Irregular fibers of complex polynomials in two variables

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ABSTRACT

For a complex polynomial in two variables we study the morphism induced in homology by the embedding of an irregular fiber in a regular neighborhood of it. We give necessary and sufficient conditions for this morphism to be injective, surjective. Particularly this morphism is an isomorphism if and only if the corresponding irregular value is regular at infinity. We apply these results to the study of vanishing and invariant cycles.

Key words: irregulars fibers, tube, vanishing cycles, invariant cycles, resolution of singularities

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Introduction

Let $f : \mathbb{C}^n \longrightarrow \mathbb{C}$ be a polynomial. The *bifurcation set* $\mathcal{B}$ for f is the minimal set of points of $\mathbb{C}$ such that $f : \mathbb{C}^n \setminus f^{-1}(\mathcal{B}) \longrightarrow \mathbb{C} \setminus \mathcal{B}$ is a locally trivial fibration. For $c \in \mathbb{C}$, we denote the fiber $f^{-1}(c)$ by F_c. The fiber F_c is *irregular* if c is in $\mathcal{B}$. If $s \notin \mathcal{B}$, then F_s is a *generic fiber* and is denoted by F_{gen}. The *tube* T_c for the value c is a neighborhood $f^{-1}(D^2_\varepsilon(c))$ of the fiber F_c, where $D^2_\varepsilon(c)$ stands for a 2-disk in $\mathbb{C}$, centered at c, of radius $\varepsilon \ll 1$. We assume that affine critical singularities are isolated. The value c is *regular at infinity* if there exists a sufficiently large compact set K of $\mathbb{C}^n$ such that the restriction of f, $f : T_c \setminus K \longrightarrow D^2_\varepsilon(c)$ is a locally trivial fibration.

Set $n = 2$. Let $j_c : H_1(F_c) \longrightarrow H_1(T_c)$ be the morphism induced by the inclusion of F_c in T_c. The first part of this work is the study of this morphism. Let G_c the dual graph of $F_c = f^{-1}(c)$, and $\bar{G}_c$ the dual graph of a compactification of the fiber F_c obtained by a resolution at infinity of f. The value c is *acyclic* if the dual graph G_c and some dual graphs $G_{c,P}$ obtained by compactification have the same number of cycles (see the full definition later). This is a combinatoric condition, for example if the fiber F_c is connected then c is acyclic if and only if $H_1(G_c)$ is isomorphic to $H_1(\bar{G}_c)$. Finally we define $j_\infty : H_1(F_c \setminus K) \longrightarrow H_1(T_c \setminus K)$ induced by inclusion (this map is independent of the sufficiently large set K).

Theorem.

(A) j_c is injective if and only if F_c is connected and c is acyclic.

(B) j_c is surjective if and only if j_∞ is surjective and c is acyclic.

(C) j_c is an isomorphism if and only if c is a regular value at infinity.

It should be noticed that the fiber F_c is not supposed to be smooth. In fact we have a stronger result for the part (A) because the rank of the kernel of j_c is:

$$\operatorname{rk}\operatorname{Ker} j_c = n(F_c) - 1 + \operatorname{rk} H_1(\bar{G}_c) - \operatorname{rk} H_1(G_c)$$

where $n(F_c)$ is the number of connected components of F_c. The acyclicity condition and the surjectivity of j_∞ can be checked from the dual graphs of resolution. E. Artal-Bartolo, Pi. Cassou-Noguès and A. Dimca have proved the part (C) in [1] for polynomials with a connected fiber F_c. For a non-connected fiber j_c is not injective (part (A)), and we could have used the result of [1] to prove part (C) but we will give a complete proof.

We apply these results to the study of neighborhoods of irregular fibers. Set $n \geqslant 2$. Let F_c° be the *smooth part* of F_c: F_c° is obtained by intersecting F_c with a large $2n$-ball and cutting out a small neighborhood of the (isolated) singularities. Then F_c° can be embedded in F_{gen}. We study the following commutative diagram that links the three elements F_c°, F_{gen}, and T_c:

$$\begin{array}{ccc} H_q(F_c^\circ) & \xrightarrow{\ j_c^\circ\ } & H_q(T_c) \\ {\scriptstyle \ell_c}\downarrow & \nearrow{\scriptstyle k_c} & \\ H_q(F_{gen}) & & \end{array}$$

where ℓ_c is the morphism induced in integral homology by the embedding; j_c° and k_c are induced by inclusions. The morphism k_c is well-known and $V_q(c) = \operatorname{Ker} k_c$ are the *vanishing cycles* for the value c. Let h_c be the monodromy induced on $H_q(F_{gen})$ by a small circle around the value c. Then we prove that the image of ℓ_c are the invariant cycles by h_c:

$$\operatorname{Ker}(h_c - \mathrm{id}) = \ell_c\big(H_q(F_c^\circ)\big).$$

This formula for the case $n = 2$ has been obtained by F. Michel and C. Weber in [6].

Finally we give a description of vanishing cycles with respect to the eigenvalues of h_c for homology with complex coefficients. For $\lambda \neq 1$ and p a large integer the characteristic space $E_\lambda = \operatorname{Ker}(h_c - \lambda\,\mathrm{id})^p$ is generated by vanishing cycles for the value c. For $\lambda = 1$ the situation is different. If $K_q(c) = V_q(c) \cap \operatorname{Ker}(h_c - \mathrm{id})$ are invariant and vanishing cycles we have

$$K_q(c) = \ell_c\big(\operatorname{Ker} j_c^\circ\big).$$

For $n = 2$ we have $E_1 = \operatorname{Ker}(h_c - \mathrm{id})^2$ and we prove that $\operatorname{Ker}(h_c - \mathrm{id})^2 \cap V_1(c) = \operatorname{Ker}(h_c - \mathrm{id}) \cap V_1(c) =: K_1(c)$ with the formula

$$\operatorname{rk} K_1(c) = r(F_c) - 1 + \operatorname{rk} H_1(\bar{G}_c),$$

where $r(F_c)$ is the number of irreducible components of F_c.

The study of vanishing cycles is motivated by the following result of [3]: the monodromy $h_\infty : H_1(F_{gen}) \longrightarrow H_1(F_{gen})$ induced by a large circle around the set $\mathcal{B}$ and Broughton's decomposition $H_1(F_{gen}) = \bigoplus_{c\in\mathcal{B}} V_1(c)$ determine the monodromy representation $\pi_1(\mathbb{C} \setminus \mathcal{B}) \longrightarrow \mathrm{Aut} H_1(F_{gen})$. The formula for $\operatorname{rk} K_1(c)$ enables to describe vanishing cycles with respect to a decomposition of the homology of the generic fiber given by the resolution of singularities.

1. Irregular fibers and tubes

1.1. Bifurcation set

We can describe the bifurcation set $\mathcal{B}$ as follows: let $\mathrm{Sing} = \big\{z \in \mathbb{C}^n \mid \mathrm{grad}_f(z) = 0\big\}$ be the set of *affine critical points* and let $\mathcal{B}_{\mathit{aff}} = f(\mathrm{Sing})$ be the set of *affine critical values.* The set $\mathcal{B}_{\mathit{aff}}$ is a subset of $\mathcal{B}$. The value $c \in \mathbb{C}$ is *regular at infinity* if there exist a disk D centered at c and a compact set K of $\mathbb{C}^n$ with a locally trivial fibration $f : f^{-1}(D) \setminus K \longrightarrow D$. The non-regular values at infinity are the *critical values at infinity* and are collected in $\mathcal{B}_\infty$. The finite set $\mathcal{B}$ of critical values is now:

$$\mathcal{B} = \mathcal{B}_{\mathit{aff}} \cup \mathcal{B}_\infty.$$

In this article we always assume that **affine singularities are isolated**, that is to say that Sing is an isolated set in $\mathbb{C}^n$. For $n = 2$ this hypothesis implies that the generic fiber is a connected set.

1.2. Preliminaries

In this paragraph $n = 2$. The inclusion of F_c in T_c induces a morphism $j_c : H_1(F_c) \longrightarrow H_1(T_c)$. We firstly recall notations and results from [1].

Let denote $F_{\mathit{aff}} = F_c \cap B^4_R$ ($R \gg 1$) and $F_\infty = \overline{F_c \setminus F_{\mathit{aff}}}$, thus $F_{\mathit{aff}} \cap F_\infty = K_c = f^{-1}(c) \cap S^3_R$ is the *link at infinity* for the value c. Similarly $T_{\mathit{aff}} = T_c \cap B^4_R$ and $T_\infty = \overline{T_c \setminus T_{\mathit{aff}}}$. We denote $j_\infty : H_1(F_\infty) \longrightarrow H_1(T_\infty)$ the morphism induced by inclusion. The inclusion $F_{\mathit{aff}} \subset T_{\mathit{aff}}$ is a homotopy equivalence so $j_{\mathit{aff}} : H_1(F_{\mathit{aff}}) \longrightarrow H_1(T_{\mathit{aff}})$ is an isomorphism. $H_1(F_{\mathit{aff}} \cap F_\infty)$ and $H_1(T_{\mathit{aff}} \cap T_\infty)$ are isomorphic.

The Mayer-Vietoris exact sequences for the decompositions $F_c = F_{\mathit{aff}} \cup F_\infty$ and $T_c = T_{\mathit{aff}} \cup T_\infty$ give the commutative diagram $(\mathcal{D})$:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H_1(F_\infty \cap F_{\mathit{aff}}) & \xrightarrow{g} & H_1(F_\infty) \oplus H_1(F_{\mathit{aff}}) & \xrightarrow{h} & H_1(F_c) & \longrightarrow & 0 \\
 & & \downarrow \cong & & \downarrow j_\infty \oplus j_{\mathit{aff}} & & \downarrow j_c & & \\
0 & \longrightarrow & H_1(T_\infty \cap T_{\mathit{aff}}) & \xrightarrow{g'} & H_1(T_\infty) \oplus H_1(T_{\mathit{aff}}) & \xrightarrow{h'} & H_1(T_c) & \longrightarrow & H_0(T_\infty \cap T_{\mathit{aff}}).
\end{array}$$

The 0 at the upper-right corner is provided by the injectivity of $H_0(F_\infty \cap F_{\mathit{aff}}) \longrightarrow H_0(F_\infty)$ hence $H_0(F_\infty \cap F_{\mathit{aff}}) \longrightarrow H_0(F_\infty) \oplus H_0(F_{\mathit{aff}})$ is injective (notice that F_c need not to be a connected set).

1.3. Resolution of singularities

To compactify the situation, for $n = 2$, we need the resolution of singularities at infinity [5]:

$$\begin{array}{ccccc}
\mathbb{C}^2 & \longrightarrow & \mathbb{C}P^2 & \xleftarrow{\pi_w} & \Sigma_w \\
f \downarrow & & \tilde{f} \downarrow & \swarrow \phi_w & \\
\mathbb{C} & \longrightarrow & \mathbb{C}P^1 & &
\end{array}$$

Here $\tilde{f}$ is a rational map coming from the homogenisation of f and is not defined at some points of the line at infinity L_∞ of $\mathbb{C}P^2$; π is the minimal blow-up at these points in order to obtain a well-defined morphism $\phi_w : \Sigma_w \longrightarrow \mathbb{C}P^1$: this is the *weak resolution*. We denote $\phi_w^{-1}(\infty)$ by D_∞, and let D_{dic} be the set of components D of $\pi_w^{-1}(L_\infty)$ that verify $\phi_w(D) = \mathbb{C}P^1$. Such a D is a *dicritical component*. The *degree* of a dicritical component D is the degree of the branched covering $\phi_w : D \longrightarrow \mathbb{C}P^1$. For the weak resolution the divisor $\phi_w^{-1}(c) \cap \pi_w^{-1}(L_\infty)$, $c \in \mathbb{C}$, is a union of bamboos (possibly empty). A *bamboo* is a divisor whose dual graph is a linear tree. The set $\mathcal{B}_\infty$ is the set of values of ϕ_w on non-empty bamboos union the set of critical values of the restriction of ϕ_w to the dicritical components.

We can blow-up more points to obtain the *total resolution*, $\phi_t : \Sigma_t \longrightarrow \mathbb{C}P^1$, such that all fibers of ϕ_t are normal crossing divisors that intersect the dicritical components transversally; moreover we solve affine singularities. Then $D_\infty = \phi_t^{-1}(\infty)$ is the same as above and for $c \in \mathcal{B}$ we denote D_c the divisor $\phi_t^{-1}(c)$.

The *dual graph* $\bar{G}_c$ of D_c is obtained as follows: one vertex for each irreducible component of D_c and one edge between two vertices for one intersection of the corresponding components. A similar construction is done for D_∞, we know that $\bar{G}_\infty$ is a tree [5]. The *multiplicity* of a component is the multiplicity of ϕ_t on this component.

1.4. Study of j_∞

See [1]. Let ϕ be the weak resolution map for f. Let denote by Dic_c the set of points P in the dicritical components, such that $\phi(P) = c$. To each $P \in \mathrm{Dic}_c$ is associated one, and only one, connected component T_P of T_∞; T_P is called the *place at infinity* for P. We have $T_\infty = \coprod_{P \in \mathrm{Dic}_c} T_P$ and we set $F_P = T_P \cap F_\infty = T_P \cap F_c$ and $K_P = \partial F_P$, finally $n(F_P)$ denotes the number of connected components of F_P. Let $\bar{F}_P$ be the strict transform of c by ϕ, intersected with T_P. The map j_∞ is the direct sum of the maps $j_P : H_1(F_P) \longrightarrow H_1(T_P)$. Let $\mathfrak{m}_P$ be the intersection multiplicity of $\bar{F}_P$ with the divisor $\pi_w^*(L_\infty)$ at P.

Case of $P \in \bar{F}_P$. The group $H_1(T_P)$ is isomorphic to $\mathbb{Z}$ and is generated by $[M_P]$, M_P being the boundary of a small disk with transversal intersection with the dicritical component. Moreover if $F_P = \coprod_{i=1}^{n(F_P)} F_P^i$ then $j_P([F_P^i]) = j_P([K_P^i]) = \mathfrak{m}_P^i[M_P]$.

Case of P being in a bamboo. The group $H_1(T_P)$ is also isomorphic to $\mathbb{Z}$ and is generated by $[M_P]$, M_P being the boundary of a small disk, with transversal intersection with the last component of the bamboo. Then $j_P[F_P^i] = j_P[K_P^i] = \mathfrak{m}_P^i.\ell_i[M_P]$. The integer ℓ_i only depends of the position where F_P^i intersects the bamboo, moreover $\ell_i \geqslant 1$ and $\ell_i = 1$ if and only if F_P^i intersects the bamboo at the last component. For a computation of ℓ_i with the help of continuous fraction associated to the bamboos, refer to [1].

As a consequence j_P is injective if and only if $n(F_P) = 1$ and j_∞ is injective if and only if $n(F_P) = 1$ for all P in Dic_c. In fact the rank of the kernel of j_∞ is the sum of the ranks of the kernels of j_P then we deduce:

$$\operatorname{rk}\ker j_\infty = \sum_{P \in \mathrm{Dic}_c} (n(F_P) - 1).$$

Finally j_∞ is surjective if and only if for all $P \in \mathrm{Dic}_c$, j_P is surjective.

1.5. Acyclicity

The value c is *acyclic* if the morphism $\psi : H_0(T_\infty \cap T_{\mathit{aff}}) \longrightarrow H_0(T_\infty) \oplus H_0(T_{\mathit{aff}})$ given by the Mayer-Vietoris exact sequence is injective.

Let give some interpretations of the acyclicity condition.

(i) The injectivity of ψ can be viewed as follows: two branches at infinity that intersect the same place at infinity have to be in different connected components of F_c.

(ii) Let G_c be the *dual graph* of F_c (one vertex for an irreducible component of F_c, two vertices are joined by an edge if the corresponding irreducible components have non-empty intersection, if a component has auto-intersection it provides a loop) and let $G_{c,P}$ be the graph obtained from G_c by adding edges to vertices that correspond to the same place at infinity T_P. In other words c is acyclic if and only if there is no new cycles in $G_{c,P}$, that is to say $H_1(G_c) \cong H_1(G_{c,P})$ for all P in Dic_c.

(iii) Another interpretation is the following: c is acyclic if and only if the morphism h' of the diagram $(\mathcal{D})$ is surjective. This can be proved with the exact sequence:

$$H_1(T_\infty) \oplus H_1(T_{\mathit{aff}}) \xrightarrow{h'} H_1(T_c) \xrightarrow{\varphi} H_0(T_\infty \cap T_{\mathit{aff}}) \xrightarrow{\psi}$$
$$\xrightarrow{\psi} H_0(T_\infty) \oplus H_0(T_{\mathit{aff}}) \longrightarrow H_0(T_c) \longrightarrow 0.$$

(iv) Let consider the above Mayer-Vietoris exact sequence in reduced homology, the morphism $\widetilde{\psi} : \widetilde{H}_0(T_\infty \cap T_{\mathit{aff}}) \longrightarrow \widetilde{H}_0(T_\infty) \oplus \widetilde{H}_0(T_{\mathit{aff}})$ is surjective because $\widetilde{H}_0(T_c) = \{0\}$. Moreover $\widetilde{\psi}$ is injective if and only if ψ is injective. As $\widetilde{\psi}$ is surjective, $\widetilde{\psi}$ is injective if and only if $\mathrm{rk}\, \widetilde{H}_0(T_\infty \cap T_{\mathit{aff}}) = \mathrm{rk}\, \widetilde{H}_0(T_\infty) + \mathrm{rk}\, \widetilde{H}_0(T_{\mathit{aff}})$, but T_∞ has exactly $\#\mathrm{Dic}_c$ connected components (see paragraph 1.4). Then c is acyclic if and only if

$$\sum_{P \in \mathrm{Dic}_c} n(F_P) \;\; - \;\; 1 = \#\mathrm{Dic}_c - 1 + n(F_c) - 1. \qquad (\star)$$

This implies the lemma:

Lemma 1.1. *j_∞ is injective $\Longleftrightarrow$ F_c is a connected set and c is acyclic.*

Proof. If j_∞ is injective then $n(F_P) = 1$ for all P in Dic_c, then $H_0(T_\infty \cap T_{\mathit{aff}}) \cong H_0(T_\infty)$ and ψ is injective, hence c is acyclic and from equality $(\star)$, we have $n(F_c) = 1$ *i.e.* F_c is a connected set. Conversely, if c is acyclic and $n(F_c) = 1$ then equality $(\star)$ gives $n(F_P) = 1$ for all P in Dic_c, thus j_∞ is injective. $\square$

Let us define a stronger notion of acyclicity. Let $\bar{G}_c$ be the dual graph of $\phi^{-1}(c)$. The graph $\bar{G}_c$ can be obtained from G_c by adding edges between vertices that belong to the same place at infinity for all P in Dic_c. The value c is *strongly acyclic* if $H_1(\bar{G}_c) \cong H_1(G_c)$. Strong acyclicity implies acyclicity, but the converse can be false. However if F_c is a connected set (that is to say G_c is a connected graph) then both conditions are equivalent. This is implicitly expressed in the next lemma, which is just a result involving graphs.

Lemma 1.2. $\operatorname{rk} H_1(\bar{G}_c) - \operatorname{rk} H_1(G_c) = \sum_{P \in \mathrm{Dic}_c} \big(n(F_P) - 1\big) \; - \; \big(n(F_c) - 1\big).$

1.6. Surjectivity

Part (B). *j_c surjective $\iff$ j_∞ surjective and c acyclic.*

Proof. Let us suppose that j_c is surjective then a version of the five lemma applied to diagram $(\mathcal{D})$ proves that j_∞ is surjective. As j_c and j_∞ are surjective, diagram $(\mathcal{D})$ implies that $h' : H_1(T_\infty) \oplus H_1(T_{\mathit{aff}}) \longrightarrow H_1(T_c)$ is surjective, that means that c is acyclic.

Conversely if j_∞ is surjective and c is acyclic then h' is surjective and diagram $(\mathcal{D})$ implies that j_c is surjective. □

1.7. Injectivity

Part (A). *j_c is injective $\iff$ F_c is a connected set and c is acyclic.*

It follows from lemma 1.1 and from the next lemma.

Lemma 1.3. *j_c injective $\iff$ j_∞ injective.*
Moreover the rank of the kernel is:

$$\operatorname{rk}\ker j_c = \operatorname{rk}\ker j_\infty = \sum_{P \in \mathrm{Dic}_c} \big(n(F_P) - 1\big) = n(F_c) - 1 + \operatorname{rk} H_1(\bar{G}_c) - \operatorname{rk} H_1(G_c).$$

Proof. The first part of this lemma can be proved by a version of the five lemma. However we shall only prove the equality of the ranks of $\ker j_c$ and $\ker j_\infty$. It will imply the lemma because we already know that $\operatorname{rk}\ker j_\infty = \sum_{P \in \mathrm{Dic}_c} \big(n(F_P) - 1\big)$ (see paragraph 1.4) and from lemma 1.2 we then have $\operatorname{rk}\ker j_\infty = n(F_c) - 1 + \operatorname{rk} H_1(\bar{G}_c) - \operatorname{rk} H_1(G_c)$.

The study of the morphism $j_c : H_1(F_c) \longrightarrow H_1(T_c)$ is equivalent to the study of the morphism $H_1(T_{\mathit{aff}}) \longrightarrow H_1(T_c)$ induced by inclusion that, by abuse, will also be denoted by j_c. To see this, it suffices to remark that F_c is obtained from $F_{\mathit{aff}} = F_c \cap B^4_R$ by gluing $F_c \cap S^3_R \times [0, +\infty[$ to its boundary $F_c \cap S^3_R$. We then have two homotopy equivalences $F_c \supset F_c \cap T_{\mathit{aff}} \subset T_{\mathit{aff}}$. Then the morphism $H_1(F_{\mathit{aff}}) \longrightarrow$

$H_1(F_c)$ induced by inclusion is an isomorphism; finally $j_{aff} : H_1(F_{aff}) \longrightarrow H_1(T_{aff})$ is also an isomorphism. The long exact sequence for the pair (T_c, T_{aff}) is:

$$H_2(T_c) \longrightarrow H_2(T_c, T_{aff}) \longrightarrow H_1(T_{aff}) \xrightarrow{j_c} H_1(T_c)$$

but $H_2(T_c) = 0$ (see [1] for example) then the rank of $\ker j_c$ is the rank of $H_2(T_c, T_{aff})$.

On the other hand, the study of $j_\infty : H_1(F_\infty) \longrightarrow H_1(T_\infty)$ is the same as the study of $H_1(\partial T_\infty) \longrightarrow H_1(T_\infty)$ induced by inclusion (and denoted by j_∞) because the morphisms $H_1(\partial F_\infty) \longrightarrow H_1(F_\infty)$ and $H_1(\partial F_\infty) \longrightarrow H_1(\partial T_\infty)$ induced by inclusions are isomorphisms. The long exact sequence for $(T_\infty, \partial T_\infty)$ is:

$$H_2(T_\infty) \longrightarrow H_2(T_\infty, \partial T_\infty) \longrightarrow H_1(\partial T_\infty) \xrightarrow{j_\infty} H_1(T_\infty).$$

As $H_2(T_\infty) = 0$ (see [1]), then the rank of $\ker j_\infty$ is equal to $\operatorname{rk} H_2(T_\infty, \partial T_\infty)$.

Finally the groups $H_2(T_\infty, \partial T_\infty)$ and $H_2(T_c, T_{aff})$ are isomorphic by excision, then the ranks of $\ker j_c$ and of $\ker j_\infty$ are equal. That completes the proof. □

1.8. Bijectivity

Part (C). *j_c is an isomorphism $\iff c \notin \mathcal{B}_\infty$*

Proof. If $c \notin \mathcal{B}_\infty$, F_c is homotopic to F_{aff} and T_c to T_{aff} then the isomorphism $j_{aff} : H_1(F_{aff}) \longrightarrow H_1(T_{aff})$ implies that j_c is an isomorphism. Let suppose that c is a critical value at infinity and that j_c is injective. We have to prove that j_c is not surjective. As j_c is injective then by lemma 1.3, j_∞ is injective. By the part (B) it suffices to prove that j_∞ is not surjective. Let P be a point of Dic_c that provides irregularity at infinity for the value c, then $n(F_P) = 1$ because j_∞ is injective. Let us prove that the morphism j_P is not surjective. For the case of $P \in \bar{F}_P$, the intersection multiplicity $\mathfrak{m}_P$ is greater than 1, then j_P is not surjective. For the second case, in which P belongs to a bamboo, we have $\mathfrak{m}_P.\ell_i > 1$ except for the situation where only one strict transform intersects the bamboo at the last component (remember that $\ell_i = 1$ if and only if we are at the last component of a bamboo). This is exactly the situation excluded by the lemma "bamboo extremity fiber" of [6]. Hence j_∞ is not surjective and j_c is not an isomorphism. □

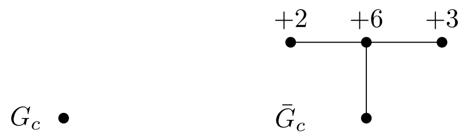
1.9. Examples

We apply the results to two classical examples.

Broughton polynomial. Let $f(x, y) = x(xy + 1)$, then $\mathcal{B}_{aff} = \varnothing$, $\mathcal{B} = \mathcal{B}_\infty = \{0\}$. Then for $c \neq 0$, j_c is an isomorphism. For the value 0, F_0 contains two connected components $(x = 0)$ and $(xy + 1 = 0)$ so G_c is composed of two vertices. The dual graph $\bar{G}_0$ contains also a vertex that corresponds to a component with multiplicity +1 of a bamboo. The value 0 is acyclic since $H_1(G_0) \cong H_1(\bar{G}_0)$. The fiber F_0 is not connected hence j_0 is not injective. As the new component of $\bar{G}_0$ is of multiplicity 1 the corresponding morphism j_∞ is surjective, hence j_0 is surjective.

Briançon polynomial. Let $f(x,y) = yp^3 + p^2 s + a_1 ps + a_0 s$ with $s = xy+1$, $p = x(xy+1)+1$, $a_1 = -\frac{5}{3}$, $a_0 = -\frac{1}{3}$, see [2]. The bifurcation set is $\mathcal{B} = \mathcal{B}_\infty = \{0, c = -\frac{16}{9}\}$, moreover all fibers are smooth and irreducible. The value 0 is not acyclic then j_0 is neither injective nor surjective (but j_∞ is surjective).

The value c is acyclic, and F_c is connected (since irreducible) then j_c is injective. The morphism j_c is not surjective: j_∞ is not surjective because the compactification of F_c does not intersect the bamboo at the last component.



2. Situation around an irregular fiber

For $f : \mathbb{C}^n \longrightarrow \mathbb{C}$ we study the neighborood of an irregular fiber.

2.1. Smooth part of F_c

Let fix a value $c \in \mathbb{C}$ and let B_R^{2n} be a large closed ball ($R \gg 1$). Let $B_1^{2n}, \ldots, B_p^{2n}$ be small open balls around the singular points (which are supposed to be isolated) of F_c : $F_c \cap \mathrm{Sing}$. We denote $B_1^{2n} \cup \ldots \cup B_p^{2n}$ by $B_\cup$. Then the *smooth part* of F_c is

$$F_c^\circ = F_c \cap (B_R^{2n} \setminus B_\cup).$$

It is possible to embed F_c° in the generic fiber F_{gen} (see [6] and [8]). We now explain the construction of this embedding by W. Neumann and P. Norbury. As F_c has transversal intersection with the balls of $B_\cup$ and with B_R^{2n}, there exists a small disk $D^2_\varepsilon(c)$ such that, for all s in this disk, F_s has transversal intersection with these balls. According to Ehresmann fibration theorem, f induces a locally trivial fibration

$$f_| : f^{-1}\big(D^2_\varepsilon(c)\big) \cap (B^{2n}_R \setminus B_\cup) \longrightarrow D^2_\varepsilon(c).$$

In fact, as $D^2_\varepsilon(c)$ is null homotopic, this fibration is trivial. Hence $F^\circ_c \times D^2_\varepsilon(c)$ is diffeomorphic to $f^{-1}(D^2_\varepsilon(c)) \cap (B^{2n}_R \setminus B_\cup)$. That provides an embedding of F°_c in F_s for all s in $D^2_\varepsilon(c)$; and for such a s with $s \neq c$, F_s is a generic fiber. The morphism induced in homology by this embedding is denoted by ℓ_c. Let j°_c be the morphism induced by the inclusion of F°_c in $T_c = f^{-1}(D^2_\varepsilon(c))$. Similarly k_c denotes the morphism induced by the inclusion of the generic fiber $F_{gen} = F_s$ (for $s \in D^2_\varepsilon(c)$, $s \neq c$) in T_c. As all morphisms are induced by natural maps we have the lemma:

Lemma 2.1. *The following diagram commutes:*

$$\begin{array}{ccc} H_q(F^\circ_c) & \xrightarrow{j^\circ_c} & H_q(T_c)\ . \\ {\scriptstyle \ell_c}\downarrow & \nearrow {\scriptstyle k_c} & \\ H_q(F_{gen}) & & \end{array}$$

2.2. Invariant cycles by h_c

Invariant cycles by the monodromy h_c can be recovered by the following property.

Proposition 2.2.
$$\operatorname{Ker}\big(h_c - \mathrm{id}\big) = \ell_c\big(H_q(F^\circ_c)\big).$$

For $n = 2$, there is a similar formula in [6], even for non-isolated singularities.

Proof. The proof uses a commutative diagram due to W. Neumann and P. Norbury [8]:

$$\begin{array}{ccc} H_q(F_{gen}, F^\circ_c) & \xrightarrow[\psi]{\sim} & V_q(c) \\ {\scriptstyle \varphi}\uparrow & & \downarrow {\scriptstyle i}\ \subset \\ H_q(F_{gen}) & \xrightarrow{\mathrm{id} - h_c} & H_q(F_{gen}) \end{array}$$

The morphism i is the inclusion and ψ is an isomorphism, so $\operatorname{Ker}(h_c - \mathrm{id})$ equals $\operatorname{Ker}\varphi$. The long exact sequence for the pair (F_{gen}, F°_c) is:

$$\cdots \longrightarrow H_q(F^\circ_c) \xrightarrow{\ell_c} H_q(F_{gen}) \xrightarrow{\varphi} H_q(F_{gen}, F^\circ_c) \longrightarrow \cdots$$

So $\operatorname{Im}\ell_c = \operatorname{Ker}\varphi = \operatorname{Ker}(h_c - \mathrm{id})$. □

We are able to applicate this result to the calculus of the rank of $\operatorname{Ker}(h_c - \mathrm{id})$ in $H_1(F_{gen})$ for $n = 2$. Let denote the number of irreducible components in F_c by $r(F_c)$, and let Sing_c be $\operatorname{Sing} \cap F_c$: the affine singularities on F_c. Let B^4_x be a small closed ball around a singular point x of F_c. Then $H_2(F_{gen}, F^\circ_c) = \bigoplus_{x \in \operatorname{Sing}_c} H_2(F_{gen} \cap B^4_x, F_{gen} \cap \partial B^4_x) = \bigoplus_{x \in \operatorname{Sing}_c} H_2(S^2)$, then the rank of $H_2(F_{gen}, F^\circ_c)$ is the cardinal of Sing_c, which is also the rank of $\operatorname{Ker} \ell_c$ (by the exact sequence of the proof of proposition 2.2). Moreover $\chi(F_c) = \operatorname{rk} H_1(F^\circ_c) - r(F_c) - \# \operatorname{Sing}_c$.

$$\begin{aligned}
\operatorname{rk} \operatorname{Ker} \big(h_c - \mathrm{id}\big) &= \operatorname{rk} \operatorname{Im} \ell_c \\
&= \operatorname{rk} H_1(F^\circ_c) - \operatorname{rk} \operatorname{Ker} \ell_c \\
&= r(F_c) - \chi(F_c) + \# \operatorname{Sing}_c - \# \operatorname{Sing}_c \\
&= r(F_c) - \chi(F_c).
\end{aligned}$$

Remark. We obtain the following fact (see [6]): if the fiber F_c ($c \in \mathcal{B}$) is irreducible then $h_c \neq \mathrm{id}$. The proof is as follows: if $r(F_c) = 1$ and $h_c = \mathrm{id}$ then from one hand $\operatorname{rk} \operatorname{Ker}(h_c - \mathrm{id}) = \operatorname{rk} H_1(F_{gen}) = 1 - \chi(F_{gen})$ and from the other hand $\operatorname{rk} \operatorname{Ker}(h_c - \mathrm{id}) = 1 - \chi(F_c)$; thus $\chi(F_c) = \chi(F_{gen})$ which is absurd for c in $\mathcal{B}$ by Suzuki formula, see [4].

2.3. Vanishing cycles

Now and until the end of this paper homology is homology with complex coefficients.

Vanishing cycles for eigenvalues $\lambda \neq 1$. Let E_λ be the space $E_\lambda = \operatorname{Ker}(h_c - \lambda \, \mathrm{id})^p$ for a large integer p.

Lemma 2.3. *If $\lambda \neq 1$ then $E_\lambda \subset V_q(c)$.*

Proof. If $\sigma \in H_q(F_{gen})$ then $h_c(\sigma) - \sigma \in V_q(c)$. This is just the fact that the cycle $h_c(\sigma) - \sigma$ corresponds to the boundary of a "tube" defined by the action of the geometrical monodromy. We remark that this fact can be generalized for $j \geqslant 1$ to

$$h^j_c(\sigma) - \sigma \in V_q(c).$$

Let p be an integer that defines E_λ, then for $\sigma \in E_\lambda$:

$$\begin{aligned}
0 = (h_c - \lambda \, \mathrm{id})^p(\sigma) &= \sum_{j=0}^{p} \binom{p}{j} (-\lambda)^{p-j} h^j_c(\sigma) \\
&= \sum_{j=0}^{p} \binom{p}{j} (-\lambda)^{p-j} \big(h^j_c(\sigma) - \sigma\big) + \sum_{j=0}^{p} \binom{p}{j} (-\lambda)^{p-j} \sigma \\
&= \sum_{j=0}^{p} \binom{p}{j} (-\lambda)^{p-j} \big(h^j_c(\sigma) - \sigma\big) + (1-\lambda)^p \sigma.
\end{aligned}$$

Each $h_c^j(\sigma)-\sigma$ is in $V_q(c)$, and a sum of such elements is also in $V_q(c)$, then $(1-\lambda)^p\sigma \in V_q(c)$. As $\lambda \neq 1$, $\sigma \in V_q(c)$. □

Vanishing cycles for the eigenvalue $\lambda = 1$. Let recall that vanishing cycles $V_q(c) = \operatorname{Ker} k_c$ for the value c, are cycles that "disappear" when the generic fiber tends to the fiber F_c. Hence cycles that will not vanish are cycles that already exist in F_c. From lemma 2.3, a cycle of E_λ with eigenvalue $\lambda \neq 1$ is a vanishing cycle. We now study what happens for cycles associated to the eigenvalue 1.

Let $(\tau_1, \ldots, \tau_p)$ be a family of $H_q(F_{gen})$ such that the matrix of h_c in this family is:

$$\begin{pmatrix} 1 & 1 & & & (0) \\ & 1 & 1 & & \\ & & 1 & \ddots & \\ & (0) & & \ddots & 1 \\ & & & & 1 \end{pmatrix}.$$

Then, the cycles $\tau_1, \ldots, \tau_{p-1}$ are vanishing cycles. It is a simple consequence of the fact that $h_c(\sigma) - \sigma \in V_q(c)$, because for $i = 1, \ldots, p-1$, we have $h_c(\tau_{i+1}) - \tau_{i+1} = \tau_i$, and then τ_i is a vanishing cycle. It remains the study of the cycle τ_p and the particular case of Jordan blocks (1) of size 1×1. We will start with the second part.

Vanishing and invariant cycles. Let $K_q(c)$ be the space of invariant and vanishing cycles for the value c: $K_q(c) = \operatorname{Ker}(h_c - \mathrm{id}) \cap V_q(c)$. Let us remark that the space $K_q(c) \oplus \bigoplus_{c' \neq c} V_q(c')$ is not equal to $\operatorname{Ker}(h_c - \mathrm{id})$. But equality holds in cohomology, see [7].

Lemma 2.4. $K_q(c) = \ell_c(\operatorname{Ker} j_c^\circ)$.

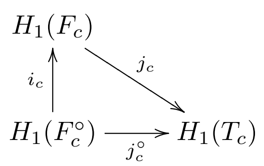
This lemma just follows from the description of invariant cycles (proposition 2.2) and from the diagram of lemma 2.1. For $n = 2$ we can calculate the dimension of $K_1(c)$.

Proposition 2.5. *For $n = 2$,* $\operatorname{rk} K_1(c) = r(F_c) - 1 + \operatorname{rk} H_1(\bar{G}_c)$.

Proof. The proof will be clear after the following remarks:

(i) $K_1(c) = \ell_c(\operatorname{Ker} j_c^\circ)$, by lemma 2.4.

(ii) $j_c^\circ = j_c \circ i_c$ with $i_c : H_1(F_c^\circ) \longrightarrow H_1(F_c)$ the morphism induced by inclusion. It is consequence of the commutative diagram:



(iii) $\operatorname{rk}\operatorname{Ker} j_c^\circ = \operatorname{rk}\operatorname{Ker} i_c + \operatorname{rk}(\operatorname{Ker} j_c \cap \operatorname{Im} i_c)$, which is a general formula for the kernel of the composition of morphisms.

(iv) $\operatorname{Ker} j_c \cap \operatorname{Im} i_c = \operatorname{Ker} j_c$; we have the composition of maps:

$$H_1(F_c^\circ) \xrightarrow{i_c \oplus 0} (\operatorname{Im} i_c \oplus H_1(G_c)) = H_1(F_c) \xrightarrow{j_c} H_1(T_c),$$

and the restriction of j_c, $j_c : H_1(G_c) \longrightarrow H_1(T_c)$ is injective, so $\operatorname{Ker} j_c \subset \operatorname{Im} i_c$.

(v) $\operatorname{rk}\operatorname{Ker} i_c = \sum_{z \in \operatorname{Sing}_c} r(F_{c,z})$, where $F_{c,z}$ denotes the germ of the curve F_c at z.

(vi) $\operatorname{rk}\operatorname{Ker} j_c = \operatorname{rk}\operatorname{Ker} j_\infty = \sum_{P \in \operatorname{Dic}_c}(n(F_P) - 1) = n(F_c) + \operatorname{rk} H_1(\bar{G}_c) - \operatorname{rk}(G_c)$, it has been proved in lemma 1.3.

(vii) $r(F_c) + \operatorname{rk} H_1(G_c) = n(F_c) + \sum_{z \in \operatorname{Sing}_c}(r(F_{c,z}) - 1)$. This a general formula for the graph G_c, the number of vertices of G_c is $r(F_c)$, the number of connected components is $n(F_c)$, the number of loops is $\operatorname{rk} H_1(G_c)$ and the number of edges for a vertex that correspond to an irreducible component F_{irr} of F_c is: $\sum_{z \in F_{irr}}(r(F_{irr,z}) - 1)$.

(viii) $\operatorname{rk} K_1(c) = \operatorname{rk}\operatorname{Ker} j_c^\circ - \#\operatorname{Sing}_c$ because $\operatorname{Ker} i_c$ is a subspace of $\operatorname{Ker} \ell_c$ so $\operatorname{rk} K_1(c) = \operatorname{rk}\operatorname{Ker} j_c^\circ - \operatorname{rk}\operatorname{Ker} \ell_c$ and the dimension of $\operatorname{Ker} \ell_c$ is $\#\operatorname{Sing}_c$ (see paragraph 2.2).

We complete the proof:

$$\begin{aligned}
\operatorname{rk} K_1(c) &= \operatorname{rk} \ell_c(\operatorname{Ker} j_c^\circ) && \text{by (i)}\\
&= \operatorname{rk}\operatorname{Ker} j_c^\circ - \operatorname{rk}\operatorname{Ker} \ell_c && \text{by (viii)}\\
&= \operatorname{rk}\operatorname{Ker} j_c \circ i_c - \#\operatorname{Sing}_c && \text{by (ii) and (viii)}\\
&= \operatorname{rk}\operatorname{Ker} i_c + \operatorname{rk}\operatorname{Ker} j_c \cap \operatorname{Im} i_c - \#\operatorname{Sing}_c && \text{by (iii)}\\
&= \operatorname{rk}\operatorname{Ker} i_c - \#\operatorname{Sing}_c + \operatorname{rk}\operatorname{Ker} j_c && \text{by (iv)}\\
&= \sum_{z \in \operatorname{Sing}_c} \big(r(F_{c,z}) - 1\big) + n(F_c) + \operatorname{rk} H_1(\bar{G}_c) - \operatorname{rk}(G_c) && \text{by (v) and (vi)}\\
&= r(F_c) - 1 + \operatorname{rk} H_1(\bar{G}_c). && \text{by (vii)} \quad \square
\end{aligned}$$

Filtration. Let ϕ be the map provided by the total resolution of f. The divisor $\phi^{-1}(c)$ is denoted by $D = \sum_i m_i D_i$ where m_i stands for the multiplicity of D_i. We associate to D_i a part of the generic fiber denoted by F_i. We briefly recall this construction (see [6]), let $V = \phi^{-1}(D_\varepsilon^2(c))$ be a tubular neighborhood of D, we will identify the generic fiber F_{gen} with $\phi^{-1}(s) \setminus \pi^{-1}(L_\infty)$ for a generic value $s \in \partial D_\varepsilon^2(c)$, π is the blow-up associated to ϕ. There is a natural deformation retraction $R : V \longrightarrow D$, and we set $F_i = R^{-1}(D_i) \cap F_{gen}$. The *filtration* of the homology of the generic fiber is the sequence of inclusions:

$$W_{-1} \subset W_0 \subset W_1 \subset W_2 = H_1(F_{gen}),$$

with

- W_{-1}: the *boundary cycles*, that is to say, if $\bar{F}_{gen}$ is the compactification of F_{gen} and $\iota_* : H_1(F_{gen}) \longrightarrow H_1(\bar{F}_{gen})$ is induced by inclusion then $W_{-1} = \operatorname{Ker} \iota_*$;
- W_0: these are *gluing cycles*: the homology group on the components of $F_i \cap F_j$ $(i \neq j)$;
- W_1: the direct sum of the $H_1(F_i)$;
- $W_2 = H_1(F_{gen})$.

The subspaces W_0 and W_1 depend on the value c.

Jordan blocks for $n = 2$. For polynomials in two variables, the size of Jordan blocks for the monodromy h_c is less or equal to 2. Let denote by σ and τ cycles of $H_1(F_{gen})$ such that $h(\sigma) = \sigma$ and $h(\tau) = \sigma + \tau$. The matrix of h_c for the family (σ, τ) is $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. We already know that the cycle σ vanishes.

A *large cycle* is a cycle of $W_2 = H_1(F_{gen})$ that has a non-trivial class in W_2/W_1. According to [6] τ is large cycle; moreover large cycles associated to the eigenvalue 1 are the embedding of $H_1(\bar{G}_c)$ in $H_1(F_{gen})$. So large cycles are not vanishing cycles. We have $\operatorname{rk} W_2/W_1$ equal to $\operatorname{rk} H_1(\bar{G}_c)$, this is also the number of Jordan 2-blocks for the eigenvalue 1.

Vanishing cycles. We are now able to describe vanishing cycles. For all the spaces W_{-1}, W_0/W_{-1}, W_1/W_0 and W_2/W_1 the cycles associated to eigenvalues different from 1 are vanishing cycles (lemma 2.3).

Proposition 2.6. *We have*

$$\operatorname{Ker}(h_c - \mathrm{id})^2 \cap V_1(c) = \operatorname{Ker}(h_c - \mathrm{id}) \cap V_1(c) =: K_1(c)$$

and vanishing cycles for the eigenvalue 1 *are dispatch as follows:*

- *for W_{-1}: $r(F_c) - 1$ cycles (*i. e. $\operatorname{rk} W_{-1} \cap K_1(c) = r(F_c) - 1$*),*
- *for W_0: $\operatorname{rk} H_1(\bar{G}_c)$ other cycles (*i. e. $\operatorname{rk}(W_0 \cap K_1(c))/(W_{-1} \cap K_1(c)) = \operatorname{rk} H_1(\bar{G}_c)$*),*
- *for W_1, no other cycle (*i. e. $\operatorname{rk}(W_1 \cap K_1(c))/(W_0 \cap K_1(c)) = 0$*),*
- *for W_2, no other cycle (*i. e. $\operatorname{rk}(W_2 \cap K_1(c))/(W_1 \cap K_1(c)) = 0$*).*

Proof. We have already remarked that large cycles (like τ) associated to $\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$ are not vanishing cycles, so vanishing cycles in W_2 are in W_1, that is to say $W_2 \cap \operatorname{Ker}(h_c - \mathrm{id})^2 \cap V_1(c) = W_1 \cap \operatorname{Ker}(h_c - \mathrm{id})^2 \cap V_1(c)$. Then the other vanishing cycles for the eigenvalue 1 are invariant cycles by h_c, in other words we proved that $\operatorname{Ker}(h_c - \mathrm{id})^2 \cap V_1(c) = K_1(c)$. We have $W_1 \cap K_1(c) = W_0 \cap K_1(c)$ because invariant cycles for W_1 that are not in W_0 correspond to the genus of the smooth part F_c° of F_c (this is due to the equality $\operatorname{Ker}(h_c - \mathrm{id}) = \ell_c(H_1(F_c^\circ)))$. As they already appear in F_c, these cycles are not vanishing cycles for the value c. Moreover there are $\operatorname{rk} H_1(\bar{G}_c)$ Jordan 2-blocks for the eigenvalue 1 that provide $\operatorname{rk} H_1(\bar{G}_c)$ vanishing cycles (like σ) in W_0/W_{-1}. Finally, proposition 2.5 proves that $\operatorname{rk} W_{-1} \cap K_1(c) = r(F_c) - 1$. □

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