum_{j=1}^{k} y_j) - z_k \right\| \leq 2^{-k}$.

Then an easy computation shows that the vector $y = \sum_{k=1}^{+\infty} y_k$ is a hypercyclic vector for the operator $I + T$.

A crucial lemma is the following ([11], Lemma 3.2):

Lemma 2.1. *Let n and r be positive integers. Consider the $2^r \times 2^r$ matrix C_n defined by $C_n = (c_{ij}(n)_{1 \leq i,j \leq 2^r})$ where $c_{ij}(n) = \binom{n}{2^r+j-i}$. Let $B_n = (b_i(n))_{1 \leq i \leq 2^r}$ be a column vector such that for every i, $1 \leq i \leq 2^r$, $b_i(n)$ is a polynomial in n of degree at most $2^r - i$. Then the equation $C_n X_n = B_n$ has a solution $X_n = (x_i(n))_{1 \leq i \leq 2^r}$ for n large enough which satisfies: there exists a constant P such that for every i, $1 \leq i \leq 2^r$, $|x_i(n)| \leq \frac{P}{n^i}$.*

Let us first construct y_1: z_1 can be written as $z_1 = z_0^{(1)} e_0 + z_1^{(1)} e_1$. We will be looking for a vector y_1 of the form $y_1 = x_1 e_3 + x_2 e_7$. Now:

$$||(I+T)^n(x_1 e_3 + x_2 e_7) - z_1|| = \left|\left|(x_1 + x_2 \, \omega_3 \, \tbinom{n}{1})e_3 + x_2 e_7\right|\right|$$

$$+\left|\left|\left(x_1 \, \omega_1 \omega_0 \, \tbinom{n}{2} + x_2 \, \omega_3 \, \omega_1 \, \omega_0 \, \tbinom{n}{3} - z_0^{(1)}\right) e_0 + \left(x_1 \, \omega_1 \, \tbinom{n}{1} + x_2 \, \omega_3 \, \omega_1 \, \tbinom{n}{2} - z_1^{(1)}\right) e_1\right|\right|.$$

Since

$$\begin{vmatrix} \omega_1 \, \omega_0 \, \tbinom{n}{2} & \omega -3 \, \omega_1 \, \omega_0 \, \tbinom{n}{3} \\ \omega_1 \, \tbinom{n}{1} & \omega_3 \, \omega_1 \, \tbinom{n}{2} \end{vmatrix} = \omega_3 \, \omega_1^2 \, \omega_0 \begin{vmatrix} \tbinom{n}{2} & \tbinom{n}{3} \\ \tbinom{n}{1} & \tbinom{n}{2} \end{vmatrix},$$

the system

$$\begin{pmatrix} \omega_1 \, \omega_0 \, \tbinom{n}{2} & \omega_3 \, \omega_1 \, \omega_0 \, \tbinom{n}{3} \\ \omega_1 \, \tbinom{n}{1} & \omega_3 \, \omega_1 \, \tbinom{n}{2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} z_0^{(1)} \\ z_1^{(1)} \end{pmatrix}$$

has a solution for n large enough with $|x_i(n)| \leq \frac{P}{n^i}$ for $i = 1, 2$. Then

$$||(I+T)^n(x_1(n)e_3 + x_2(n)e_7) - z_1 = ||(x_1(n) + x_2(n) \, \omega_3 \binom{n}{1})e_3 + x_2(n)e_7|| \underset{+\infty}{=} O(\frac{1}{n}).$$

If n_1 is big enough and $y_1 = x_1(n_1)e_3 + x_2(n_1)e_7$, then y_1 satisfies conditions (2) and (3) for $k = 1$.

Suppose now that $y_1, \ldots y_{k-1}, n_1, \ldots n_{k-1}$ have already been constructed, and let r be a positive integer such that $2^r - 1 \geq k$ and for every $i = 1 \ldots k - 1$, $\text{supp}(y_i) \subseteq \{0 \ldots 2^r - 1\}$.

Notation. For $l \geq 0$, the sequence $(u_n^{(2l)})_{n \geq 0}$ is the sequence defined by

$$u_0^{(2l)} = 2l \text{ and for } n \geq 0, \ u_{n+1}^{(2l)} = 2u_n^{(2l)} + 1.$$

Let I_{2l} be the set of cardinality 2^r containing the 2^r first terms of the sequence $(u_n^{(2l)})_{n \geq 0}$:

$$I_{2l} = \{u_0^{(2l)}, u_1^{(2l)}, \ldots, u_{2^r-1}^{(2l)}\}.$$

It is easy to see that the sets I_{2l} are disjoint. Denote by Π_{2l} the orthogonal projection onto the space spanned by the vectors $e_v, v \in I_{2l}$.

With this notation, $Te_{u_{n+1}^{(2l)}} = \omega_{u_n^{(2l)}} e_{u_n^{(2l)}}$ for every $n \geq 0$. Moreover, $\{u_j^{(2l)}\}$ is the support of the vector $T^{-j}e_{u_0^{(2l)}}$ for every $j \geq 1$, and

$$e_{u_j^{(2l)}} = \left(\prod_{i=0}^{j-1} \omega_{u_i^{(2l)}}\right) T^{-j}e_{u_0^{(2l)}}.$$

We will be looking for a vector y_k of the form $y_k = \sum_{l=0}^{s}(\sum_{i=1}^{2^r} x_i^{(2l)} e_{u_{2^r+i-1}^{(2l)}})$ where the coordinates $x_i^{(2l)}$ have to be determined. This vector y_k must satisfy $s+1$ systems of equations: for every $l = 0 \ldots s$ and for every $p = 0 \ldots 2^r - 1$,

$$\langle (I+T)^n \left(\sum_{l=0}^{s} \left(\sum_{i=1}^{2^r} x_i^{(2l)} e_{u_{2^r+i-1}^{(2l)}} \right) + \sum_{j=1}^{k-1} y_j \right) - z_k, e_{u_p^{(2l)}} \rangle = 0.$$

Since the sets $I_0, I_2, \ldots, I_{2s}$ do not overlap and since the vector space spanned by the vectors $e_v, v \in I_{2l}$, is invariant by T, the equations above are equivalent to the following $s+1$ independent systems:

$$(\mathcal{S}_{2l}) \begin{cases} \text{for every } p = 0 \ldots 2^r - 1, \\ \langle (I+T)^n \left(\sum_{i=1}^{2^r} x_i^{(2l)} e_{u_{2^r+i-1}^{(2l)}} \right), e_{u_p^{(2l)}} \rangle \\ \qquad = \langle \Pi_{2l} \left(z_k - (I+T)^n \sum_{j=1}^{k-1} y_j \right), e_{u_p^{(2l)}} \rangle \end{cases}$$

for $l = 0 \ldots s$.

Now

$$\langle (I+T)^n \left(\sum_{i=1}^{2^r} x_i^{(2l)} e_{u_{2^r+i-1}^{(2l)}} \right), e_{u_p^{(2l)}} \rangle = \langle \sum_{i=1}^{2^r} x_i^{(2l)} \sum_{j=0}^{n} \binom{n}{j} T^j e_{u_{2^r+i-1}^{(2l)}}, e_{u_p^{(2l)}} \rangle$$

$$= \langle \sum_{i=1}^{2^r} x_i^{(2l)} \sum_{j=0}^{n} \binom{n}{j} \left(\prod_{m=2^r+i-j-1}^{2^r+i-2} \omega_{u_m^{(2l)}} \right) e_{u_{2^r+i-j-1}^{(2l)}}, e_{u_p^{(2l)}} \rangle.$$

Thus the system $(\mathcal{S}_{2l})$ has the form $C_n^{(2l)} X_n^{(2l)} = B_n^{(2l)}$ with $X_n^{(2l)} = (x_p^{(2l)}(n))_{1 \leq p \leq 2^r}$,

$$B_n^{(2l)} = (b_p^{(2l)}(n))_{1 \leq p \leq 2^r} = (\langle \Pi_{2l}(z_k - (I+T)^n \sum_{j=1}^{k-1} y_j), e_{u_{p-1}^{(2l)}} \rangle)_{1 \leq p \leq 2^r}$$

and $C_n^{(2l)} = (c_{ij}^{(2l)}(n))_{1 \leq i,j \leq 2^r}$ where $c_{ij}^{(2l)}(n) = \left(\prod_{m=2^r+i-j-1}^{2^r+i-2} \omega_{u_m^{(2l)}} \right) \binom{n}{j}$. Moreover,

$$\det C_n^{(2l)} = \left(\prod_{q=0}^{2^r-1} \prod_{m=q}^{2^r-1} \omega_{u_m^{(2l)}} \right) \left(\prod_{q=0}^{2^r-1} \prod_{m=0}^{q} \omega_{u_m^{(2l)}} \right) \det C_n,$$

where C_n is the matrix of size $2^r \times 2^r$ considered in Lemma 2.1. Thus in order to apply Lemma 2.1, we only have to check that for $p = 1 \ldots 2^r$, $b_p^{(2l)}(n)$ is a polynomial in n of degree at most $2^r - p$. The projection onto I_{2l} of the vector $\sum_{j=1}^{k-1} y_j$ can be written as $\Pi_{2l} \left(\sum_{j=1}^{k-1} y_j \right) = \sum_{i=0}^{2^r-1} \alpha_i^{(2l)} e_{u_i^{(2l)}}$, and since $\Pi_{2l}(I+T)^n \sum_{j=1}^{k-1} y_j = (I+T)^n \Pi_{2l} \sum_{j=1}^{k-1} y_j$, this yields that

$$\langle \Pi_{2l}(I+T)^n \sum_{j=1}^{k-1} y_j, e_{u_{p-1}^{(2l)}} \rangle = \sum_{i=0}^{2^r-1} \alpha_i^{(2l)} \sum_{q=0}^{n} \binom{n}{q} \langle T^q e_{u_i^{(2l)}}, e_{u_{p-1}^{(2l)}} \rangle.$$

Since $\langle T^q e_{u_i^{(2l)}}, e_{u_{p-1}^{(2l)}} \rangle$ is non-zero if and only if $q = i - p + 1$, the expression above can be reduced to

$$b_p^{(2l)}(n) = \langle z_k, e_{u_{p-1}^{(2l)}} \rangle - \sum_{i=0}^{2^r-1} \alpha_i^{(2l)} \binom{n}{i-p+1} \langle T^{i-p+1} e_{u_i^{(2l)}}, e_{u_{p-1}^{(2l)}} \rangle.$$

This shows that $b_p^{(2l)}(n)$ is a polynomial in n of degree at most $2^r - p$. We can therefore apply Lemma 2.1 and obtain a solution $(x_i^{(2l)}(n))_{1 \le i \le 2^r}$ of the system $(\mathcal{S}_{2l})$ with

$$|x_i^{(2l)}(n)| \le \frac{P^{(2l)}}{n^i}$$

for n large enough, where $P^{(2l)}$ is some constant depending only on l. Setting

$$y_k(n) = \sum_{l=0}^{s} \left(\sum_{i=1}^{2^r} x_i^{(2l)}(n) e_{u_{2^r+i-1}^{(2l)}} \right)$$

we have

$$(I+T)^n \left(\sum_{j=1}^{k-1} y_j + y_k(n) \right) - z_k = \sum_{q=0}^{+\infty} \langle (I+T)^n \left(\sum_{j=1}^{k-1} y_j + y_k(n) \right) - z_k, e_q \rangle e_q$$

$$= \sum_{q \notin I_0 \cup I_2 \cup ... \cup I_{2s}} \langle (I+T)^n \left(\sum_{j=1}^{k-1} y_j + y_k(n) \right) - z_k, e_q \rangle e_q.$$

Now $\sum_{j=1}^{k-1} y_j$ and z_k belong to the space spanned by the vectors $e_v, v \in I_0 \cup I_2 \cup ... \cup I_{2s}$, which is invariant by $(I+T)^n$, and thus the last equality becomes:

$$(I+T)^n \left(\sum_{j=1}^{k-1} y_j + y_k(n) \right) - z_k = \sum_{q \notin I_0 \cup I_2 \cup ... \cup I_{2s}} \langle (I+T)^n (y_k(n)), e_q \rangle e_q$$

$$= \sum_{l=0}^{s} \sum_{i=1}^{2^r} x_i^{(2l)}(n) \sum_{q \notin I_0 \cup I_2 \cup ... \cup I_{2s}} \sum_{p=0}^{n} \binom{n}{p} \langle T^p e_{u_{2^r+i-1}^{(2l)}}, e_q \rangle e_q.$$

Since q does not belong to I_{2l}, $\langle T^p e_{u_{2^r+i-1}^{(2l)}}, e_q \rangle$ is zero as soon as $2^r + i - 1 - p < 2^r$, i.e. $i - 1 < p$, and we obtain:

$$(I+T)^n \left(\sum_{j=1}^{k-1} y_j + y_k(n) \right) - z_k = \sum_{l=0}^{s} \sum_{i=1}^{2^r} x_i^{(2l)}(n) \sum_{p=0}^{i-1} \binom{n}{p} \sum_{q \notin I_0 \cup I_2 \cup ... \cup I_{2s}} \langle T^p e_{u_{2^r+i-1}^{2l}}, e_q \rangle e_q$$

and

$$\left\| (I+T)^n \left(\sum_{j=1}^{k-1} y_j + y_k(n) \right) - z_k \right\| \underset{+\infty}{=} O\left(\sum_{i=1}^{2^r} |x_i^{(2l)}(n)| \sum_{j=0}^{i-1} \binom{n}{j} \right) \underset{+\infty}{=} O(\frac{1}{n}).$$

Taking n_k large enough and $y_k = y_k(n_k)$ yields (1), (2) and (3). This terminates the proof of Theorem 1.3.

2.2. Proof of Theorem 1.2

This is a mere rewriting of the arguments of [2]. If $(x_i, x_i^*)_{i \geq 0}$ is a bounded biorthogonal system of the Banach space X and if $\omega_i, i \geq 0$, are positive numbers such that $\sum_{i \geq 0} \left\| x_{2i+1}^* \right\| \omega_i \|x_i\| < +\infty$, the operator

$$\widetilde{T}(x) = \sum_{i \geq 0} x_{2i+1}^*(x)\omega_i x_i$$

is a nuclear operator on X. If T is defined on the space ℓ_1 with canonical basis $(e_i)_{i \geq 0}$ by $Te_{2i} = 0$ and $Te_{2i+1} = \omega_i e_i$ for $i \geq 0$, then $I + \widetilde{T}$ is a quasi-extension of $I + T$ in the sense of [2]. Since $I + T$ is hypercyclic on ℓ_1, $I + \widetilde{T}$ is hypercyclic on X, and $I + \widetilde{T}$ is equal to the identity on the infinite dimensional closed subspace spanned by the vectors $x_{2i}, i \geq 0$. Theorem 1.2 is proved.

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