On reduced pairs of bounded closed convex sets

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Recibido: 12 de Septiembre de 2003
Aceptado: 23 de Febrero de 2002

ABSTRACT

In this paper certain criteria for reduced pairs of bounded closed convex set are presented. Some examples of reduced and not reduced pairs are enclosed.

2000 Mathematics Subject Classification: 52A07, 26A27.
Key words: Convex analysis, pairs of convex sets

Let $X = (X, \tau)$ be a topological vector space over the field $\mathbb{R}$. Let $\mathcal{K}(X)\,[\mathcal{B}(X)]$ be a family of all nonempty compact [bounded closed] convex subsets of X. For any $A, B \subset X$ the Minkowski sum is defined by $A + B = \{a + b \mid \ a \in A \ \text{ and } \ b \in B\}$. Since $A + B$ is not always closed [4],[9] we define $A \dot{+} B = \overline{A + B}$ for $A, B \in \mathcal{B}(X)$. It was showed in [9] that for $A, B, C \in \mathcal{B}(X)$ the inclusion $A \dot{+} B \subset B \dot{+} C$ implies $A \subset C$. From this it follows that $\mathcal{B}(X)$ together with " $\dot{+}$" is a semigroup satisfying the law of cancellation, i.e. $A \dot{+} B = B \dot{+} C$ implies $A = C$.

For $(A, B), (C, D) \in \mathcal{B}^2(X)$, let $(A, B) \sim (C, D)$ if and only if $A \subset C,\ B \subset D$ and $(A, B) \sim (C, D)$. The relation "$\sim$" is an equivalence relation in $\mathcal{B}^2(X)$ and "$\leq$" is an ordering in the equivalence class $[A, B]$ of any pair (A, B). It should be mentioned that the space $\mathcal{K}(X)/_\sim$, $\mathcal{K}(X) = \{A \in \mathcal{B}(X) \mid \ A \text{ is compact}\}$, plays important role in quasidifferential calculus [2].

The set $A \in \mathcal{B}(X)$ is called a *polytope* if A is convex hull of a finite set. If $A, B \in \mathcal{B}(X)$ then $A \vee B$ is the convex hull of $A \cup B$.

It was proved in [6] that if $A, B \in \mathcal{K}(X)$, then there exists minimal element (C, D) in $[A, B]$ such that $(C, D) \leq (A, B)$. From [3], [8] we know that if $(A, B), (C, D) \in \mathcal{K}^2(X)$, are two minimal pairs in $[A, B]$ and $\dim X \leq 2$ then $C + x,\ D = B + x$.

Let $(A, B) \in \mathcal{B}^2(X)$. The pair (A, B) is called *reduced* if for any $(C, D) \in [A, B]$ there exists $M \in \mathcal{B}(X)$ such that $C = A \dot{+} M$ and $D = B \dot{+} M$. Let us notice that every reduced pair is minimal. Every minimal pair is reduced in $X = \mathbb{R}$ (see, [6]).

Let $A \in \mathcal{K}(X)$, $f \in X^*$. Then $H_f A = \{x \in A \mid f(x) = \max_{y \in A} f(y)\}$.

The set $A \in \mathcal{B}(X)$ is called a *summand* of $B \in \mathcal{B}(X)$ if there exists $M \in \mathcal{B}(X)$ such that $B = A \dot{+} M$.

W. Weil has proved in [11] the following lemma.

Lemma. *Let $A, B \in \mathcal{K}(\mathbb{R}^n)$ and A be a convex polytope.Then A is a summand of B if an only if each one-dimensional face $H_f A$ is contained in a translate of the corresponding face $H_f B$.*

Theorem 1. *Let A, $B \in \mathcal{K}(\mathbb{R}^n)$ and A be a convex polytope such that* $\operatorname{card} H_f B = 1$ *for each one-dimensional face $H_f A$. Then the pair (A, B) is reduced.*

Proof. Let $(C, D \in [A, B]$. Then $A + D = B + C$. Let $f \in (\mathbb{R}^n)^*$ and $H_f A$ be one-dimensional face of A. Then, by virtue of the formula of the addition of faces, we have

$$H_f A + H_f D = H_f B + H_f C.$$

According to the assumption, $H_f B = \{b\}$ for some $b \in \mathbb{R}^n$. Then $H_f A \subset b - d + H_f C$, where $d \in H_f D$. Applying Lemma, we obtain that $C = A + M$ for some $M \in \mathcal{K}(\mathbb{R}^n)$. Hence, from the law of cancellation, it follows that $D = B + M$. ■

Theorem 2. *Let $A, B \in \mathcal{K}(\mathbb{R}^2)$ be a reduced pair. Then* $\operatorname{card} H_f B = 1$ *for each one-dimensional face $H_f A$.*

Proof. Let us assume that $\dim H_f B = \dim H_f A = 1$ for some $f \in (\mathbb{R}^2)^*$. Then there exists an interval I and a triangle T such that length of I is not greater than both lengths of $H_f A$ and $H_f B$, and $H_{-f} T = I$. If $H_f T = \{b\}$ then $H_f(A+T) = H_f A + b$, $H_{-f}(A+T) = H_{-f} A + I$, $H_f(B+T) = H_f B + b$ and $H_{-f}(B+T) = H_{-f} B + I$. Hence I is a summand of both $A+T$ and $B+T$, and $A+T = A' + I$, $B+T = B' + I$ for some $A', B' \in \mathcal{K}(\mathbb{R}^2)$. Then $A', B') \in [A, B]$, and since $H_f A$ is not a summand of $H_f A'$ then A is not a summand of A'. Therefore, (A, B) is not reduced. ■

Proposition 1. *Let $(A, B), (C, D), (E, F) \in \mathcal{B}^2(X)$ and $A = C \dot{+} E$, $B = D \dot{+} F$. If the pair (A, B) is reduced then both (C, D) and (E, F) are reduced.*

Proof. Let $(C', D') \in [C, D]$. Then $C' \dot{+} D = C \dot{+} D'$, and we have

$$A \dot{+} D \dot{+} F \dot{+} D' = A \dot{+} B \dot{+} D' = C \dot{+} E \dot{+} B \dot{+} D' = E \dot{+} B \dot{+} C' \dot{+} D.$$

Hence $A \dot{+} F \dot{+} D' = B \dot{+} E \dot{+} C'$. From the assumption, it follows that $E \dot{+} C' = A \dot{+} M$ and $F \dot{+} D' = B \dot{+} M$ for some $M \in \mathcal{B}(X)$. Then $E \dot{+} C' = C \dot{+} E \dot{+} M$ and $F \dot{+} D' = D \dot{+} F \dot{+} M$. Hence $C' = C \dot{+} M$ and $D' = D \dot{+} M$. ■

Proposition 2. *Let $A, B \in \mathcal{B}(X)$. If the pair $(A \vee B, A+B)$ is reduced then $(A \vee B, B)$ is also reduced.*

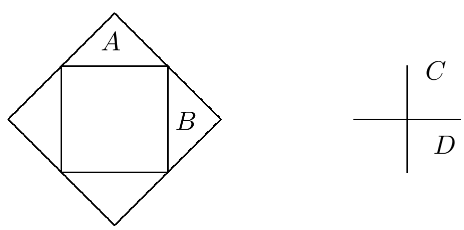
Proof. Since $(A \vee B, A+B) = (A \vee B, B) + (\{0\}, A)$ then applying Proposition 1 we obtain our Proposition. ■

Let $A, B \in \mathcal{B}(X)$. We call the pair (A, B) *convex* if $A \cup B$ is convex. We call (A, B) *convexly reduced* if for any convex pair (C, D) in $[A, B]$ there exists $M \in \mathcal{B}(X)$ such that $C = A \dot{+} M$ and $D = B \dot{+} M$.

Theorem 3. *The convex pair $(A, B) \in \mathcal{B}^2(X)$ is convexly reduced if and only if $(A \cap B, A \cup B)$ is reduced.*

Proof. $\Rightarrow$) Let the pair (A, B) be convexly reduced and $(F, G) \in [A \cap B, A \cup B]$. From [4],[10] it follows that there exists $(A_0, B_0) \in [A, B]$ such that $A_0 \cap B_0 = F$ and $A_0 \cup B_0 = G$. From the assumption, $A_0 = A \dot{+} M$ and $B_0 = B \dot{+} M$ for some $M \in \mathcal{B}(X)$. Then $F = A_0 \cap B_0 = A \cap B \dot{+} M$ and $G = A_0 \cup B_0 = A \cup B \dot{+} M$. Therefore, the pair $(A \cap B, A \cup B)$ is reduced.

$\Leftarrow$) Let $(A \cap B, A \cup B)$ be reduced, $(C, D) \in [A, B]$ and $C \cup D$ be convex. Then $A \dot{+} D = B \dot{+} C = A \cap B \dot{+} C \cup D = C \cap D \dot{+} A \cup B$, [see [10]]. Hence $C \cap D = A \cap B \dot{+} M$ and $C \cup D = A \cup B \dot{+} M$ for some $M \in \mathcal{B}(X)$. From the law of cancellation, we obtain $C = A \dot{+} M$ and $D = B \dot{+} M$. ■



The pair (A, B) is convexly reduced and $(A, B) \sim (C, D)$.

Theorem 4. *Let $A, B \in \mathcal{B}(X)$. If $(A \vee B, B)$ is a reduced pair then the pair (A, B) is reduced.*

Proof. Let $(C, D) \in [A, B]$. Then $A \dot{+} D = B \dot{+} C$. Therefore,

$$D \dot{+} A \vee B = (A \dot{+} D) \vee (B \dot{+} D) = (B \dot{+} C) \vee (B \dot{+} D) = B \dot{+} C \vee D.$$

Since the pair $(A \vee B, B)$ is reduced then $D = B \dot{+} M$ for some $M \in \mathcal{B}(X)$. From the law of cancellation ([9]) $C = A \dot{+} M$. ■

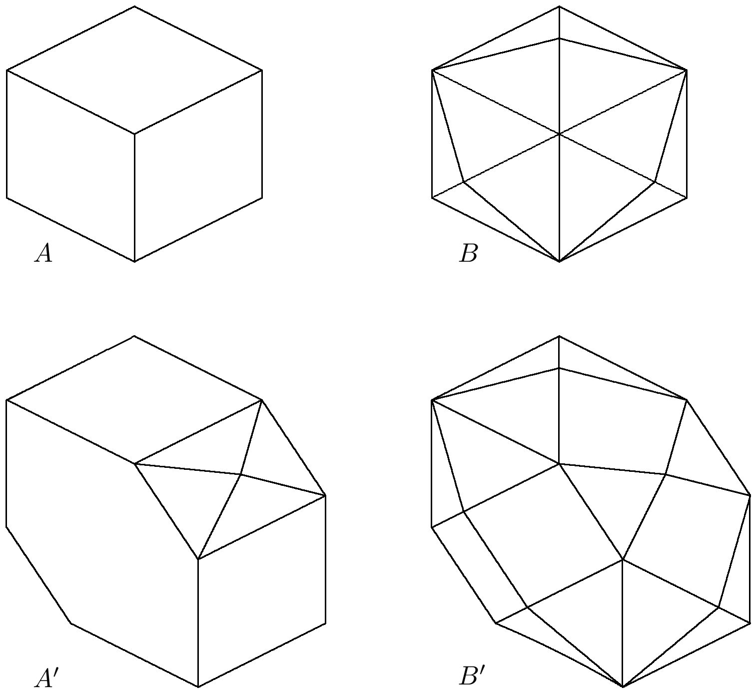
The pair (A, B) is convexly reduced and $(A, B) \sim (C, D)$. The pair (A, B) is also reduced and the class $[A, B]$ is convex, that is $C \cup D$ is convex for any $(C, D) \in [A, B]$ ([4]).

In [5] the following theorem was proved:

Theorem 5. *Let $A, B \in \mathcal{K}(\mathbb{R}^n)$ and A be a polytope with nonempty interior. Let* $\operatorname{card} H_f B = 1$ *for each face $H_f A$ such that* $\dim H_f A = n - 1$. *Then the pair (A, B) is minimal.*

For $n = 2$, Theorem 1 and Theorem 5 have equivalent assumptions , hence Theorem 1 is stronger than Theorem 5. For $n = 3$, the assumption of Theorem 5 is weaker than the assumption of Theorem 1. The following example shows that generally we cannot replace the assumption in Theorem 1 with the assumption from Theorem 5.

Example. Let $A = [-1, 1]^3$ and
$B = A \vee (0, 0, 3/2) \vee (0, 0, -3/2) \vee (0, 3/2, 0) \vee (0, -3/2, 0) \vee (3/2, 0, 0) \vee (-3/2, 0, 0)$. Let us notice that if $\dim H_f A = 2$ then $\operatorname{card} H_f B = 1$. Let $I = (1, 0, 0) \vee (0, 1, 0)$. Let $A' = (A + I) \vee (5/3, 5/3, 0)$ and $B' = (B + I) \vee (5/3, 5/3, 0)$. We have $(A', B') \sim (A + I, B + I) \sim (A, B)$. Let us notice that $H_f A' = (5/3, 5/3, 0)$ and $H_f A = (1, 1, -1) \vee (1, 1, 1)$ for $f(x, y, z) = x + y$. Then A is not a summand of A'. The pair (A, B) is not reduced.



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