On the existence of weak solutions to the Cauchy problem for a class of quasilinear hyperbolic equations with a source term

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ABSTRACT

Following the ideas of D. Serre and J. Shearer in [16], we prove in this paper the existence of a weak solution of the Cauchy problem for the second order quasilinear hyperbolic equation

$$\phi_{tt} - \sigma'(\phi_x)\phi_{xx} + F(\phi) = 0, \quad (x,t) \in \mathbb{R} \times [0, +\infty[,$$

where $F(\phi)$ is a suitable source term.

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1. Introduction and main results.

This paper presents a study of the initial values problem for the second order quasilinear equation

$$\phi_{tt} - \sigma'(\phi_x)\phi_{xx} + F(\phi) = 0, \qquad (x,t) \in \mathbb{R} \times [0, +\infty[, \tag{1}$$

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following the work of J. P. Dias and M. Figueira, who studied this problem in [4], considering particular F and σ, namely

$$F(\phi) = \phi^3 \text{ and } \sigma(u) = u + \frac{u^3}{3}. \tag{2}$$

Previously, P. Marcati and R. Natalini proved in [9] a result of existence of a Lipschitz continuous solution to the Cauchy problem for equation (1) with bounded, compactly supported initial data, in the L^∞ framework, by using an approximating scheme of Lax-Friedrichs kind, and imposing some restrictions on F, namely $F(0) = 0$ and F' bounded.

Here, we generalize these authors' work and we prove the existence of weak solution for equation (1), with initial data

$$\phi(x,0) = \phi_0(x) \in H^3(\mathbb{R}), \quad \phi_t(x,0) = \phi_1(x) \in H^2(\mathbb{R}).$$

To this purpose, we follow the method of D. Serre and J. Shearer ([16]), who proved, by using the compensated compactness method developed by F. Murat, L. Tartar and R. DiPerna ([11], [18], [5]) and L^η Young measures, the existence of weak solution to the Cauchy problem for the hyperbolic system of conservation laws

$$\begin{cases} u_t - v_x = 0, \\ v_t - \sigma'(u)u_x = 0. \end{cases} \tag{3}$$

We consider $F : \mathbb{R} \longrightarrow \mathbb{R}$ a smooth function such that $F(0) = 0$, $F'(\phi) \geq 0$, $\forall \phi \in \mathbb{R}$, and $|F(\phi)| \leq c_1|\phi^p|$, for some $c_1 > 0$, $p \geq 1$. We put $G(\phi) = \int_0^\phi F(\theta)d\theta$.

The function $\sigma : \mathbb{R} \longrightarrow \mathbb{R}$ is in the same conditions of [16], a smooth function such that $\sigma(0) = 0$ and satisfying the following hypotheses:

H1 $\exists\, c > 0 : \sigma'(u) \geq c$, $\forall\, u \in \mathbb{R}$;

H2 $\sigma''(\lambda) \neq 0$, $\forall \lambda \in \mathbb{R}$, or $\exists\ \lambda_0 \in \mathbb{R} : \sigma''(\lambda_0) = 0$, $\sigma''(\lambda) \neq 0$, $\forall \lambda \neq \lambda_0$;

H3 $\dfrac{\sigma''}{(\sigma')^{5/4}}, \dfrac{\sigma'''}{(\sigma')^{7/4}} \in L^2(\mathbb{R})$; $\dfrac{\sigma''}{(\sigma')^{3/2}}, \dfrac{\sigma'''}{(\sigma')^2} \in L^\infty(\mathbb{R})$;

H4 We define $\Sigma(u) = \int_0^u \sigma(s)ds$. $\dfrac{\sigma(u)}{\Sigma(u)} \longrightarrow 0$, $|u| \to +\infty$ and there are m and q, $q > 1/2$, such that $(\sigma'(u))^q \leq m(1 + \Sigma(u))$.

We point out that, under these hypotheses, $G(\phi) \geq 0$, $\forall \phi$, and $\Sigma(u) \geq c\frac{u^2}{2}$. It is easy to check that the functions F and σ defined by (2) satisfy all these conditions and that H3–H4 hold for any σ with a suitable polynomial like behaviour.

The Cauchy problem for equation (1) will be considered in the following equivalent formulation: we put $u = \phi_x$, $v = \phi_t$; then (1) reduces to the quasilinear system

$$\begin{cases} u_t - v_x = 0, \\ v_t - \sigma'(u)u_x + F(\phi) = 0, \quad \phi(x,t) = \int_0^t v(x,\tau)d\tau + \phi_0(x). \end{cases} \tag{4}$$

We consider the Cauchy problem for this system with initial data

$$u(\cdot,0) = \phi_{0x}(\cdot,0) = u_0, \qquad v(\cdot,0) = \phi_1(\cdot,0) = v_0, \tag{5}$$

$$\phi_0 \in H^3(\mathbb{R}), \qquad u_0, v_0 \in H^2(\mathbb{R}). \tag{6}$$

Let

$$E(u,v) = \int_{\mathbb{R}} \frac{v^2(x)}{2} + \Sigma(u(x))dx$$

be the energy functional and, setting $\eta(u,v) = \dfrac{v^2}{2} + \Sigma(u)$, we consider

$$L^\eta = \left\{(u,v) \in (L^1_{loc}(\mathbb{R}))^2 \; : E(u,v) < +\infty\right\}$$

the space of functions with finite energy. Let $L^\infty([0,+\infty[; L^\eta)$ be the space of the pairs of functions (u,v), defined a. e. and measurable in $[0,+\infty[\times\mathbb{R}$, such that $(u(t),v(t)) \in L^\eta$, a. e. $t \in [0,+\infty[$, and $\operatorname*{ess\,sup}_{[0,+\infty[} E(u(t),v(t)) < +\infty$.

A pair of functions $(u,v) \in L^\infty([0,+\infty[; L^\eta)$ is called a **weak solution** of the Cauchy problem (4), (5), if

$$\int_{\mathbb{R}} \int_0^{+\infty} (u\varphi_t - v\varphi_x)dxdt + \int_{\mathbb{R}} u_0\varphi(x,0)dx + \\ \int_{\mathbb{R}} \int_0^{+\infty} (v\psi_t - \sigma(u)\psi_x - F(\phi)\psi)dxdt + \int_{\mathbb{R}} v_0\psi(x,0)dx = 0, \tag{7}$$

for any $\varphi, \psi \in C_0^\infty(\mathbb{R} \times [0,+\infty[)$.

A pair of functions $p, q \; : \; \mathbb{R}^2 \longrightarrow \mathbb{R}$ is an **entropy-entropy flux pair** for the system (4), if all smooth solutions (u,v) of (4) also satisfy

$$p(u,v)_t + q(u,v)_x + \nabla p \cdot (0, F(\phi)) = 0.$$

It is sufficient that p and q satisfy

$$\nabla p(u,v) \cdot \nabla f(u,v) = \nabla q(u,v), \quad \forall \; (u,v) \in \mathbb{R}^2, \tag{8}$$

where $f(u,v) = (-v, -\sigma(u))$.

We call (u,v) a **weak entropy solution** of (4), (5), if (u,v) is a weak solution that also satisfies

$$p(u,v)_t + q(u,v)_x + \nabla p(u,v)\cdot(0,F(\phi)) \leq 0, \tag{9}$$

in the sense of distributions in $\mathbb{R}\times]0,+\infty[$, for any convex entropy p of flux q.

We present now the main result of this work:

Theorem 1.1. *We assume the above conditions for F and σ. If u_0 and v_0 satisfy* (6) *and $(u_0,v_0)\in L^\eta$, then there is a global weak solution (u,v) of the Cauchy problem* (4), (5) *in $L^\infty([0,+\infty[;L^\eta)$ that satisfies the entropy inequality* (9) *for the entropy-entropy flux pair defined by*

$$p(u,v) = \eta(u,v) = \frac{v^2}{2} + \Sigma(u), \quad q(u,v) = -v\sigma(u). \tag{10}$$

To prove this result, we consider a sequence of viscosity functions $(u_\varepsilon,v_\varepsilon)$, solutions of the approximated system

$$\begin{cases} u_{\varepsilon t} - v_{\varepsilon x} = 0, \\ v_{\varepsilon t} - \sigma'(u_\varepsilon)u_{\varepsilon x} + F(\phi_\varepsilon) = \varepsilon\Delta v_\varepsilon, \quad \phi_\varepsilon(x,t) = \int_0^t v_\varepsilon(x,\tau)d\tau + \phi_0(x), \end{cases} \tag{11}$$

which is obtained by adding the viscosity parameter $\varepsilon\Delta\phi_t$ to the second member of (1).

In section 2 we prove the existence of global solution $(u_\varepsilon,v_\varepsilon)$ in $C([0,+\infty[;H^2(\mathbb{R})^2)\cap C^1([0,+\infty[;L^2(\mathbb{R})^2)$ of the Cauchy problem for system (11), with initial data

$$u_\varepsilon(\cdot,0) = \phi_{0x} = u_0, \quad v_\varepsilon(\cdot,0) = \phi_1 = v_0, \tag{12}$$

In section 3 we derive energy estimates for the approximated solutions u_ε and v_ε, which allow us to conclude that the sequence $(u_\varepsilon,v_\varepsilon)_\varepsilon$ is bounded in $L^2_{loc}(\mathbb{R}\times[0,+\infty[)$ and so we may consider a subsequence $(u_{\varepsilon'},v_{\varepsilon'})_{\varepsilon'}$ converging weakly to $(u,v)\in(L^2_{loc}(\mathbb{R}\times[0,+\infty[))^2$. Our aim is to prove that the pair (u,v) is a global weak solution of the Cauchy problem (4), (5).

If we write the weak formulation of (11), (12),

$$\int_{\mathbb{R}}\int_0^{+\infty}(u_\varepsilon\varphi_t - v_\varepsilon\varphi_x)dxdt + \int_{\mathbb{R}} u_0\,\varphi(x,0)dx + \int_{\mathbb{R}}\int_0^{+\infty}(v_\varepsilon\psi_t - \sigma(u_\varepsilon)\psi_x - F(\phi_\varepsilon)\psi)dxdt + \int_{\mathbb{R}} v_0\,\psi(x,0)dx = -\varepsilon\int_{\mathbb{R}}\int_0^{+\infty} v_\varepsilon\psi_{xx}, \tag{13}$$

we see that, if $(u_{\varepsilon'}, v_{\varepsilon'}) \rightharpoonup (u, v)$, weakly in $L^2_{loc}(\mathbb{R}\times[0,+\infty[)^2$, the linear terms in the previous equation clearly converge to the correspondent terms in the equation (7). But the uniform bound in L^2 is not enough to warrant the strong local convergence of the subsequence $(u_{\varepsilon'}, v_{\varepsilon'})_{\varepsilon'}$, and the weak convergence doesn't allow us to pass to the limit the nonlinear terms $\sigma(u_\varepsilon)$ and $F(\phi_\varepsilon)$. We use the associated Young measure to represent the weak limit of the nonlinear compositions $g(u_\varepsilon, v_\varepsilon)$, of continuous functions g with $(u_\varepsilon, v_\varepsilon)$. Since L^∞ estimates are not available in this case, we follow Serre and Shearer's method ([16]), who used L^η Young measures and a class of slowly growing entropy-entropy flux pairs to prove the existence of solution of the Cauchy problem for equation (3) with physical viscosity. The Young measure gives a criteria to know when the weak convergence is, in fact, strong, which happens if the measure is a Dirac mass. The theory of compensated compactness provides the compacity conditions to conclude the strong local convergence of $(u_{\varepsilon'}, v_{\varepsilon'})_{\varepsilon'}$. By applying Murat's lemma and div-curl lemma, we derive Tartar's equation. The results obtained by Serre and Shearer imply the reduction of the support of the Young measure.

2. The approximated problem.

In this section we consider the Cauchy problem for the approximated system (11), with initial data defined by (12), where $\phi_0 \in H^3(\mathbb{R})$, $\phi_1 \in H^2(\mathbb{R})$, and σ and F as described above.

We will prove that the Cauchy problem for the nonlinear parabolic equation

$$\phi_{tt} - \sigma'(\phi_x)\phi_{xx} + F(\phi) = \varepsilon\triangle\phi_t, \quad x\in\mathbb{R}, \quad t\geq 0, \tag{14}$$

with initial data

$$\phi(\cdot,0) = \phi_0, \quad \phi_t(\cdot,0) = \phi_1, \tag{15}$$

has a unique global solution

$$\phi_\varepsilon \in C([0,+\infty[;H^3(\mathbb{R}))\cap C^1([0,+\infty[;H^2(\mathbb{R}))\cap C^2([0,+\infty[;L^2(\mathbb{R})).$$

In this conditions, if we put $u_\varepsilon = \phi_{\varepsilon x}$, $v_\varepsilon = \phi_{\varepsilon t}$, we conclude that $(u_\varepsilon, v_\varepsilon) \in C([0,+\infty[;H^2(\mathbb{R})^2)\cap C^1([0,+\infty[;L^2(\mathbb{R})^2)$ is the unique solution of the Cauchy problem (11), (12).

The proof that we present here generalizes to $\mathbb{R}$ the results obtained by J. Greenberg, R. Mac Camy and V. Mizel ([8]) for the viscoelasticity equations in the interval $[0,1]$, and follows these authors and J. P. Dias' ideas, who proves in [3] a result of global existence of strong solution for a similar problem in two space dimensions, considering radial symmetric initial data.

By using a classical fix point method, we begin to prove the following result of local existence:

Theorem 2.1. *Let $\phi_0 \in H^3(\mathbb{R})$ and $\phi_1 \in H^2(\mathbb{R})$. Then, there exists $T_0 > 0$ such that the Cauchy problem* (14), (15) *has a unique solution in $C([0,T_0];H^3(\mathbb{R})) \cap C^1([0,T_0];H^2(\mathbb{R})) \cap C^2([0,T_0];L^2(\mathbb{R}))$.*

Proof. For simplicity, we consider $\varepsilon = 1$. Let us assume that $\phi_0 \in H^3(\mathbb{R})$, $\phi_1 \in H^2(\mathbb{R})$, and let $(S(t))_{t\geq 0}$ be the semigroup of operators of $H^{-1}(\mathbb{R})$ associated to the heat equation in $\mathbb{R}$.

We will use the following result (cf. [2], [13]):

If $\varphi \in H^1(\mathbb{R})$, there exists $c > 0$ such that

$$\phi(t) = S(t)\varphi \in C([0,+\infty[;H^1(\mathbb{R})) \cap C^1([0,+\infty[;H^{-1}(\mathbb{R}))$$

satisfies

$$\|\nabla\phi(t)\|_{L^2(\mathbb{R})} \leq \frac{c}{\sqrt{2t}}\|\varphi\|_{L^2(\mathbb{R})}, \qquad \forall\, t > 0, \tag{16}$$

$$\|\Delta\phi(t)\|_{L^2(\mathbb{R})} \leq \frac{c}{\sqrt{2t}}\|\nabla\varphi\|_{L^2(\mathbb{R})}, \qquad \forall\, t > 0. \tag{17}$$

Let us put, for $t > 0$,

$$\widetilde{\psi}(t) = \int_0^t S(\tau)\phi_1 d\tau + \phi_0.$$

We have

$$\widetilde{\psi}_t = S(t)\phi_1, \quad \widetilde{\psi}_x = \int_0^t S(\tau)\phi_{1x} d\tau + \phi_{0x},$$

$$\widetilde{\psi}_{xx} = \int_0^t S(\tau)\phi_{1xx} d\tau + \phi_{0xx}$$

and, since $\phi_{1x} \in H^1(\mathbb{R})$,

$$\begin{aligned}\Delta\widetilde{\psi}_x(t) &= \int_0^t \Delta(S(\tau)\phi_{1x})d\tau + \phi_{0xxx} = \int_0^t \frac{\partial}{\partial\tau}(S(\tau)\phi_{1x})d\tau + \phi_{0xxx}\\ &= S(t)\phi_{1x} - \phi_{1x} + \phi_{0xxx} \quad \text{(cf. [2])}.\end{aligned}$$

Hence, $\widetilde{\psi} \in C([0,+\infty[;H^3(\mathbb{R}))$, $\widetilde{\psi}_x \in C([0,+\infty[;H^2(\mathbb{R}))$ and $\widetilde{\psi}_t \in C([0,+\infty[;H^2(\mathbb{R}))$.

Let us consider, for $T > 0$,

$$X_T = \{\psi \in C([0,T];H^3(\mathbb{R})) \cap C^1([0,T];H^2(\mathbb{R})) \,:\, \|\psi - \widetilde{\psi}\|_{X_T} \leq M\},$$

where $\|\psi\|_{X_T} = \max_{[0,T]}\|\psi(t)\|_{H^3(\mathbb{R})} + \max_{[0,T]}\|\psi_t(t)\|_{H^2(\mathbb{R})}$ and M is a positive constant such that $\|\widetilde{\psi}\| \leq M$. We will prove that there exists $T_0 > 0$ such that the problem

$$\begin{cases}\dfrac{\partial}{\partial t}\phi_t - \Delta\phi_t = f(\phi), \quad f(\phi) = \sigma'(\phi_x)\phi_{xx} - F(\phi),\\ \phi(\cdot,0) = \phi_0, \quad \phi_t(\cdot,0) = \phi_1,\end{cases} \tag{18}$$

has a solution $\phi \in X_{T_0}$.

In order to do this, we consider, for a given $\psi \in X_T$, the linear problem in X_T

$$\begin{cases} \dfrac{\partial}{\partial t}\phi_t - \Delta\phi_t = f(\psi), \\ \phi(\cdot,0) = \phi_0,\ \phi_t(\cdot,0) = \phi_1. \end{cases} \tag{19}$$

Since $(f(\psi))_x = \sigma'(\psi_x)\psi_{xxx} + \sigma''(\psi_x)\psi_{xx}^2 - F'(\psi)\psi_x$, and due to the inclusion $H^1(\mathbb{R}) \subseteq L^\infty(\mathbb{R})$, we conclude that $f(\psi) \in C([0,T]; H^1(\mathbb{R}))$ and so (19) has a unique solution $\phi = \mathcal{T}(\psi)$ in $[0,T]$,

$$\phi(t) = \int_0^t \phi_t(\tau)d\tau \,+\, \phi_0,$$

where

$$\phi_t(t) = S(t)\phi_1 \,+\, \int_0^t S(t-\tau)\,(f(\psi))\,(\tau)d\tau.$$

Next, we prove that there exists $T' > 0$ such that, for each $T < T'$, $\mathcal{T}(X_T) \subseteq X_T$. Let $\psi \in X_T$ and $0 < t \leq T$. For $\phi = \mathcal{T}(\psi)$ defined as above, we conclude from (16) and (17) that

$$\begin{aligned} \|\phi_t(t) - \widetilde{\psi}_t(t)\|_{H^2(\mathbb{R})} &= \left\| \int_0^t S(t-\tau)\,(f(\psi))\,(\tau)d\tau \right\|_{H^2(\mathbb{R})} \\ &\leq \int_0^t \frac{1}{\sqrt{2(t-\tau)}} \|\,(f(\psi))\,(\tau)\|_{H^1(\mathbb{R})} d\tau \leq g(t)\,C(M) \end{aligned} \tag{20}$$

and

$$\begin{aligned} \|\phi(t) - \widetilde{\psi}(t)\|_{H^2(\mathbb{R})} &= \left\| \int_0^t (\phi_t(\tau) - \widetilde{\psi}_t(\tau))d\tau \right\|_{H^2(\mathbb{R})} \\ &\leq \int_0^t \|\phi_t(\tau) - \widetilde{\psi}_t(\tau)\|_{H^2(\mathbb{R})} d\tau \leq g(t)\,C(M), \end{aligned} \tag{21}$$

where g is an increasing continuous function such that $g(0) = 0$ and $C(M)$ is a continuous function of M.

In order to estimate $\|\phi_x(t) - \widetilde{\psi}_x(t)\|_{H^2(\mathbb{R})}$, we point out that

$$\frac{\partial}{\partial t}\left[\left(\frac{\partial}{\partial t} - \Delta\right)\phi_x\right] = \frac{\partial}{\partial x}\left[\left(\frac{\partial}{\partial t} - \Delta\right)\phi_t\right] \,=\, (f(\psi))_x\,,$$

and so

$$\phi_{xt} - \Delta\phi_x = \int_0^t (f(\psi))_x\,(\tau)d\tau + \phi_{1x} - \Delta\phi_{0x}.$$

As a consequence of the above considerations, we obtain

$$
\begin{aligned}
(\phi_x - \widetilde{\psi}_x) - \Delta(\phi_x - \widetilde{\psi}_x) &= \phi_x - \phi_{xt} + \phi_{1x} - \Delta\phi_{0x} + \int_0^t (f(\psi))_x(\tau) d\tau \\
&\quad - \int_0^t S(\tau)\phi_{1x} d\tau - \phi_{0x} + S(t)\phi_{1x} - \phi_{1x} + \Delta\phi_{0x} \\
&= \phi_x - \phi_{0x} + S(t)\phi_{1x} - \phi_{xt} + \int_0^t (f(\psi))_x(\tau) d\tau - \int_0^t S(\tau)\phi_{1x} d\tau \\
&= \phi_x - \phi_{0x} - \int_0^t S(t-\tau)(f(\psi))_x(\tau) d\tau \\
&\quad + \int_0^t (f(\psi))_x(\tau) d\tau - \int_0^t S(\tau)\phi_{1x} d\tau = h,
\end{aligned}
$$

which allow us to conclude that $(\phi_x - \widetilde{\psi}_x) - \Delta(\phi_x - \widetilde{\psi}_x) \in L^2(\mathbb{R})$, because $h \in L^2(\mathbb{R})$, since, again by (16) and (17), we deduce that

$$
\begin{aligned}
&\|(\phi_x - \widetilde{\psi}_x) - \Delta(\phi_x - \widetilde{\psi}_x)\|_{L^2(\mathbb{R})} = \|h\|_{L^2(\mathbb{R})} \leq \\
&\quad \leq \|\phi_x - \phi_{0x}\|_{L^2(\mathbb{R})} + \int_0^t \|S(t-\tau)(f(\psi))_x(\tau)\|_{L^2(\mathbb{R})} d\tau \\
&\quad\quad + \int_0^t \|(f(\psi))_x(\tau)\|_{L^2(\mathbb{R})} d\tau + \int_0^t \|S(\tau)\phi_{1x}\|_{L^2(\mathbb{R})} d\tau \\
&\quad \leq g(t)C(M),
\end{aligned} \tag{22}
$$

because

$$
\begin{aligned}
\|\phi_x - \phi_{0x}\|_{L^2(\mathbb{R})} &= \left\| \int_0^t \phi_{tx}(\tau) d\tau \right\|_{L^2(\mathbb{R})} \leq \int_0^t \|\phi_{tx}(\tau)\|_{L^2(\mathbb{R})} d\tau \leq \\
&\leq \int_0^t \|\phi_{tx}(\tau) - S(\tau)\phi_{1x}\|_{L^2(\mathbb{R})} d\tau + \int_0^t \|S(\tau)\phi_{1x}\|_{L^2(\mathbb{R})} d\tau \\
&\leq g(t)C(M).
\end{aligned}
$$

By Fourier transform we obtain

$$
\|\phi_x - \widetilde{\psi}_x\|_{H^2(\mathbb{R})} \leq c\|(\phi_x - \widetilde{\psi}_x) - \Delta(\phi_x - \widetilde{\psi}_x)\|_{L^2(\mathbb{R})} \leq g(t)C(M). \tag{23}
$$

We can now choose $T' > 0$ such that $g(T')C(M) \leq M$ and, from (20), (21) and (23), we obtain

$$
\|\phi(t) - \widetilde{\psi}(t)\|_{H^3(\mathbb{R})} \leq M, \quad \|\phi_t(t) - \widetilde{\psi}_t(t)\|_{H^2(\mathbb{R})} \leq M,
$$

for all $0 < t < T'$. Hence, if $0 < T < T'$, $\mathcal{T}(X_T) \subseteq X_T$.

Now we have that, for given $\psi, \overline{\psi} \in X_T$ $(T < T')$, $\phi = \mathcal{T}(\psi)$ and $\overline{\phi} = \mathcal{T}(\overline{\psi})$ satisfy

$$\begin{aligned}
&\|\phi(t) - \overline{\phi}(t)\|_{H^2(\mathbb{R})} + \|\phi_t(t) - \overline{\phi}_t(t)\|_{H^2(\mathbb{R})} \leq \\
&\quad \leq \int_0^t \|\phi_t(\tau) - \overline{\phi}_t(\tau)\|_{H^2(\mathbb{R})} d\tau + \int_0^t \frac{1}{\sqrt{2(t-\tau)}} \|(f(\psi)(\tau) - f(\overline{\psi})(\tau))\|_{H^1(\mathbb{R})} d\tau \\
&\quad \leq g(T)\, C(M) \left(\max_{[0,T]} \|\psi(t) - \overline{\psi}(t)\|_{H^3(\mathbb{R})} + \max_{[0,T]} \|\psi_t(t) - \overline{\psi}_t(t)\|_{H^2(\mathbb{R})} \right).
\end{aligned}$$

If we proceed in the same way that we did to obtain (22), we get that

$$\begin{aligned}
&\|\phi_x(t) - \overline{\phi}_x(t)\|_{H^2(\mathbb{R})} \leq \\
&\qquad g(T)\, C(M) \left(\max_{[0,T]} \|\psi(t) - \overline{\psi}(t)\|_{H^3(\mathbb{R})} + \max_{[0,T]} \|\psi_t(t) - \overline{\psi}_t(t)\|_{H^2(\mathbb{R})} \right). \qquad (24)
\end{aligned}$$

Hence

$$\begin{aligned}
&\max_{[0,T]} \|\phi(t) - \overline{\phi}(t)\|_{H^3(\mathbb{R})} + \max_{[0,T]} \|\phi_t(t) - \overline{\phi}_t(t)\|_{H^2(\mathbb{R})} \\
&\qquad \leq g(T)\, C(M) \left(\max_{[0,T]} \|\psi(t) - \overline{\psi}(t)\|_{H^3(\mathbb{R})} + \max_{[0,T]} \|\psi_t(t) - \overline{\psi}_t(t)\|_{H^2(\mathbb{R})} \right),
\end{aligned}$$

and we can choose $T_0 < T'$ such that $g(T_0)C(M) < 1$ and so $\mathcal{T} : X_{T_0} \longrightarrow X_{T_0}$ is a strict contraction in the complete normed space X_{T_0}, hence it has a unique fix point $\phi = \mathcal{T}(\phi)$, which is the unique solution of the Cauchy problem (14), (15). □

Remark. Using the same notations as above, we point out that T_0 depends only on M which depends only on the initial data ϕ_0 and ϕ_1. In consequence, since $T_0 < T'$, $g(T_0)C(M) < 1$ and $g(T')C(M) \leq M$, we conclude that there is a minimal instant $T_M > 0$ such that the Cauchy problem for equation (14) has solution in $[0, T_M]$, whatever the functions ϕ_0 and ϕ_1 such that $\|\phi_0\| \leq M$, $\|\phi_1\| \leq M$ that we consider for initial data are.

We present now the main result of this section:

Theorem 2.2. *Given $\phi_0 \in H^3(\mathbb{R})$ and $\phi_1 \in H^2(\mathbb{R})$, the Cauchy problem* (14), (15) *has a unique solution in* $C([0,+\infty[; H^3(\mathbb{R})) \cap C^1([0,+\infty[; H^2(\mathbb{R})) \cap C^2([0,+\infty[; L^2(\mathbb{R}))$.

In order to prove this result we will obtain the following estimate for a solution ϕ of (14), (15):

$$\|\phi(t)\|_{H^3(\mathbb{R})} + \|\phi_t(t)\|_{H^2(\mathbb{R})} + \|\phi_{tt}(t)\|_{L^2(\mathbb{R})} \leq c(t), \qquad (25)$$

where $c(t)$ is a positive continuous function.

Let $\phi \in C([0,T[; H^3(\mathbb{R})) \cap C^1([0,T[; H^2(\mathbb{R})) \cap C^2([0,T[; L^2(\mathbb{R}))$ be a solution of (14), (15) in $[0,T[$.

By multiplying equation (14) by ϕ_t, integrating in $\mathbb{R}$, integrating by parts and integrating in $[0,t]$, $(0<t<T)$, we obtain

$$\int_{\mathbb{R}} \left(\frac{{\phi_t}^2}{2} + \Sigma(\phi_x) + G(\phi) \right)(x,t)dx + \int_0^t \int_{\mathbb{R}} {\phi_{tx}}^2(x,\tau)dxd\tau = C, \tag{26}$$

where C depends only on the initial data ϕ_0 and ϕ_1.

We now assume that $\phi \in C^2([0,T[;H^2(\mathbb{R}))$ (cf. [10]). By multiplying equation (14) by ϕ_{xx}, integrating in $\mathbb{R}$ and integrating by parts, we get

$$-\int_{\mathbb{R}} \frac{d}{dt}(\phi_{tx}\phi_x) + \int_{\mathbb{R}} {\phi_{tx}}^2 - \int_{\mathbb{R}} \sigma'(\phi_x){\phi_{xx}}^2 - \int_{\mathbb{R}} F'(\phi){\phi_x}^2 = \frac{d}{dt}\left(\int_{\mathbb{R}} \frac{{\phi_{xx}}^2}{2} \right).$$

Integrating in $[0,t]$, we obtain

$$\int_{\mathbb{R}} \frac{{\phi_{xx}}^2}{2}(x,t)dx = \\ -\int_{\mathbb{R}} (\phi_{tx}\phi_x)(x,t)dx + \int_0^t \int_{\mathbb{R}} \left({\phi_{tx}}^2 - \sigma'(\phi_x){\phi_{xx}}^2 - F'(\phi){\phi_x}^2 \right) dxd\tau + C.$$

Hence, by (26) and since $\sigma'(u) > 0$, $F'(\phi) \geq 0$, $\forall\ u$, $\forall\ \phi$,

$$\begin{aligned} \int_{\mathbb{R}} \frac{{\phi_{xx}}^2}{2} &\leq -\int_{\mathbb{R}} \phi_{tx}\phi_x + C = \int_{\mathbb{R}} \phi_t\phi_{xx} + C \\ &\leq \int_{\mathbb{R}} {\phi_t}^2 + \int_{\mathbb{R}} \frac{{\phi_{xx}}^2}{4} + C \leq \int_{\mathbb{R}} \frac{{\phi_{xx}}^2}{4} + C, \end{aligned}$$

and so

$$\int_{\mathbb{R}} {\phi_{xx}}^2(x,t)dx \ \leq\ C. \tag{27}$$

As we have

$$c\int_{\mathbb{R}} \frac{{\phi_x}^2}{2} \ \leq\ \int_{\mathbb{R}} \Sigma(\phi_x),$$

from (26) and (27) we deduce that

$$\|\phi_x(\cdot,t)\|_{H^1(\mathbb{R})} \ \leq\ C \quad \text{ and so } \quad \|\phi_x(\cdot,t)\|_{L^\infty(\mathbb{R})} \ \leq\ C.$$

From (26) we also obtain that $\phi(t) = \displaystyle\int_0^t \phi_t(\tau)d\tau + \phi_0$ is such that $\|\phi\|_{L^2\mathbb{R}} \ \leq\ c(t)$, and then

$$\|\phi(\cdot,t)\|_{H^2(\mathbb{R})} \ \leq\ c(t). \tag{28}$$

Now we derivate equation (14) in order to t, multiply by ϕ_{tt}, integrate in $\mathbb{R}$ and integrate by parts. We get

$$\frac{d}{dt}\left(\int_{\mathbb{R}} \frac{{\phi_{tt}}^2}{2}\right) + \int_{\mathbb{R}} \sigma'(\phi_x)\phi_{tx}\phi_{ttx} = -\int_{\mathbb{R}} F'(\phi)\phi_t\phi_{tt} - \int_{\mathbb{R}} \phi_{ttx}^2.$$

By the previous estimates and from (26) and (28) we obtain

$$\begin{aligned}\left|\int_{\mathbb{R}} \sigma'(\phi_x)\phi_{tx}\phi_{ttx}\right| &\leq \int_{\mathbb{R}} (\sigma'(\phi_x))^2 \frac{{\phi_{tx}}^2}{2} + \int_{\mathbb{R}} \frac{{\phi_{ttx}}^2}{2}\\ &\leq c_1 \int_{\mathbb{R}} \frac{{\phi_{tx}}^2}{2} + \int_{\mathbb{R}} \frac{{\phi_{ttx}}^2}{2},\\ \left|\int_{\mathbb{R}} F'(\phi)\phi_t\phi_{tt}\right| &\leq \int_{\mathbb{R}} \frac{{\phi_{tt}}^2}{2} + \int_{\mathbb{R}} \frac{(F'(\phi))^2{\phi_t}^2}{2}\\ &\leq \int_{\mathbb{R}} \frac{{\phi_{tt}}^2}{2} + c(t),\end{aligned}$$

and, again by (26),

$$\frac{d}{dt}\int_{\mathbb{R}} \frac{{\phi_{tt}}^2}{2} + \int_{\mathbb{R}} \phi_{tx}\phi_{ttx} \leq c(t) + \int_{\mathbb{R}} {\phi_{tt}}^2 + c_1 \int_{\mathbb{R}} {\phi_{tx}}^2.$$

Integrating the above inequality in $[0, t]$, we have

$$\int_{\mathbb{R}} \frac{{\phi_{tt}}^2}{2} + \int_{\mathbb{R}} \frac{{\phi_{tx}}^2}{2} \leq c(t) + \int_0^t \left(\int_{\mathbb{R}} \frac{{\phi_{tt}}^2}{2} + c_1 \int_{\mathbb{R}} \frac{{\phi_{tx}}^2}{2}\right),$$

and by Gronwall's lemma we conclude that

$$\int_R \frac{{\phi_{tt}}^2}{2}(x,t) + \int_R \frac{{\phi_{tx}}^2}{2}(x,t) \leq c(t). \tag{29}$$

Since ϕ is a solution of equation (14) and $|F(\phi)| \leq c_1|\phi|^p$ $(p \geq 1)$, we have

$$\int_{\mathbb{R}} (F(\phi))^2 \leq {c_1}^2 \int_{\mathbb{R}} \phi^{2p} \leq c(t)$$

and then

$$\int_{\mathbb{R}} {\phi_{txx}}^2(x,t) \leq c(t). \tag{30}$$

We estimate now ϕ_{xxx}. In order to do this, we use the following result, due to Gagliardo and Nirenberg (cf. [7]):

If $\phi \in H^3(\mathbb{R})$, then $\phi_{xx} \in L^4(\mathbb{R})$ and

$$\|\phi_{xx}\|_{L^4(\mathbb{R})} \leq c\|\phi_{xxx}\|_{L^2(\mathbb{R})}{}^{1/4}\|\phi_{xx}\|_{L^2(\mathbb{R})}{}^{3/4}.$$

If we derivate equation (14) in order to x, multiply by ϕ_{xxx}, integrate in $\mathbb{R}$ and integrate by parts, we obtain

$$-\frac{d}{dt}\left(\int_{\mathbb{R}}\phi_{txx}\phi_{xx}\right)+\int_{\mathbb{R}}{\phi_{txx}}^2-\int_{\mathbb{R}}\sigma''(\phi_x){\phi_{xx}}^2\phi_{xxx} -\int_{\mathbb{R}}\sigma'(\phi_x){\phi_{xxx}}^2+\int_{\mathbb{R}}F'(\phi)\phi_x\phi_{xxx}=\frac{d}{dt}\left(\int_{\mathbb{R}}\frac{{\phi_{xxx}}^2}{2}\right),$$

and so

$$\frac{d}{dt}\left(\int_{\mathbb{R}}\frac{{\phi_{xxx}}^2}{2}\right)\leq-\frac{d}{dt}\left(\int_{\mathbb{R}}\phi_{txx}\phi_{xx}\right) +\int_{\mathbb{R}}{\phi_{txx}}^2+\int_{\mathbb{R}}F'(\phi)\phi_x\phi_{xxx}-\int_{\mathbb{R}}\sigma''(\phi_x){\phi_{xx}}^2\phi_{xxx}. \tag{31}$$

Now, from (27) and Gagliardo-Nirenberg inequality, we have

$$\left|\int_{\mathbb{R}}\sigma''(\phi_x){\phi_{xx}}^2\phi_{xxx}\right|\leq\|\sigma''(\phi_x)\|_{L^\infty(\mathbb{R})}\|\phi_{xx}\|^2_{L^4(\mathbb{R})}\|\phi_{xxx}\|_{L^2(\mathbb{R})} \leq c\|\phi_{xxx}\|^{3/2}_{L^2(\mathbb{R})}\|\phi_{xx}\|^{3/2}_{L^2(\mathbb{R})}\leq c\left(1+\|\phi_{xxx}\|^2_{L^2(\mathbb{R})}\right).$$

By integrating inequality (31) in $[0,t]$, we obtain

$$\int_{\mathbb{R}}\frac{{\phi_{xxx}}^2}{2}(x,t)\leq c(t)\int_0^t\int_{\mathbb{R}}{\phi_{xxx}}^2(x,\tau)dxd\tau+c(t),$$

and, again by Gronwall's lemma,

$$\int_{\mathbb{R}}\frac{{\phi_{xxx}}^2}{2}(x,t)\leq c(t). \tag{32}$$

From (26), (28), (29), (30) and (32) we deduce (25).

Proof of Theorem 2.2. Let $T^*=\sup\{T>0\,:\,\exists\,\phi\in X_T,\text{ solution of (14), (15)}\}$. By theorem 2.1, $T^*>0$, and by the property of unicity we can consider a maximal solution of (14), (15),

$$\phi\in C([0,T^*[;H^3(\mathbb{R}))\cap C^1([0,T^*[;H^2(\mathbb{R}))\cap C^2([0,T^*[;L^2(\mathbb{R})).$$

If $T^*<+\infty$, from (25), we have that $\forall\ 0<t<T^*$,

$$\|\phi(t)\|_{H^3(\mathbb{R})}+\|\phi_t(t)\|_{H^2(\mathbb{R})}+\|\phi_{tt}(t)\|_{L^2(\mathbb{R})}\ \leq\ c(t)\ \leq\ M^*,$$

where $M^*=\max\limits_{[0,T^*]}c(t)$. According to the remark that follows the proof of Theorem 2.1, there exists T_{M^*} such that, for all $0<t<T_{M^*}$, the Cauchy problem for equation (14) with initial data $\phi(\cdot,t)$, $\phi_t(\cdot,t)$, has a solution in $[0,T_{M^*}]$. In these conditions, it is possible to extend the solution ϕ into a bigger time interval, which contradicts the definition of T^*. Hence, $T^*=+\infty$. □

3. Young measures and reduction of their support.

We begin this section with the following energy estimates:

Lemma 3.1. *The approximated solutions u_ε and v_ε satisfy, for all $t > 0$,*

$$\int_{\mathbb{R}} \left(\frac{{v_\varepsilon}^2}{2} + \Sigma(u_\varepsilon) + G(\phi_\varepsilon) \right)(x,t)dx \leq \int_{\mathbb{R}} \left(\frac{{v_0}^2}{2} + \Sigma(u_0) + G(\phi_0) \right)(x)dx, \tag{33}$$

$$\begin{aligned} \varepsilon \int_0^t \int_{\mathbb{R}} (\sigma'(u_\varepsilon){u_{\varepsilon x}}^2 + {v_{\varepsilon x}}^2)(x,\tau)dxd\tau \leq \\ 3 \int_{\mathbb{R}} \left(\frac{{v_0}^2}{2} + \Sigma(u_0) + G(\phi_0) \right)(x)dx + \varepsilon^2 \int_{\mathbb{R}} {u_{0x}}^2(x)dx. \end{aligned} \tag{34}$$

Proof. By multiplying the first equation of (11) by $\sigma(u_\varepsilon)$, the second by v_ε and adding both equations, we obtain, since $v_\varepsilon = \phi_{\varepsilon t}$,

$$\frac{d}{dt}\left(\frac{{v_\varepsilon}^2}{2} \right) + \frac{d}{dt}\left(\Sigma(u_\varepsilon) \right) - (\sigma'(u_\varepsilon)v_\varepsilon)_x + \frac{d}{dt}(G(\phi_\varepsilon)) = \varepsilon \Delta v_\varepsilon v_\varepsilon.$$

Integrating the above equation in $\mathbb{R}$ and then by parts, we get

$$\int_{\mathbb{R}} \frac{d}{dt} \left(\frac{{v_\varepsilon}^2}{2} + \Sigma(u_\varepsilon) + G(\phi_\varepsilon) \right) + \varepsilon \int_{\mathbb{R}} {v_{\varepsilon x}}^2 = 0.$$

If we now integrate this equation in $[0,t]$, we obtain

$$\begin{aligned} \int_{\mathbb{R}} \left(\frac{{v_\varepsilon}^2}{2} + \Sigma(u_\varepsilon) + G(\phi_\varepsilon) \right)(x,t)dx + \varepsilon \int_0^t \int_{\mathbb{R}} {v_{\varepsilon x}}^2(x,\tau)dxd\tau = \\ \int_{\mathbb{R}} \left(\frac{{v_0}^2}{2} + \Sigma(u_0) + G(\phi_0) \right)(x)dx, \end{aligned} \tag{35}$$

and (33) follows.

In order to prove (34), we follow Serre and Shearer's ideas ([16]). Since $v_{\varepsilon x} = u_{\varepsilon t}$, we have $\Delta v_\varepsilon = u_{\varepsilon xt}$ and $\phi_{\varepsilon xx} = u_{\varepsilon x}$. Hence, if we multiply the second equation of (11) by $u_{\varepsilon x}$ and integrate in $\mathbb{R} \times [0,t]$, we have

$$\int_0^t \int_{\mathbb{R}} (u_{\varepsilon x} v_{\varepsilon t} - \sigma'(u_\varepsilon){u_{\varepsilon x}}^2) = \int_0^t \int_{\mathbb{R}} F'(\phi_\varepsilon){\phi_{\varepsilon x}}^2 + \varepsilon \int_0^t \int_{\mathbb{R}} \frac{d}{dt}\left(\frac{{u_{\varepsilon x}}^2}{2} \right),$$

and, since $u_{\varepsilon xt} = v_{\varepsilon xx}$, we get

$$\int_0^t \int_{\mathbb{R}} u_{\varepsilon x} v_{\varepsilon t} = \int_0^t \int_{\mathbb{R}} ((v_\varepsilon u_{\varepsilon x})_t - v_\varepsilon u_{\varepsilon xt}) = \int_{\mathbb{R}} v_\varepsilon u_{\varepsilon x}|_0^t + \int_0^t \int_{\mathbb{R}} {v_{\varepsilon x}}^2,$$

and then

$$\int_{\mathbb{R}} v_\varepsilon u_{\varepsilon x}|_0^t + \int_0^t \int_{\mathbb{R}} {v_{\varepsilon x}}^2 - \frac{\varepsilon}{2} \int_{\mathbb{R}} {u_{\varepsilon x}}^2|_0^t = \int_0^t \int_{\mathbb{R}} \sigma'(u_\varepsilon) {u_{\varepsilon x}}^2 + \int_0^t \int_{\mathbb{R}} F'(\phi_\varepsilon) {\phi_{\varepsilon x}}^2.$$

Since $F'(\phi) \geq 0$, from the above equality we have

$$\int_0^t \int_{\mathbb{R}} \sigma'(u_\varepsilon) {u_{\varepsilon x}}^2 \leq \int_0^t \int_{\mathbb{R}} \sigma'(u_\varepsilon) {u_{\varepsilon x}}^2 + \int_0^t \int_{\mathbb{R}} F'(\phi_\varepsilon) {\phi_{\varepsilon x}}^2 \leq$$
$$\left(\int_{\mathbb{R}} {u_{\varepsilon x}}^2(t)\right)^{(1/2)} \left(\int_{\mathbb{R}} {v_\varepsilon}^2(t)\right)^{(1/2)} - \int_{\mathbb{R}} v_0 u_{0x} + \int_0^t \int_{\mathbb{R}} {v_{\varepsilon x}}^2 - \frac{\varepsilon}{2} \int_{\mathbb{R}} {u_{\varepsilon x}}^2|_0^t \leq$$
$$\frac{1}{2\varepsilon} \int_{\mathbb{R}} {v_\varepsilon}^2(t) + \frac{1}{2\varepsilon} \int_{\mathbb{R}} {v_0}^2 + \varepsilon \int_{\mathbb{R}} {u_{0x}}^2 + \int_0^t \int_{\mathbb{R}} {v_{\varepsilon x}}^2.$$

Hence,

$$\varepsilon \int_0^t \int_{\mathbb{R}} \sigma'(u_\varepsilon) {u_{\varepsilon x}}^2 \leq \frac{1}{2} \int_{\mathbb{R}} {v_\varepsilon}^2(t) + \frac{1}{2} \int_{\mathbb{R}} {v_0}^2 + \varepsilon^2 \int_{\mathbb{R}} {u_{0x}}^2 + \varepsilon \int_0^t \int_{\mathbb{R}} {v_{\varepsilon x}}^2.$$

The estimate (34) follows then from (35). □

We now present the theorem of existence of Young measures. For the proof and more details concerning this subject, we refer to [1] and [17].

Let $\mathcal{M}(\Omega)$ be the space of finite real Radon measures on Ω.

Theorem 3.2 (Young measures and representation of weak limits). *Let $\eta : \mathbb{R}^m \longrightarrow \mathbb{R}$ be a continuous positive function such that $\frac{1}{\eta(\lambda)} \to 0$, $|\lambda| \to +\infty$, and $U_\varepsilon = (U_{1\varepsilon}, \dots, U_{m\varepsilon})$ a sequence defined a. e. in $\mathbb{R} \times [0, +\infty[$ such that, for all compact set $K \subseteq \mathbb{R} \times [0, +\infty[$, $\exists\ C_K > 0 : \int_K \eta(U_\varepsilon(x,t)) dx dt \leq C_K$. Then there is a subsequence $(U_{\varepsilon'})_{\varepsilon'}$ and a weakly measurable family of nonnegative measures of $\mathcal{M}(\mathbb{R}^m)$, $\{\nu_{x,t}\}_{(x,t) \in \mathbb{R} \times [0,+\infty[}$, with mass equal to one a. e. $(x,t) \in \mathbb{R} \times [0, +\infty[$, such that*

(i) *For any continuous function $g : \mathbb{R}^m \longrightarrow \mathbb{R}$ such that $\frac{g(\lambda)}{\eta(\lambda)} \to 0$, $|\lambda| \to +\infty$, let*

$$\bar{g}(x,t) = \int_{\mathbb{R}^m} g(\lambda) d\nu_{x,t}(\lambda).$$

Then $\bar{g} \in L^1_{loc}(\mathbb{R} \times [0, +\infty[)$ and $g(U_{\varepsilon'}) \rightharpoonup \bar{g}$ in the weak topology of $L^1_{loc}(\mathbb{R} \times [0, +\infty[)$ induced by $C_c(\mathbb{R} \times [0, +\infty[)$, the space of continuous functions with compact support in $\mathbb{R} \times [0, +\infty[$.

(ii) *If $\frac{|\lambda|^q}{\eta(\lambda)} \to 0$, $|\lambda| \to +\infty$, and if the support of $\nu_{x,t}$ is a point a. e. $(x,t) \in \mathbb{R} \times [0, +\infty[$, then $U_{\varepsilon'} \to \bar{U}(x,t) = \int_{\mathbb{R}^m} \lambda d\nu_{x,t}(\lambda)$ in $L^q_{loc}(\mathbb{R} \times [0, +\infty[)$, $\nu_{x,t} = \delta_{\bar{U}(x,t)}$ and, if g is in the same conditions as above, $g(U_{\varepsilon'}) \to g(\bar{U})$ in $L^1_{loc}(\mathbb{R} \times [0, +\infty[)$.*

Let $\eta(u,v) = \dfrac{v^2}{2} + \Sigma(u),\ \forall\ u,v \in \mathbb{R}$. Since the approximated solutions $u_\varepsilon, v_\varepsilon$ satisfy the energy estimate (33), for all $t > 0$, we can we apply the Young measures theorem and associate to a subsequence $(u_{\varepsilon'}, v_{\varepsilon'})_{\varepsilon'}$ a family of Young measures $\{\nu_{x,t}\}_{x,t\in\mathbb{R}\times[0,+\infty[}$ that verify (i) and (ii) of theorem 3.2.

Since
$$\frac{u^2}{2} + \frac{v^2}{2} \leq \frac{1}{c}\Sigma(u) + \frac{v^2}{2},$$
it follows from (33) that $(u_{\varepsilon'}, v_{\varepsilon'})_{\varepsilon'}$ is bounded in $L^2_{loc}(\mathbb{R}\times[0,+\infty[)$, and then we may consider a subsequence, which will still be called $(u_{\varepsilon'}, v_{\varepsilon'})_{\varepsilon'}$, converging weakly in $L^2_{loc}(\mathbb{R}\times[0,+\infty[)$ to functions $(u,v) \in (L^2_{loc}(\mathbb{R}\times[0,+\infty[))^2$. Now, again from the above inequality, we see that, if the Young measures $\nu_{x,t}$ are Dirac measures, then, by (ii) of theorem 3.2, $\nu_{x,t} = \delta_{(u(x,t),v(x,t))}$ and $(u_{\varepsilon'}, v_{\varepsilon'}) \to (u,v)$, strongly in $L^q_{loc}(\mathbb{R}\times[0,+\infty[)$, for all $q < 2$.

Following [16], we state now Tartar's equation for two classes of entropy-entropy flux pairs, solutions of a Goursat problem for system (39). Since we don't have L^∞ estimates for the approximated solutions $(u_\varepsilon, v_\varepsilon)$, we can only use the above L^η Young measures and, in particular, in Tartar's equation below, we are restricted to use entropy-entropy flux pairs (p,q) that verify (ii) of theorem 3.2, which means that $|p/\eta|, |q/\eta| \to 0$.

Let (p,q) be an entropy-entropy flux pair. We have
$$\begin{cases} p_u + q_v = 0, \\ \sigma'(u)p_v + q_u = 0. \end{cases} \tag{36}$$

Since $\sigma'(u) \geq c > 0$, we can define a smooth increasing function
$$z(u) = \int_0^u \sqrt{\sigma'(s)}ds.$$
We change to a Riemann coordinate system (w_1, w_2) by defining
$$w_1(u,v) = v + z(u), \quad w_2(u,v) = v - z(u).$$
As in [17] we also consider the change of variables $(p,q) \longrightarrow (P,Q)$, defined by
$$p = \frac{1}{2}(\sigma')^{-1/4}(P+Q), \tag{37}$$
$$q = \frac{1}{2}(\sigma')^{1/4}(P-Q), \tag{38}$$
and rewrite equation (36) in the new coordinates:
$$\begin{cases} P_{w_1} = aQ, \\ Q_{w_2} = -aP, \end{cases} \tag{39}$$

where $a = a(w_1 - w_2) = \sigma''(z^{-1}(\frac{w_1-w_2}{2}))/8(\sigma'(z^{-1}(\frac{w_1-w_2}{2})))^{3/2}$.

We consider entropy-entropy flux pairs (p, q), given by (37), (38), where P and Q are solutions of a Goursat problem related to equation (39). The Goursat problem consists in solving system (39), with data in the lines $w_1 = \overline{w}_1$ and $w_2 = \overline{w}_2$, $(\overline{w}_1, \overline{w}_2) \in \mathbb{R}^2$:

$$P(\overline{w}_1, w_2) = g(w_2), \quad Q(w_1, \overline{w}_2) = h(w_1).$$

If g and h are regular then the Goursat problem has a unique solution (P, Q) with the same regularity and if g has his support contained in the set $\{w_2 \in \mathbb{R} : w_2 > \overline{w}_2\}$, $\overline{w}_2 \in \mathbb{R}$, then P and Q have their supports contained in the halfplane $\{(w_1, w_2) \in \mathbb{R}^2 : w_2 \geq \overline{w}_2\}$. For details concerning Goursat problem we refer [14] and [16].

We now state Tartar's equation, which is deduced by applying div-curl lemma to $p_1(u_\varepsilon, v_\varepsilon)$, $q_1(u_\varepsilon, v_\varepsilon)$, $p_2(u_\varepsilon, v_\varepsilon)$ and $q_2(u_\varepsilon, v_\varepsilon)$, where (p_1, q_1) and (p_2, q_2) are entropy-entropy flux pairs associated to P and Q, solutions of a Goursat problem for the system (39) with continuous, compactly supported Goursat data, or solutions of a Cauchy problem for this system with continuous, compactly supported initial data on the line $w_1 - w_2 = \xi_0$, ξ_0 constant.

Let (p, q) be an entropy-entropy flux pair. In order to apply div-curl lemma, we must prove that $(p(u_\varepsilon, v_\varepsilon))_t + (q(u_\varepsilon, v_\varepsilon))_x$ lies in a compact subset of $H^{-1}_{loc}(\mathbb{R} \times [0, +\infty[)$. Multiplying system (11) by $(p_u(u_\varepsilon, v_\varepsilon), p_v(u_\varepsilon, v_\varepsilon))$, we obtain

$$\begin{aligned}(p(u_\varepsilon, v_\varepsilon))_t + (q(u_\varepsilon, v_\varepsilon))_x &= \varepsilon p_v v_{\varepsilon xx} - p_v F(\phi_\varepsilon) \\ &= \varepsilon(p_v v_{\varepsilon x})_x - \varepsilon(p_{uv} u_{\varepsilon x} v_{\varepsilon x} + p_{vv} {v_{\varepsilon x}}^2) - p_v F(\phi_\varepsilon),\end{aligned}$$

where, in the second member, the derivatives refer to the point $(u_\varepsilon, v_\varepsilon)$.

To use Murat's lemma (cf. [6]) we need to have the following conditions:

M1 $(p(u_\varepsilon, v_\varepsilon) + q(u_\varepsilon, v_\varepsilon))_\varepsilon$ is uniformly bounded in $L^p_{loc}(\mathbb{R} \times [0, +\infty[)$, for some $p > 2$;

M2 $(\varepsilon(p_v v_{\varepsilon x})_x)_\varepsilon$ is precompact in $H^{-1}_{loc}(\mathbb{R} \times [0, +\infty[)$;

M3 $(\varepsilon(p_{uv} u_{\varepsilon x} v_{\varepsilon x} + p_{vv} {v_{\varepsilon x}}^2))_\varepsilon$ is uniformly bounded in $L^1_{loc}(\mathbb{R} \times [0, +\infty[)$;

M4 $(p_v F(\phi_\varepsilon))_\varepsilon$ is uniformly bounded in $L^1_{loc}(\mathbb{R} \times [0, +\infty[)$.

We remark that, if M1 holds, then $((p(u_\varepsilon, v_\varepsilon))_t + (q(u_\varepsilon, v_\varepsilon))_x)_\varepsilon$ is uniformly bounded in $W^{-1,p}_{loc}(\mathbb{R} \times [0, +\infty[)$, and, in M3 and M4, the bound in $L^1_{loc}(\mathbb{R} \times [0, +\infty[)$ implies a bound in $\mathcal{M}(\omega)$, for any open bounded set ω of $\mathbb{R} \times [0, +\infty[$. Then, if M1–M4 hold, we can apply Murat's lemma to $((p(u_\varepsilon, v_\varepsilon))_t + (q(u_\varepsilon, v_\varepsilon))_x)_\varepsilon$.

Theorem 3.3 (Tartar's equation). *Let (p_1, q_1) and (p_2, q_2) be entropy-entropy flux pairs, given by* (37), (38), *where P_1, Q_1, P_2 and Q_2 are either solutions of a Goursat problem for system* (39), *with continuous, compactly supported Goursat data, or are solutions of a Cauchy problem for the same system, with continuous,* *compactly supported initial data on the line $w_1 - w_2 = \xi_0$, ξ_0 constant. Then p_1, q_1, p_2 and q_2 satisfy Tartar's equation*

$$< \nu, p_1 q_2 - p_2 q_1 >=< \nu, p_1 >< \nu, q_2 > - < \nu, p_2 >< \nu, q_1 >, \tag{40}$$

where $\nu = \nu_{x,t}$ is the Young measure associated to the subsequence $(u_{\varepsilon'}, v_{\varepsilon'})_{\varepsilon'}$ of the approximated solutions, and $< \nu, p >= \int p(\lambda) d\nu(\lambda)$.

Proof. In the case of entropy-entropy flux pairs solutions of the Goursat problem, we have to prove M1–M4 and apply Murat's lemma and then div-curl lemma. The proof of M1–M3 is the same as in [17]. To obtain M4 we consider a compact set $K \subseteq \mathbb{R} \times [0, +\infty[$. If $t > 0$, we have

$$\begin{aligned} \|\phi_\varepsilon(\cdot, t)\|^2_{L^2(\mathbb{R})} = \int_{\mathbb{R}} {\phi_\varepsilon}^2(x, t) dx &\le C \int_{\mathbb{R}} \left(\int_0^t v_\varepsilon(x, \tau) d\tau \right)^2 + {\phi_0}^2 dx \\ &\le C \|\phi_0\|^2_{L^2(\mathbb{R})} + C \int_{\mathbb{R}} \int_0^t {v_\varepsilon}^2(x, \tau) dx d\tau \\ &\le C + c(t) \sup_{[0,t]} \|v_\varepsilon(\cdot, \tau)\|_{L^2(\mathbb{R})}. \end{aligned}$$

Then, from (33) follows that $\|\phi_\varepsilon(\cdot, t)\|_{L^2(\mathbb{R})} \le c(t)$, where c is a continuous function. Since ${\phi_\varepsilon}_x = u_\varepsilon$, we also obtain from this estimate that $\|{\phi_\varepsilon}_x(\cdot, t)\|_{L^2(\mathbb{R})} \le c$. Then we have $\|\phi_\varepsilon(\cdot, t)\|_{H^1(\mathbb{R})} \le c(t)$ and $\|\phi_\varepsilon\|_{L^\infty(K)} \le c(t)$. Since F is continuous, we have

$$\int_K |F(\phi_\varepsilon)| dx dt \le C,$$

hence $(F(\phi_\varepsilon))_\varepsilon$ is uniformly bounded in $L^1_{loc}(\mathbb{R} \times [0, +\infty[)$, and, since $(p_v(u_\varepsilon, v_\varepsilon))_\varepsilon$ is uniformly bounded in $L^\infty(\mathbb{R} \times [0, +\infty[)$ (cf. [17]), we obtain M4.

If M1–M4 hold, then Murat's lemma and div-curl lemma imply Tartar's equation.

To prove the case where P and Q are solutions of the Cauchy problem, we refer to [16]. □

Now, as in [17] for the case where σ'' is never null, or as in [16] for the case where σ'' is null only once, we have the following result:

Theorem 3.4 (Reduction of the support of ν). *The Young measure $\nu_{x,t}$ is a point mass.*

For the proof, see the references indicated above.

4. Convergence of the approximated solutions; Proof of theorem 1.1.

Let $(u_\varepsilon, v_\varepsilon) \in C([0, +\infty[; H^2(\mathbb{R})^2) \cap C^1([0, +\infty[; L^2(\mathbb{R})^2)$ be the solution of the Cauchy problem for the approximated system (11), with initial data (12).

Let us consider φ and $\psi \in C_0^\infty(\mathbb{R}\times[0,+\infty[)$. By multiplying the first equation of the system (11) by φ, the second by ψ, adding the resulting equations and integrating by parts in $\mathbb{R}\times[0,+\infty[$, we obtain that u_ε and v_ε satisfy the weak formulation of the Cauchy problem (11), (12),

$$\int_{\mathbb{R}}\int_0^{+\infty}(u_\varepsilon\varphi_t - v_\varepsilon\varphi_x)dxdt + \int_{\mathbb{R}} u_0\,\varphi(x,0)dx + \int_{\mathbb{R}}\int_0^{+\infty}(v_\varepsilon\psi_t - \sigma(u_\varepsilon)\psi_x - F(\phi_\varepsilon)\psi)dxdt + \int_{\mathbb{R}} v_0\,\psi(x,0)dx = -\,\varepsilon\int_{\mathbb{R}}\int_0^{+\infty} v_\varepsilon\psi_{xx}dxdt. \tag{12}$$

We want to pass to the limit the above equation.

From the previous section we have that the support of the Young measures $\nu_{x,t}$ is reduced to a point. Let, for $(x,t)\in\mathbb{R}\times[0,+\infty[$, $(\overline{u}(x,t),\overline{v}(x,t))$ be the support of the Young measure $\nu_{x,t}$. Let $p<2$. Since

$$\eta(u,v) \geq c\frac{v^2}{2} + \frac{u^2}{2},$$

we have

$$0 \leq \frac{|u|^p + |v|^p}{\eta(u,v)} \leq C\frac{|u|^p+|v|^p}{v^2+u^2} \to 0, \quad |u|+|v|\to+\infty.$$

Then, from property (ii) of the Young measures theorem, we have

$$\overline{u}(x,t) = \int_{\mathbb{R}^2}\lambda_1 d\nu_{x,t}(\lambda_1,\lambda_2), \quad \overline{v}(x,t) = \int_{\mathbb{R}^2}\lambda_2 d\nu_{x,t}(\lambda_1,\lambda_2) \in L^p_{loc}(\mathbb{R}\times[0,+\infty[)$$

and $(u_{\varepsilon'},v_{\varepsilon'}) \to (\overline{u},\overline{v})$, strongly in $(L^p_{loc}(\mathbb{R}\times[0,+\infty[))^2$. We had previously seen that a subsequence $(u_{\varepsilon'},v_{\varepsilon'})_{\varepsilon'}$ converged, weakly in $(L^2_{loc}(\mathbb{R}\times[0,+\infty[))^2$, to a function $(u,v)\in L^2_{loc}(\mathbb{R}\times[0,+\infty[)^2$, and so, by the unicity of weak limit, we may conclude that $(\overline{u},\overline{v}) = (u,v) \in (L^2_{loc}(\mathbb{R}\times[0,+\infty[))^2$.

Since σ satisfies H4, we have

$$\frac{\sigma(u)}{\eta(u,v)} \to 0, \quad \text{if } |u|+|v|\to+\infty,$$

and again from (ii) we conclude that $\sigma(u_{\varepsilon'}) \to \sigma(\overline{u})$ in $L^1_{loc}(\mathbb{R}\times[0,+\infty[)$.

Due to what was exposed above, it follows immediately that

$$\lim_{\varepsilon'\to0}\int_{\mathbb{R}}\int_0^{+\infty}(u_{\varepsilon'}\varphi_t - v_{\varepsilon'}\varphi_x)dxdt = \int_{\mathbb{R}}\int_0^{+\infty}(\overline{u}\varphi_t - \overline{v}\varphi_x)dxdt, \tag{41}$$

$$\lim_{\varepsilon'\to0}\int_{\mathbb{R}}\int_0^{+\infty} v_{\varepsilon'}\psi_t dxdt = \int_{\mathbb{R}}\int_0^{+\infty}\overline{v}\psi_t dxdt, \tag{42}$$

$$\lim_{\varepsilon'\to0}\int_{\mathbb{R}}\int_0^{+\infty}\sigma(u_{\varepsilon'})\psi_x dxdt = \int_{\mathbb{R}}\int_0^{+\infty}\sigma(\overline{u})\psi_x dxdt. \tag{43}$$

Now, since

$$\begin{aligned}\left|\varepsilon' \int_{\mathbb{R}} \int_0^{+\infty} v_{\varepsilon'} \psi_{xx} dxdt\right| &\leq \|\psi_{xx}\|_{L^\infty} \varepsilon' \int_{\sup(\psi)} |v_{\varepsilon'}| dxdt \\ &\leq \|\psi_{xx}\|_{L^\infty} (\mathrm{m}(\sup(\psi)))^{1/2} \varepsilon' \left(\int_{\sup(\psi)} {v_{\varepsilon'}}^2 dxdt\right)^{1/2},\end{aligned}$$

we obtain, provided that, as a consequence of (33), $(v_{\varepsilon'})_{\varepsilon'}$ is uniformly bounded in $L^2(\sup(\psi))$,

$$\lim_{\varepsilon' \to 0} \varepsilon' \int_{\mathbb{R}} \int_0^{+\infty} v_{\varepsilon'} \psi_{xx} dxdt = 0. \tag{44}$$

To show that $(\overline{u}, \overline{v})$ is a weak solution of the problem (4), (5), we now study the limit of

$$\int_{\mathbb{R}} \int_0^{+\infty} F(\phi_{\varepsilon'}) \psi \, dxdt. \tag{45}$$

Let $\phi = \displaystyle\int_0^t \overline{v}(x,\tau) d\tau + \phi_0$ and $K \subseteq [a,b] \times [0,T]$ be a compact set of $\mathbb{R} \times [0, +\infty[$.

$$\begin{aligned}\left|\int_K \phi_{\varepsilon'}(x,t) - \phi(x,t) dxdt\right| &= \left|\int_K \int_0^t v_{\varepsilon'}(x,\tau) - \overline{v}(x,\tau) d\tau \, dxdt\right| \\ &\leq \int_a^b \int_0^T \int_0^T |v_{\varepsilon'}(x,\tau) - \overline{v}(x,\tau)| d\tau \, dxdt \\ &= T \int_a^b \int_0^T |v_{\varepsilon'}(x,\tau) - \overline{v}(x,\tau)| dxd\tau \\ &\leq T T^{1/q} (b-a)^{1/q} \|v_{\varepsilon'} - \overline{v}\|_{L^p([a,b]\times[0,T])} \to 0,\end{aligned}$$

and so $\phi_{\varepsilon'} \to \phi$ in $L^1_{loc}(\mathbb{R} \times [0, +\infty[)$. Hence, there exists a subsequence, that we still call $\phi_{\varepsilon'}$, which converges pointwise, a. e. $(x,t) \in \mathbb{R} \times [0, +\infty[$, to ϕ. Since F is continuous, $F(\phi_{\varepsilon'}(x,t)) \to F(\phi(x,t))$, a. e. $(x,t) \in \mathbb{R} \times [0, +\infty[$.

On the other hand, for $t > 0$, we show, as we did to obtain property M4 in section 3, that

$$\|\phi_{\varepsilon'}(\cdot,t)\|_{L^2(\mathbb{R})} \leq c(t), \quad \|{\phi_{\varepsilon'}}_x(\cdot,t)\|_{L^2(\mathbb{R})} \leq c,$$

which implies that $\|\phi_{\varepsilon'}(\cdot,t)\|_{H^1(\mathbb{R})} \leq c(t)$ and $\|\phi_{\varepsilon'}\|_{L^\infty(\mathbb{R}\times[0,t])} \leq c(t)$.

Now, we can apply dominated convergence theorem to (45) to obtain

$$\lim_{\varepsilon' \to 0} \int_{\mathbb{R}} \int_0^{+\infty} F(\phi_{\varepsilon'}) \psi \, dxdt = \int_{\mathbb{R}} \int_0^{+\infty} F(\phi) \psi \, dxdt. \tag{46}$$

From (41), (42), (43), (44) and (46) we have that $\overline{u}$ and $\overline{v}$ satisfy the weak formulation of the Cauchy problem (4), (5), and, from (33), it follows that u_ε and v_ε also satisfy

$$\int_{\mathbb{R}} \left(\frac{{v_\varepsilon}^2}{2} + \Sigma(u_\varepsilon) \right)(x,t) \le C, \quad \forall\ t > 0.$$

By passing the above inequality to the limit, we obtain

$$\int_{\mathbb{R}} \left(\frac{\overline{v}^2}{2} + \Sigma(\overline{u}) \right)(x,t) \le C, \quad \forall\ t > 0,$$

and so $(\overline{u}, \overline{v}) \in L^\infty([0,+\infty[; L^\eta)$ is a weak solution of the Cauchy problem (4), (5).

To complete the proof of theorem 1.1, we show that the entropy inequality (9) is satisfied by the entropy-entropy flux pair defined by (10).

Since $\nabla p(u,v) \cdot \nabla f(u,v) = \nabla q(u,v)$, $\forall\ (u,v) \in \mathbb{R}^2$, $(f(u,v) = (-v, -\sigma(u)))$, if we multiply system (11) by $(\nabla p)(u_\varepsilon, v_\varepsilon) = (p_u(u_\varepsilon, v_\varepsilon), p_v(u_\varepsilon, v_\varepsilon))$, since $p_{uv} = 0$, we conclude that

$$\begin{aligned} &p(u_\varepsilon, v_\varepsilon)_t + q(u_\varepsilon, v_\varepsilon)_x + \nabla p\,(u_\varepsilon, v_\varepsilon) \cdot (0, F(\phi)) = \\ &\qquad \varepsilon(p_v(u_\varepsilon, v_\varepsilon) v_{\varepsilon x})_x - \varepsilon(p_v(u_\varepsilon, v_\varepsilon))_x v_{\varepsilon x} = \varepsilon(p_v(u_\varepsilon, v_\varepsilon) v_{\varepsilon x})_x - \varepsilon p_{vv}(u_\varepsilon, v_\varepsilon) {v_{\varepsilon x}}^2. \end{aligned}$$

Since the second derivative in the equation above is positive, we have that, for $\psi \in \mathcal{D}(\mathbb{R} \times]0, +\infty[)$, $\psi \ge 0$,

$$\int_{\mathbb{R}} \int_0^{+\infty} (p(u_\varepsilon, v_\varepsilon)\psi_t + q(u_\varepsilon, v_\varepsilon)\psi_x - p_v(u_\varepsilon, v_\varepsilon) F(\phi_\varepsilon)\psi)\, dxdt \\ - \varepsilon \int_{\mathbb{R}} \int_0^{+\infty} (p_v(u_\varepsilon, v_\varepsilon)) v_{\varepsilon x} \psi_x \ge 0.$$

Now, $p_v(u_\varepsilon, v_\varepsilon) = v_\varepsilon$ and $\varepsilon |v_\varepsilon v_{\varepsilon x}| = \varepsilon^{1/2} \varepsilon^{1/2} |v_\varepsilon v_{\varepsilon x}| \le \varepsilon^{1/2} \big(\frac{{v_\varepsilon}^2}{2} + \frac{\varepsilon {v_{\varepsilon x}}^2}{2} \big)$. Hence, from (33) and (34) follows that the second term in the above inequality converges to 0. Since p and q are continuous, by passing both members of the above inequality to the limit, we obtain

$$\int_{\mathbb{R}} \int_0^{+\infty} (p(\overline{u}, \overline{v})\psi_t + q(\overline{u}, \overline{v})\psi_x - p_v(\overline{u}, \overline{v}) F(\phi)\psi)\, dxdt \ge 0.$$

This finishes the proof of theorem 1.1.

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