On some nonlinear elliptic systems with coercive perturbations in $\mathbb{R}^N$

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ABSTRACT

A nonlinear elliptic system involving the p-Laplacian is considered in the whole $\mathbb{R}^N$. Existence of nontrivial solutions is obtained by applying critical point theory; also a regularity result is established.

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1. Introduction

In this paper, we give some existence and regularity results concerning the following class of nonlinear elliptic systems in $\mathbb{R}^N$, $N \geq 2$

$$(S)\begin{cases} -\Delta_p u + a(x) \mid u \mid^{p-2} u = f(x) \mid u \mid^{\alpha-1} u \mid v \mid^{\beta+1} & \text{in } \mathbb{R}^N \\ -\Delta_q v + b(x) \mid v \mid^{q-2} v = f(x) \mid u \mid^{\alpha+1} \mid v \mid^{\beta-1} v & \text{in } \mathbb{R}^N \\ \lim_{|x|\to\infty} u(x) = \lim_{|x|\to\infty} v(x) = 0 \end{cases}$$

where $1 < p < N$, $1 < q < N$ and α, β are real constants satisfying

$$(1.1) \quad 0 < \alpha \leq p-1,\ 0 < \beta \leq q-1 \text{ and } \max\left(\tfrac{N-p}{N}, \tfrac{N-q}{N}\right) < \tfrac{\alpha+1}{p^*} + \tfrac{\beta+1}{q^*} < 1.$$

f and the perturbations a and b are measurable functions satisfying some conditions which ensure the existence and regularity of solutions.

Let us mention that many results in the scalar case have appeared for this kind of problems involving Laplacian and p-Laplacian operators in unbounded domains. Costa in [3], among others, obtained a nontrivial solution to the problem $-\Delta u+a(x)u = f(x,u)$ in $\mathbb{R}^N$, under the assumption that a is coercive and that the potential $F(x,u) = \int_0^u f(x,s)\,ds$ is nonquadratic at infinity. In [4], this result was generalized by the same author to a class of elliptic systems with respect to the potential and subcritical growth.
Other problems have been considered in this direction, see e.g. P. Drabek [5] and Lao Sen Yu in [10].
Concerning systems of the type (S), several studies have been devoted to investigation recently both in bounded and unbounded domains-we refer to [2, 4, 6, 7, 13, 15, 17]. The case of unbounded domains becomes more complicate, generally the main difficulty lies in the loss of Sobolev compact imbedding.
In this paper, by establishing sufficient condition, we give an extension and we complement some results of the scalar case to a system of elliptic equations. In particular, to treat variationally this class of problems, we assume a lower regularity condition on the function f (not necessary in L^∞, see assumption (H_2) below), so that a nontrivial solution can be obtained via Mountain Pass Theorem and local minimization of energy functional associated to our problem respectively in connection with cases $\frac{\alpha+1}{p} + \frac{\beta+1}{q} > 1$ and $\frac{\alpha+1}{p} + \frac{\beta+1}{q} < 1$.
On the other hand, to overcome the lack of compactness that has arisen from the critical exponent and the unboundedness of the domain, we use a compact imbedding result essentially given by the coerciveness of the perturbations a and b (see assumption (H_1)).
In the second section, we establish a regularity result, more precisely, we prove that such solutions (u,v) belong to $L^{p_1} \times L^{q_1}$ for any $p_1 \in [p^*, \infty[$ and $q_1 \in [q^*, \infty[$. In general, this regularity cannot reach the space $\text{Ł}^\infty \times \text{Ł}^\infty$ taking into account our argument developed here.

Through this paper, we use standard notation:
$W^{1,p} := W^{1,p}(\mathbb{R}^N)$ is the ordinary Sobolev space, $L^p = L^p(\mathbb{R}^N)$ is the Lebesgue space equipped with the norm $||\ .\ ||_p$ and $p^* = \frac{Np}{N-p}$ is the critical Sobolev exponent, the Lebesgue integral in $\mathbb{R}^N$ will be denoted by the symbol $\int$ whenever the integration is carried out over all $\mathbb{R}^N$; $\mathcal{C}_0^\infty := \mathcal{C}_0^\infty(\mathbb{R}^N)$ is the space of all functions with compact support in $\mathbb{R}^N$ with continuous derivatives of arbitrary order.

Let us formulate the assumptions on the perturbations a and b and on the function $f = f(x)$.
(H_1) $a, b : \mathbb{R}^N \to \mathbb{R}$ are continuous functions satisfying

$$a(x) \geq a_0 > 0,\ b(x) \geq b_0 > 0,\ \text{i.e. in } \mathbb{R}^N,$$

which are coercive, that is

$$\lim_{|x|\to+\infty} a(x) = \lim_{|x|\to+\infty} b(x) = +\infty,$$

(H_2) the function $f : \mathbb{R}^N \to \mathbb{R}$ is a nonnegative and

$$f \in L^{\omega} \cap L^{\frac{\omega}{1-\delta}}, \text{ with } \omega = \frac{p^* q^*}{p^* q^* - (\alpha+1)q^* - (\beta+1)p^*}.$$

Here $0 < \delta < 1$ is a small positive real. We introduce the following function space

$$E = \{(u,v) \in W^{1,p} \times W^{1,q} / \int (|\nabla u|^p + a(x)|u|^p + |\nabla v|^q + b(x)|v|^q)\, dx < \infty\}$$

endowed with the norm

$$\begin{aligned} \|(u,v)\|_E &= \left(\int |\nabla u|^p + a(x)|u|^p\ dx\right)^{\frac{1}{p}} + \left(\int |\nabla v|^q + b(x)|v|^q\ dx\right)^{\frac{1}{q}} \\ &= \|u\|_1 + \|v\|_2 \end{aligned}$$

where $\|u\|_1 = (\int |\nabla u|^p + a(x)|u|^p\ dx)^{1/p}$ and $\|v\|_2 = (\int |\nabla v|^q + b(x)|v|^q\ dx)^{1/q}$.

Since $a(x) \geq a_0 > 0$ and $b(x) \geq b_0 > 0$, we clearly see that the Banach space E is continuously embedded in $W^{1,p} \times W^{1,q}$. We also conclude from Sobolev's Theorem the continuous imbedding $E \hookrightarrow L^{p_1} \times L^{q_1}$, for all $p \leq p_1 \leq p^*$ and $q \leq q_1 \leq q^*$.

Definition 1.1. *We say that a pair $(u,v) \in E$ is a weak solution of (S) if*

$$(SV) \begin{cases} \int |\nabla u|^{p-2} \nabla u \nabla \varphi\, dx + \int a(x)|u|^{p-2} u\varphi\, dx = \int f(x)|u|^{\alpha-1} u\varphi\, |v|^{\beta+1}\, dx \\ \int |\nabla v|^{q-2} \nabla v \nabla \psi\, dx + \int b(x)|v|^{q-2} v\psi\, dx = \int f(x)|u|^{\alpha+1}|v|^{\beta-1} v\psi\, dx \end{cases}$$

holds for all $(\varphi,\psi) \in E$.

Let us remark that the assumptions (H_1) and (H_2) guarantee that integrals given in (SV) are well defined.

Now, we state the main results of this paper.

Theorem 1.1. *Suppose $1 < p, q < N$, (1.1), (H_1) and (H_2) are satisfied. Then the system (S) has at least one nontrivial weak solution $(u,v) \in E$.*

Theorem 1.2. *Let (u,v) be a solution of (S). Then $(u,v) \in L^{\sigma_1} \times L^{\sigma_2}$, with $p^* \leq \sigma_1 < \infty$ and $q^* \leq \sigma_2 < \infty$. Moreover $u, v > 0$ in $\mathbb{R}^N$ and $\lim_{|x|\to\infty} u(x) = \lim_{|x|\to\infty} v(x) = 0$.*

2. Preliminaries

Let us consider the functional $I : E \to \mathbb{R}$ given by

$$I(u,v) = \frac{\alpha+1}{p}\int \mid \nabla u \mid^p + a(x) \mid u \mid^p \; dx + \frac{\beta+1}{q}\int \mid \nabla v \mid^q + b(x) \mid v \mid^q \; dx$$
$$- \int f(x) \mid u \mid^{\alpha+1} \mid v \mid^{\beta+1} \; dx.$$

By assumption (H_2) and Sobolev's inequality, we can see that the functional K defined by

$$K(u,v) = \int f(x) \mid u \mid^{\alpha+1} \mid v \mid^{\beta+1} \; dx$$

is indeed well defined and of class $\mathcal{C}^1$ on the space E with

$$< K'(u,v); (\varphi,\psi) >= (\alpha+1)\int f(x) \mid u \mid^{\alpha-1} u\varphi \mid v \mid^{\beta+1} \; dx$$
$$+(\beta+1)\int f(x) \mid u \mid^{\alpha+1} \mid v \mid^{\beta-1} v\psi \, dx,$$

for all (u,v) and $(\varphi,\psi) \in E$; where $<;>$ denotes the duality symbol from E to E^*.

Therefore, a weak solution of a system (S) is a critical point (u,v) of I, i.e

$$I'(u,v)(\varphi,\psi) = 0 \qquad \forall (\varphi,\psi) \in E.$$

In order to prove our main result, we will use the following basic properties.

Lemma 2.1. *There exists constant $C > 0$ such that*

$$f(x) \mid u \mid^{\alpha+1} \mid v \mid^{\beta+1} \leq C(f(x)^{\frac{\omega}{1-\delta}} + \mid u \mid^{m_1} + \mid v \mid^{m_2})$$

with $m_1 \in]1, p^*[$ and $m_2 \in]1, q^*[$.

Proof. Since $\alpha + 1 < p^*$ and $\beta + 1 < q^*$, there exists $0 < \delta < 1$ small enough such that

$$p^* > \bar{p} \text{ and } q^* > \bar{q},$$

where

$$\bar{p} = \alpha + 1 + \frac{\delta}{\omega(\frac{1}{p^*} + \frac{1}{q^*})}, \; \bar{q} = \beta + 1 + \frac{\delta}{\omega(\frac{1}{p^*} + \frac{1}{q^*})}.$$

Then the lemma follows from Young's inequality with

$$m_1 = \frac{p^*}{\bar{p}} \text{ and } m_2 = \frac{q^*}{\bar{q}}.$$

Lemma 2.2. *Suppose* (H_1) *and* (H_2), *then*
i) E *is compactly embedded in* $L^p \times L^q$.
ii) K' *is a compact map from* E *to* E^*.

Proof. *i*) Without loss of generality, we will show that $(u_n, v_n) \to (0,0)$ strongly in E for such sequence $(u_n, v_n) \in E$ which converges weakly to $(0,0)$.
Indeed, we have $\| (u_n, v_n) \|_E \leq C$ for some constant $C > 0$. From (H_1), for a given $\varepsilon > 0$ and $R > 0$ such that

$$a(x) \geq 4\frac{C^p}{\varepsilon} \text{ and } b(x) \geq 4\frac{C^q}{\varepsilon} \text{ for all } \mid x \mid \geq R,$$

we have

$$(u_n, v_n) \rightharpoonup (0,0) \text{ weakly in } W^{1,p}(B_R) \times W^{1,q}(B_R),$$

where B_R is the Ball of radius R centered at origin. By using the compact imbedding $W^{1,p}(B_R) \times W^{1,q}(B_R) \hookrightarrow L^p(B_R) \times L^q(B_R)$, we get

$$\int_{B_R} (\mid u_n \mid^p + \mid v_n \mid^q)\, dx \leq \frac{\varepsilon}{2} \quad \forall n \geq n_0 \tag{2.1}$$

for some $n_0 \in \mathbb{N}$. We also have

$$\frac{4}{\varepsilon}\int_{\mathbb{R}^N - B_R} (\mid u_n \mid^p + \mid v_n \mid^q)\, dx \leq \int_{\mathbb{R}^N - B_R} (\frac{a(x)}{C^p} \mid u_n \mid^p + \frac{b(x)}{C^q} \mid v_n \mid^q)\, dx \leq 2 \tag{2.2}$$

since $\| (u_n, v_n) \|_E^p \geq \int_{\mathbb{R}^N - B_R} \mid u_n \mid^p \; dx$ and $\| (u_n, v_n) \|_E^q \geq \int_{\mathbb{R}^N - B_R} \mid v_n \mid^q \; dx$.
Combining (2.1) and (2.2), we obtain that

$$\int_{\mathbb{R}^N} (\mid u_n \mid^p + \mid v_n \mid^q)\, dx \leq \varepsilon, \;\; \forall n \geq n_0.$$

ii) The compactness of K' follows from the estimate

$$< K'(u_n, v_n) - K'(u_0, v_0); (\varphi, \psi) >= I_1 - I_2,$$

where

$$I_1 = \int f(x)(\mid u_n \mid^{\alpha-1} u_n \mid v_n \mid^{\beta+1} - \mid u_0 \mid^{\alpha-1} u_0 \mid v_0 \mid^{\beta+1})\varphi\, dx$$

and

$$I_2 = \int f(x)(\mid u_n \mid^{\alpha+1} + \mid v_n \mid^{\beta-1} v_n - \mid u_0 \mid^{\alpha+1} \mid v_0 \mid^{\beta-1} v_0)\psi\, dx.$$

The objective is to prove that $I_1 \to 0$ and $I_2 \to 0$. On one hand, we have $I_1 \leq I_{11} + I_{12}$ where

$$I_{11} = \int f(x)(\mid u_n \mid^{\alpha-1} u_n - \mid u_0 \mid^{\alpha-1} u_0)(\mid v_n \mid^{\beta+1} \varphi)\, dx$$

and
$$I_{12} = \int f(x) \mid u_0 \mid^{\alpha-1} u_0(\mid v_n \mid^{\beta+1} - \mid v_0 \mid^{\beta+1})\varphi\, dx.$$
By choosing δ sufficiently small such that $x = \frac{1}{\frac{\alpha}{p^*}+\frac{\delta}{\omega}} > 1$ and $p \leq \alpha x < p^*$, we obtain in view of (H_2) the following estimate
$$I_{11} \leq || f ||_{L^{\frac{\omega}{1-\delta}}} || \mid u_n \mid^{\alpha-1} u_n - \mid u_0 \mid^{\alpha-1} u_0 ||_{L^x} || v_n ||^{\beta+1}_{L^{\frac{q^*}{\beta+1}}} || \varphi ||_{L^{p^*}} .$$
On the other hand, since the imbedding $E \hookrightarrow L^p \times L^q$ is compact, it follows from the interpolation inequality i.e.
$$\mid u \mid_{L^t} \leq \mid u \mid^{\sigma}_{L^p} \mid u \mid^{1-\sigma}_{L^{p^*}}, \ \ \forall u \in L^p \cap L^{p^*},$$
where $\frac{1}{t} = \frac{\sigma}{p} + \frac{1-\sigma}{p^*}$, that the imbedding $E \hookrightarrow L^{p_1} \times L^{q_1}$ is compact for $p \leq p_1 < p^*$ and $q \leq q_1 < q^*$. Hence, we get $I_{11} \to 0$ (strongly) as n goes to infinity, since $p \leq \alpha x < p^*$. Now we estimate I_{12} :
$$\begin{aligned} I_{12} &= \int f(x) \mid u_0 \mid^{\alpha-1} u_0(\mid v_n \mid^{\beta+1} - \mid v_0 \mid^{\beta+1})\varphi\, dx \\ &\leq || f ||_{L^{\frac{\omega}{1-\delta}}} || u_0 ||^{\alpha}_{L^{\frac{p^*}{\alpha}}} || \mid v_n \mid^{\beta+1} - \mid v_0 \mid^{\beta+1} ||_{L^y} || \varphi ||_{L^{p^*}} \end{aligned}$$
where $y = \frac{1}{\frac{\beta+1}{q^*}+\frac{\delta}{\omega}} > 1$. A simple calculation shows that $q \leq (\beta+1)y \leq q^*$ for δ small enough. Consequently, we conclude that $I_{12} \to 0$ (strongly) as $n \to \infty$.
Finally, using the same argument to estimate I_2, we get $I_2 \to 0$ (strongly) as $n \to \infty$. Therefore
$$K'(u_n, v_n) \to K'(u_0, v_0) \text{ strongly in } E^*.$$
as n tends to infinity.
This ends the proof of Lemma 2.2.

Recall that $(u_n, v_n) \in E$ is a Palais-Smale sequence if there exists $M > 0$ such that,
$I(u_n, v_n) \leq M$ and $I'(u_n, v_n) \to 0$ strongly in E^* as n goes to infinity.

Remark 2.1. 1) Let us remark that an optimal value of the generic constant δ is taken here. This allows us also to consider more lower regularity condition on the function f.
2) Note that the assumption (H_1) gives a compact imbedding result which is used only to prove that the Palais Smale sequence obtained by Mountain Pass type argument converges to a weak nontrivial solution.

Lemma 2.3. *Suppose* $\frac{\alpha+1}{p} + \frac{\beta+1}{q} > 1$, *let* (u_n, v_n) *be a Palais-Smale sequence. Then* (u_n, v_n) *possesses a subsequence which converges strongly in* E.

Proof. Let $(u_n, v_n) \in E$ be a Palais-Smale sequence. We have

$I(u_n, v_n) - < I'(u_n, v_n); (\frac{u_n}{p}, \frac{v_n}{q}) >= (-1 + (\frac{\alpha+1}{p} + \frac{\beta+1}{q})) \int f(x) \mid u_n \mid^{\alpha+1} \mid v_n \mid^{\beta+1} \, dx,$

since

$$\begin{aligned} < I'(u_n, v_n); (\frac{u_n}{p}, \frac{v_n}{q}) > &= \frac{\alpha+1}{p} \int \mid \nabla u_n \mid^p + a(x) \mid u_n \mid^p \, dx \\ &+ \frac{\beta+1}{q} \int \mid \nabla v_n \mid^q + b(x) \mid v_n \mid^q \, dx \\ &- (\frac{\alpha+1}{p} + \frac{\beta+1}{q}) \int f(x) \mid u_n \mid^{\alpha+1} \mid v_n \mid^{\beta+1} \, dx. \end{aligned}$$

Hence, we come to the conclusion that

$$\begin{aligned} (1 - \frac{1}{\frac{\alpha+1}{p} + \frac{\beta+1}{q}})(\frac{\alpha+1}{p} \int \mid \nabla u_n \mid^p + a(x) \mid u_n \mid^p \, dx \\ + \frac{\beta+1}{q} \int \mid \nabla v_n \mid^q + b(x) \mid v_n \mid^q \, dx) \\ \leq M - < I'(u_n, v_n); (\frac{u_n}{p}, \frac{v_n}{q}) > . \end{aligned}$$

From this inequality, we easily find that (u_n, v_n) is a bounded sequence in E, Since $\frac{\alpha+1}{p} + \frac{\beta+1}{q} > 1$. Consequently, there exists a subsequence still denoted by (u_n, v_n) such that (u_n, v_n) converges weakly in E.
Now, we claim that (u_n, v_n) converges strongly in E. Indeed, for any pair integer (i, j), we have

$$\begin{aligned} \int (\mid \nabla u_i \mid^{p-2} \nabla u_i &- \mid \nabla u_j \mid^{p-2} \nabla u_j)(\nabla u_i - \nabla u_j) \\ &+ (a(x) \mid u_i \mid^{p-2} u_i - a(x) \mid u_j \mid^{p-2} u_j)(u_i - u_j) \\ \\ &= I'(u_i, v_i) - < I'(u_j, v_j); (u_i - u_j, 0) > \\ &+ \int f(x)(\mid u_i \mid^{\alpha-1} u_i \mid v_i \mid^{\beta+1} \\ &- \mid u_j \mid^{\alpha-1} u_j \mid v_j \mid^{\beta+1})(u_i - u_j) \, dx. \end{aligned}$$

By Palais-Smale condition, it is easy to see that

$$I'(u_i, v_i) - < I'(u_j, v_j); (u_i - u_j, 0) > \to 0$$

as i and j tend to infinity.
From the Lemma 2.2 (K' is compact), we have

$$\int f(x)(|\, u_i\,|^{\alpha-1} u_i \,|\, v_i\,|^{\beta+1} - |\, u_j\,|^{\alpha-1} u_j \,|\, v_j\,|^{\beta+1})(u_i - u_j)\, dx \to 0.$$

as i and j tend to infinity. From the following algebraic relation

$$|\, \xi_1 - \xi_2\,|^r \leq ((|\, \xi_1\,|^{r-2} \xi_1 - |\, \xi_2\,|^{r-2} \xi_2)(\xi_1 - \xi_2))^{s/2}(|\, \xi_1\,|^r + |\, \xi_2\,|^r)^{1-s/2}$$

with $s = r$, for $1 < r \leq 2$ and $s = 2$ for $2 < r$, we deduce that (u_n) is a Cauchy sequence in E, therefore it converges strongly. By the same argument, we show also that (v_n) converges strongly.
This concludes the proof of Lemma 2.3.

3. Proof of the main results

In this section, we give the proof of the existence results, we apply Mountain Pass Lemma and local minimization to find nontrivial solution. For that reason, we will separately distinguish two cases related to our study: $\boldsymbol{\frac{\alpha+1}{p} + \frac{\beta+1}{q} > 1}$ and $\boldsymbol{\frac{\alpha+1}{p} + \frac{\beta+1}{q} < 1}$. After that, we use an iterative method to prove the regularity result.

Lemma 3.1. *Suppose* $(H_1), (H_2)$ *and* $\frac{\alpha+1}{p} + \frac{\beta+1}{q} > 1$, *then*
1) There exist γ, ρ, *such that* $I(u,v) \geq \gamma$, *for* $\|\, (u,v)\,\|_E = \rho$.
2) $I(t(u,v)) \to -\infty$ *as* $t \to +\infty$.

Proof. *1*) From lemma 2.1, we have

$$I(u,v) \geq \frac{\alpha+1}{p} \|\, u\,\|_1^p + \frac{\beta+1}{q} \|\, v\,\|_2^q - C_0(\|\, u\,\|_{m_1}^{m_1} + \|\, v\,\|_{m_2}^{m_2}),$$

with $p \leq m_1 < p^*$ and $q \leq m_2 < q^*$.

Denoting by θ and η respectively $\|\, u\,\|_1$ and $\|\, v\,\|_2$, we therefore obtain the following minoration of J for any $(u,v) \in E$

$$I(u,v) \geq \theta^p(\frac{\alpha+1}{p} - C'\theta^{m_1-p}) + \eta^q(\frac{\beta+1}{q} - C'\eta^{m_2-q}).$$

Which implies that there exists γ, $\rho > 0$ such that $I(u,v) \geq \gamma > 0$ for all $\|\, (u,v)\,\|_E = \rho$.
2) From the expression

$$I(t^{1/p}u, t^{1/q}v) = \tfrac{t(\alpha+1)}{p} \|\, u\,\|_1^p + \tfrac{t(\beta+1)}{q} \|\, v\,\|_2^q - t^{\frac{\alpha+1}{p} + \frac{\beta+1}{q}} \int f(x) \,|\, u\,|^{\alpha+1} |\, v\,|^{\beta+1}\, dx,$$

it follows that

$$I(t^{1/p}u, t^{1/q}v) \to -\infty \text{ as } t \to +\infty$$

Since $\frac{\alpha+1}{p} + \frac{\beta+1}{q} > 1$. Hence, in view of Lemmas 2.3 and 3.1, we can apply the Mountain-Pass Theorem (c.f. [1]) which guarantees the existence of nontrivial weak solutions of (S).

On the other hand, in the case $\frac{\alpha+1}{p} + \frac{\beta+1}{q} < 1$, we may use the local minimization of the functional I to prove the existence result. Indeed, by hypothesis (H_2), the functional I is weakly lower semi continuous differentiable. Moreover, I is bounded below. In fact, we have

$$I(u,v) \geq \frac{\alpha+1}{p} \mid\mid u \mid\mid_1^p + \frac{\beta+1}{q} \mid\mid v \mid\mid_2^q - C \mid\mid f \mid\mid_\omega \mid\mid u \mid\mid_{p^*}^{\alpha+1} \mid\mid v \mid\mid_{q^*}^{\beta+1}.$$

Since $\frac{\alpha+1}{p} + \frac{\beta+1}{q} < 1$, there exist $\gamma_1 \in]1,p[, \gamma_2 \in]1,q[$ such that

$$\frac{\alpha+1}{\gamma_1} + \frac{\beta+1}{\gamma_2} = 1,$$

which implies that

$$I(u,v) \geq \frac{\alpha+1}{p} \mid\mid u \mid\mid_1^p + \frac{\beta+1}{q} \mid\mid v \mid\mid_2^q - C \mid\mid f \mid\mid_\omega (\mid\mid u \mid\mid_{p^*}^{\gamma_1} + \mid\mid v \mid\mid_{q^*}^{\gamma_2})$$

since $\mid\mid u \mid\mid_{p^*} \leq D_1 \mid\mid u \mid\mid_1$ and $\mid\mid v \mid\mid_{q^*} \leq D_2 \mid\mid v \mid\mid_2$ for some constants $D_1, D_2 > 0$. Then, we have

$$I(u,v) \geq \frac{\alpha+1}{p} \mid\mid u \mid\mid_1^p + \frac{\beta+1}{q} \mid\mid v \mid\mid_2^q - C \mid\mid f \mid\mid_\omega (\mid\mid u \mid\mid_1^{\gamma_1} + \mid\mid v \mid\mid_2^{\gamma_2}).$$

It follows from here that I is bounded below. Thus I has a critical point $(\bar{u}, \bar{v})$

$$I(\bar{u}, \bar{v}) = \inf\{I(u,v) : (u,v) \in E\},$$

which is solution of the system (S). We note that (u,v) must be nontrivial since

$$I(s\varphi, t\psi) = s^p \frac{\alpha+1}{p} \mid\mid u \mid\mid_1^p + t^q \frac{\beta+1}{q} \mid\mid v \mid\mid_2^q - s^{\alpha+1} t^{\beta+1} \int f(x) \mid \varphi \mid^{\alpha+1} \mid \psi \mid^{\beta+1} \, dx$$

for some $\varphi, \psi \in \mathcal{C}_0^\infty$. Hence, since $\alpha+1 < p$, $\beta+1 < q$, we get $I(s\varphi, t\psi) < 0$ for small s, t.
This concludes the proof of Theorem 1.1.

Proof of Theorem 1.2. In this section, we may choose $u, v \geq 0$ since we can show that argument developed here is true for u^+ and u^-.
Set $u_k(x) = \min\{u(x), k\}, k \in \mathbb{N}$. For any real $i \geq 1, (u_k^i, v) \in E$. We have

$$\int \mid \nabla u \mid^{p-2} \nabla u \nabla \varphi + a(x) \mid u \mid^{p-2} u\varphi \, dx = \int f(x) \mid u \mid^{\alpha-1} u\varphi \mid v \mid^{\beta+1} \, dx$$

for all $\varphi \in E$.
Substituting $\varphi = (u_k)^i$ in this equation, we obtain the following estimate

$$i\int (u_k)^{i-1} \mid \nabla u_k \mid^p \leq \int f(x) \mid u \mid^{\alpha+i} \mid v \mid^{\beta+1} \, dx.$$

Due to the fact $(u_k)^{i-1} \mid \nabla u_k \mid^p = (\frac{p}{i+p-1})^p \mid \nabla (u_k)^{\frac{(i+p-1)}{p}} \mid^p$ and Sobolev's inequality, we get

$$(\int (u_k)^{\frac{N}{N-p}(i+p-1)})^{\frac{N-p}{N}} \leq C \int f(x) \mid u \mid^{\alpha+i} \mid v \mid^{\beta+1} \, dx$$

for some constant $C > 0$.
Setting $i = i_0 = 1 + p^* \frac{\delta}{\omega}$, $s_0 = \frac{n}{n-p}(i_0 + p - 1) = \frac{N}{N-p}(p + p^* \frac{\delta}{\omega})$. Letting $k \to \infty$, we conclude that $u \in L^{s_0}$ since

$$\frac{1-\delta}{\omega} + \frac{\alpha + i_0}{p^*} + \frac{\beta+1}{q^*} = 1.$$

Setting now $i_1 = 1 + p^* \frac{\delta}{\omega} + \frac{N}{N-p} p^* \frac{\delta}{\omega} = i_0 + \frac{N}{N-p} p^* \frac{\delta}{\omega}$, repeating the same argument, we get

$$u \in L^{s_1}, \text{ where } s_1 = \frac{N}{N-p}(i_1 + p - 1),$$

since $\frac{\alpha+i_1}{p^*} + \frac{\beta+1}{q^*} + (\frac{1-\delta}{\omega} - \frac{N}{N-p}\frac{\delta}{\omega}) = 1$ and $f \in L^{\overline{\frac{\omega}{1-\delta(1+\frac{N}{N-p})}}}$ for δ small enough. Iterating this process gives

$$u \in L^{s_j} \text{ where } s_j = \frac{N}{N-p}(i_j + p - 1),$$

with $i_j = 1 + p^* \frac{\delta}{\omega} + \frac{N}{N-p} p^* \frac{\delta}{\omega} + \dots\dots + (\frac{N}{N-p})^j p^* \frac{\delta}{\omega}$. Hence, it follows that

$$u \in L^{\sigma_1} \text{ for all, } \frac{Np}{N-p} \leq \sigma_1 < \infty.$$

On the other hand, by using the same argument as above, we prove that

$$v \in L^{\sigma_2}, \frac{Nq}{N-q} \leq \sigma_2 < \infty.$$

In order to complete the proof of Theorem 1.2, we need the following result.

Claim. $f v^{\beta+1} \in L^{\frac{N}{p-\varepsilon}}$ and $f u^{\alpha+1} \in L^{\frac{N}{q-\varepsilon}}$ for some $0 < \varepsilon < 1$ small enough.

Proof. Let $\varepsilon \in]0, min\{1, p\delta, q\delta\}[$; using Hölder inequality we obtain

$$\int (fv^{\beta+1})^{\frac{N}{p-\varepsilon}}\, dx \le (\int f(x)^{\frac{\omega p}{p-\varepsilon}})^{\frac{N}{p\omega}} (\int v^{\beta+1 \frac{N}{p-\varepsilon} \frac{p}{p-\frac{N}{\omega}}})^{\frac{p-\frac{N}{\omega}}{p}},$$

Indeed, in virtue of (1.1) (in particular, we have $\frac{\alpha+1}{p^*} + \frac{\beta+1}{q^*} > \frac{N-p}{N}$), it follows that $\frac{N}{p\omega} > 1$. Let us remark also, for reader's convenience, that $p - N + \frac{N(\alpha+1)}{p^*} \le 0$ (since $\alpha + 1 \le p$), which implies

$$(\beta + 1)\frac{N}{p-\varepsilon}\frac{p}{p-\frac{N}{\omega}} > q^* \tag{3.1}$$

Hence, we deduce from (3.1) and due to the fact that $v \in L^{\sigma_1}(\sigma_1 \ge q^*)$, that $fv^{\beta+1} \in L^{\frac{N}{p-\varepsilon}}$ since $\frac{\omega p}{p-\varepsilon} \in [\omega, \omega/(1-\delta)]$. Similarly, we prove that $fu^{\alpha+1} \in L^{\frac{N}{q-\varepsilon}}$.

Letting now $\varphi = u^-$ as a test function in the first equation of (SV) implies $u \ge 0$ in $\mathbb{R}^N$. Hence, in view of previous Claim , the positivity of solutions follows immediately from the weak Harnack type inequality proved in Trudinger [18, Theorem 1.2]. Finally, the decay of u and v follows directly from the result (Theorem 1) of Serrin [12].

4. Concluding remarks

Remark 4.1. When $\frac{\alpha+1}{p} + \frac{\beta+1}{q} = 1$, one can show, by the same argument used in the case $\frac{\alpha+1}{p} + \frac{\beta+1}{q} < 1$ that there exists λ_* such that for all λ verifying $0 < \lambda < \lambda_*$, the following eigenvalue problem

$$\begin{cases} -\Delta_p u + a(x) \mid u \mid^{p-2} u = \lambda f(x) \mid u \mid^{\alpha-1} u \mid v \mid^{\beta+1} & \text{in } \mathbb{R}^N \\ -\Delta_q v + b(x) \mid v \mid^{q-2} v = \lambda f(x) \mid u \mid^{\alpha+1} \mid v \mid^{\beta-1} v & \text{in } \mathbb{R}^N \\ \lim_{|x|\to\infty} u(x) = \lim_{|x|\to\infty} v(x) = 0 \end{cases}$$

has at least one nontrivial solution in E.

Remark 4.2. The existence result is also obtained for systems of the form

$$(S') \begin{cases} -\Delta_p u + a(x) \mid u \mid^{p-2} u = \sum_I f_i(x) \mid u \mid^{\alpha_i-1} u \mid v \mid^{\beta_i+1} & \text{in } \mathbb{R}^N \\ -\Delta_q v + b(x) \mid v \mid^{q-2} v = \sum_I f_i(x) \mid u \mid^{\alpha_i+1} \mid v \mid^{\beta_i-1} v & \text{in } \mathbb{R}^N \end{cases}$$

under the more general assumptions:
$f_i \in L^{m_i}$ positive, $m_i \in [r_i, \frac{r_i}{1-\delta}]$ where

$$r_i = \frac{1}{1 - (\frac{\alpha_i+1}{p^*} + \frac{\beta_i+1}{q^*})},$$

$\frac{\alpha_i+1}{p^*} + \frac{\beta_i+1}{q^*} < 1, \alpha_i > 0, \beta_i > 0$ and $0 < \delta < 1$ is a small positive real.

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