On two recent geometrical characterizations of hyperellipticity

Antonio F. Costa and Ana M. Porto

Departamento de Matemáticas Fundamentales
Facultad de Ciencias, UNED,
28040 Madrid, Spain
acosta@mat.uned.es
asilva@mat.uned.es

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ABSTRACT

We obtain short and unified new proofs of two recent characterizations of hyperellipticity given in [4] and [6], as well as a way of establishing a relation between them.

Key words: Riemann surface, hyperelliptic involution, Fuchsian group, hyperbolic polygon.

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1. Introduction

We offer a new point of view of two recent geometrical characterizations of hyperellipticity of Riemann surfaces of genus $g \geq 2$ (see [4] and [6]). We introduce two special types of fundamental regions for surface Fuchsian groups, i. e. Fuchsian groups without elliptic elements uniformizing closed Riemann surfaces. With this approach, the Maskit characterization of hyperellipticity is equivalent to the rotational symmetry for a type of hyperbolic polygon that is a fundamental region for surface Fuchsian groups. The Schaller characterization of hyperellipticity is equivalent to the existence of a fundamental hyperbolic polygon for surface Fuchsian groups where the elements of the group induce the identification of opposite sides of the polygon. In this way, we obtain short and unified new proofs of the two recent characterizations of hyperellipticity and a way of establishing a relation between them. In order to have basic information about hyperelliptic Riemann surfaces see for instance [1] or [5].

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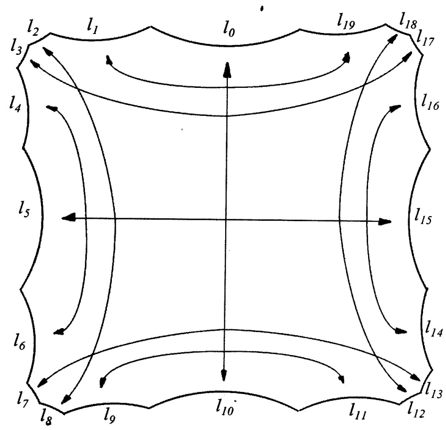

Figure 1: Prehyperelliptic polygon (g=3).

2. Prehyperelliptic and centralhyperelliptic fundamental polygons for Fuchsian groups

In this work we shall consider Riemann surfaces of genus $g \geq 2$ equipped with a metric of constant curvature -1. We start by defining two special types of fundamental polygons for surface Fuchsian groups.

Definition 2.1 (Prehyperelliptic fundamental polygon). Let Γ be a surface Fuchsian group of genus $g \geq 2$. A fundamental polygon P for Γ is said to be prehyperelliptic if and only if (see Figure 1):

(i) P has $m = 8g-4$ sides with labelling $l_{\overline{0}}, l_{\overline{1}}, \ldots, l_{\overline{m-1}}$ where $\{\overline{0}, \overline{1}, \overline{2}, \ldots, \overline{m-1}\} = \mathbb{Z}/m\mathbb{Z}$.

(ii) There are hyperbolic transformations in Γ sending $l_{\overline{2i}}$ to $l_{\overline{\frac{m}{2}-2i}}$ and $l_{\overline{2i+1}}$ to $l_{\overline{-2i-1}}$.

(iii) Let $\widehat{V_{\overline{i}}}$ denote the measure of the interior angle of P with sides $l_{\overline{i}}$ and $l_{\overline{i+1}}$. The following conditions must be satisfied:

$$\widehat{V_{\overline{i}}} = \widehat{V_{\overline{i+\frac{m}{2}}}}, \quad \widehat{V_{\overline{i}}} + \widehat{V_{\overline{-i-1}}} = \pi, \quad \widehat{V_{\overline{i}}} + \widehat{V_{\overline{-i-1+\frac{m}{2}}}} = \pi.$$

Remark that by condition (iii) every prehyperelliptic fundamental polygon is convex.

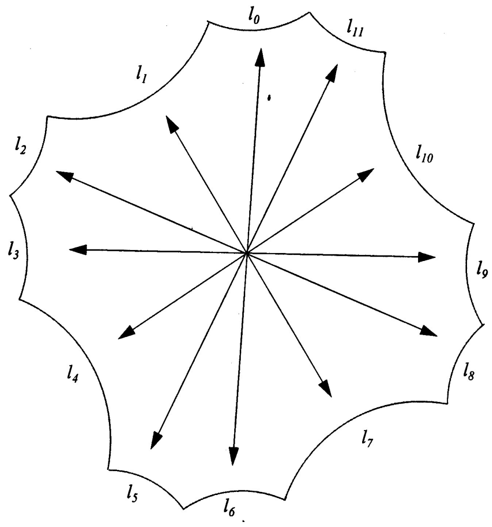

Figure 2: Centralhyperelliptic polygon (g=3).

Definition 2.2 (Centralhyperelliptic fundamental polygon). Let Γ be a surface Fuchsian group of genus $g \geq 2$. A fundamental polygon P for Γ is said to be centralhyperelliptic if and only if (see Figure 2):

(i) P has $4g$ sides with labelling $l_{\overline{0}}, l_{\overline{1}}, \ldots, l_{\overline{4g-1}}$, where $\{\overline{0}, \overline{1}, \overline{2}, \ldots, \overline{4g-1}\} = \mathbb{Z}/4g\mathbb{Z}$.

(ii) There are hyperbolic transformations in Γ sending $l_{\overline{i}}$ to $l_{\overline{i+2g}}$, $i = 0, \ldots, 2g-1$.

(iii) The interior angles of P satisfy $\widehat{V_{\overline{i}}} = \widehat{V_{\overline{i+2g}}}$, $V_{\overline{i}} = l_{\overline{i}} \cap l_{\overline{i+1}}$, $i = 0, \ldots, 2g-1$.

Since Γ is a surface group remark that $\sum\limits_{i=0}^{2g-1} \widehat{V_{\overline{i}}} = 2\pi$.

Proposition 2.3. *Every centralhyperelliptic fundamental polygon P has rotational symmetry of order two, i. e. there is an order two elliptic transformation r such that $r(P) = P$.*

Proof. We recall the following well-known property about the congruency of hyperbolic polygons:

Let P and P' be two hyperbolic polygons. Let $LV = \{l_i, V_i\}$ and $LV' = \{l'_i, V'_i\}$ be the sets of sides and vertices of P and P' respectively. Let $\phi : LV \to LV'$ be a bijection sending vertices to vertices, sides to sides and preserving the adjacency relation. Let W be a subset of LV having three elements. Assume that length(l_i) = length$(\phi(l_i))$ and $\widehat{V_i} = \widehat{\phi(V_i)}$, where $\widehat{V_i}$ (resp. $\widehat{\phi(V_i)}$) is measure of the interior angle of P (resp. P') with vertex in V_i (resp. $\phi(V_i)$) and l_i, $V_i \in LV - W$. Then the bijection ϕ is induced by an hyperbolic transformation.

Now let P be a centralhyperelliptic fundamental polygon of a surface Fuchsian group. Let d be a diagonal joining two opposite vertices of P. The diagonal d divides the polygon P in two polygons C_1 and C_2. Let r be the elliptic element of order two with fixed point the middle point of the diagonal d. Let P' be the polygon obtained by the union of C_1 and $r(C_1)$. The polygons P and P' have the vertices of C_1 in common. We can establish a bijection ϕ between the sets of vertices and sides of P and P' that preserves adjacency and fixes the sides and vertices of C_1. Then condition (ii) of the definition of centralhyperelliptic polygons implies that, if l is a side in P, l and $\phi(l)$ have the same measure. And if V_1 and V_2 are the vertices at the ends of d, it follows by condition (iii) that $\widehat{V} = \widehat{\phi(V)}$, V different from V_1 and V_2. Therefore the polygons P and P' are congruent and, since C_1 is common to P and P', then P and P' are the same polygon and thus r is a rotational symmetry of order two of P. □

The following result is the main Theorem of this work:

Theorem 2.4. *Let Γ be a surface Fuchsian group of genus $g \geq 2$. The following three conditions are equivalent:*

(i) $\mathbb{D}/\Gamma$ *is a hyperelliptic Riemann surface.*

(ii) Γ *has a prehyperelliptic fundamental polygon having a rotational symmetry of order two.*

(iii) Γ *has a centralhyperelliptic fundamental polygon.*

Proof. ((i)$\Longrightarrow$(ii)) Assume that $\mathbb{D}/\Gamma$ is a hyperelliptic Riemann surface of genus g, i. e. there is a covering $f : \mathbb{D}/\Gamma \to \widehat{\mathbb{C}}$ (the Riemann sphere) with $2g+2$ branched points. It follows that there is a Fuchsian group Δ with signature $(0, [2, \overset{2g+2}{\ldots}, 2])$ containing Γ as an index two subgroup and let $p : \mathbb{D} \to \mathbb{D}/\Delta = \widehat{\mathbb{C}}$ be the natural projection. The group Δ admits a canonical presentation:

$$\left\langle x_1, \ldots, x_{2g+2}; x_1^2 = \cdots = x_{2g+2}^2 = x_1 \cdots x_{2g+2} = 1 \right\rangle.$$

The group Δ has a fundamental region Q that is a convex polygon with $2g+2$ sides, $\lambda_1, \ldots, \lambda_{2g+2}$, and the middle point m_i of each side λ_i is the fixed point of the elliptic generator x_i, i. e. Q is the canonical fundamental polygon for Δ. The existence of such canonical fundamental polygon for Δ is a classic result by Fricke-Klein (pp. 294–320 of [2] and p. 241 of [3]). The vertices of Q belong to one orbit of Δ.

In order to be selfcontained we shall sketch a construction of the polygon Q. Let $W_1, \dots, W_{2g+2}$ be the fixed points of the generators $x_1, \dots, x_{2g+2}$. The points $W_1, \dots, W_{2g+2}$ do not lie on a hyperbolic line (in the contrary case all the elements of Δ left invariant such hyperbolic line λ, but then all the hyperbolic elements in Δ have λ as axis and Δ cannot contain a surface subgroup). The convex hull Π of $W_1, \dots, W_{2g+2}$ has nonempty interior. Let Σ be the set of all hyperbolic lines joining fixed points of elements of Δ. The set Σ is countable and then there is a point $W \in \overset{\circ}{\Pi} - \bigcup_{r\in\Sigma} r$. Now there are geodesic segments $\gamma_1, \dots, \gamma_{2g+2}$ joining $f(W)$ with the branched points of f. The result of cutting the orbifold $\mathbb{D}/\Delta$ by $\gamma_1, \dots, \gamma_{2g+2}$ and the lifting by p produces the fundamental convex polygon Q.

Let δ_i be a hyperbolic segment in Q joining m_i with m_{i+1}, $i = 1, \dots, 2g+1$. Each δ_i determines a triangle T_i with vertices m_i, m_{i+1}, and a vertex of Q. We define

$$R = (Q - (\bigcup_{i=1}^{2g+1} T_i)) \cup (\bigcup_{i=1}^{2g+1} x_i \cdots x_1(T_i)).$$

The polygon R is a fundamental region for Δ.

The polygon R has $4g$ sides:

$$\delta_1 \cup \delta_1', \delta_2, \delta_3, \dots, \delta_{2g}, \delta_{2g+1} \cup \delta_{2g+1}', \delta_{2g}', \delta_{2g-1}', \dots, \delta_2',$$

where $\delta_i' = x_i \cdots x_1(\delta_i)$.

Let $\widehat{V_i}$ (resp. $\widehat{V_i'}$) be the measure of the interior angle with vertex $\delta_i \cap \delta_{i+1}$ (resp. $\delta_i' \cap \delta_{i+1}'$). By the construction of R we have that $\widehat{V_i} + \widehat{V_i'} = \pi$.

The polygon $P = R \cup x_{2g+2}(R)$ is a fundamental region for Γ and x_{2g+2} is a rotational symmetry of order two for P. Since $\widehat{V_{2g+1}} + \widehat{V_{2g+1}'} = \pi$ the polygon P has $8g-4$ sides. The equalities $\delta_i' = x_i \cdots x_1(\delta_i)$, $\widehat{V_i} + \widehat{V_i'} = \pi$ and the fact that x_{2g+2} is a symmetry imply the prehyperelliptic quality of P.

((ii)$\Longrightarrow$(iii)) Let P be a prehyperelliptic fundamental polygon with rotational symmetry r of order two with fixed point O. Let $l_{\overline{1}}, \dots, l_{\overline{m}}$ be the sides of P and $V_{\overline{1}}, \dots, V_{\overline{m}}$ be the vertices: $l_{\overline{i}} \cap l_{\overline{i+1}} = V_i$, satisfying the conditions of Definition 2.1. Let d_i be the diagonal of P joining $V_{\overline{i}}$ with $V_{\overline{i+\frac{m}{2}}}$. By the rotational symmetry of P, all the diagonals d_i intersect in O. Let $p : \mathbb{D} \to \mathbb{D}/\Gamma$ be the natural projection and $S_{2g} = \{\, p(d_i) : i = 1, \dots, 2g\,\}$. The set S_{2g} consists of $2g$ closed geodesics which intersect in $p(O)$ and does not intersect in other point (i. e. S_{2g} is a $2g$-star, see the Definition 2.8 below). Cutting $\mathbb{D}/\Gamma$ by S_{2g} we obtain a centralhyperelliptic fundamental polygon for Γ.

((iii)$\Longrightarrow$(i)) Let C be a centralhyperelliptic polygon. Then, by Proposition 2.3, C has a rotational symmetry r of order two. The identifications induced by the elements of Γ on the sides of C are compatible with the symmetry r. Hence r induces an order two automorphism ϕ of $\mathbb{D}/\Gamma$. Let $p : \mathbb{D} \to \mathbb{D}/\Gamma$ be the natural projection. The number of fixed points of ϕ is $2g+2$: the image by p of the fixed point of r, the image by p of the $2g$ middle points of the sides of C and the point which is the projection of the vertices of C by p. □

Remark 2.5. From the above proof we have also the following geometrical result: a prehyperelliptic polygon P has rotational symmetry of order two if and only if the diagonals of P joining opposite vertices cut in a point.

We obtain from Theorem 2.4 the characterizations of hyperellipticity given in [4] and [6].

Definition 2.6 (Necklace). A (geodesic) necklace on a Riemann surface of genus g is a cyclically ordered set of $2g+2$ simple nonseparating closed geodesics $L_1, \dots, L_{2g+2}$, where each L_i intersects L_{i-1} exactly one, intersects L_{i+1} exactly one, and is otherwise disjoint from every other geodesic in the necklace.

Corollary 2.7 (B. Maskit [4]). *A Riemann surface X is hyperelliptic if and only if X has an evenly spaced geodesic necklace.*

Proof. Assume that X is a hyperelliptic Riemann surface. Then $X = \mathbb{D}/\Gamma$ where Γ is a surface Fuchsian group admitting a prehyperelliptic fundamental polygon P with a rotational order two symmetry (Theorem 2.4 (ii)). The sides of the polygon P and the axis of the hyperbolic generators of Γ identifying l_0 with $l_{\frac{m}{2}}$ and $l_{\frac{m}{4}}$ with $l_{\frac{3m}{4}}$ produce a necklace in X. The fact that the polygon P is symmetric implies that the necklace is evenly spaced.

If we have an evenly spaced necklace on X, cutting X by such necklace we obtain a prehyperelliptic fundamental polygon for a Fuchsian group uniformizing X. The condition of being evenly spaced gives that $\text{length}(l_i) = \text{length}(l_{i+\frac{m}{2}})$. By a similar argument to the one used in the proof of Proposition 3, it results that the polygon P has a rotational symmetry of order two and then by the Theorem 4, X is hyperelliptic. □

Definition 2.8 (Star). A (geodesic) k-star in a Riemann surface is a set S_k of k simple closed geodesics which all intersect in the same point and such that among the elements of S_k there are no further intersection points.

Corollary 2.9 (P. S. Schaller [6]). *A Riemann surface X is hyperelliptic if and only if X has a $(2g-2)$-star.*

Proof. Assume that X is hyperelliptic. According to Theorem 2.4 (iii), X can be uniformized by a surface Fuchsian group admitting a centralhyperelliptic fundamental polygon. The sides of such polygon produce on X a $2g$-star which contains, obviously, a $(2g-2)$-star.

Suppose now that X has a $(2g-2)$-star S_{2g-2}. Cutting the surface X by S_{2g-2} we obtain a surface T_1 homeomorphic to a torus with a boundary component and such boundary component is piecewise geodesic. Since T_1 is hyperelliptic (all surfaces with this topological type are hyperelliptic), let p_1 and p_2 be two of the fixed points of the hyperelliptic involution h of T_1 and V be one of the vertices in ∂T_1. Let a_i, $i = 1, 2$, be two disjoint geodesic arcs joining V with p_i and containing no other fixed points of h. The set $S_{2g} = S_{2g-2} \cup \{a_1 \cup h(a_1)\} \cup \{a_2 \cup h(a_2)\}$ is a $2g$-star. Cutting X by S_{2g} we obtain a centralhyperelliptic fundamental polygon for a surface Fuchsian group uniformizing X. Hence, by Theorem 2.4, X is hyperelliptic. □

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