# The Range of a Contractive Projection in $L_p(H)$

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**ABSTRACT**

We show that the range of a contractive projection on a Lebesgue-Bochner space of Hilbert valued functions $L_p(H)$ is isometric to a $\ell_p$-direct sum of Hilbert-valued $L_p$-spaces. We explicit the structure of contractive projections. As a consequence for every $1 < p < \infty$ the class $\mathcal{C}_p$ of $\ell_p$-direct sums of Hilbert-valued $L_p$-spaces is axiomatizable (in the class of all Banach spaces).

*Key words:* Contractive projections, Vector-valued $L_p$-spaces.

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## Introduction

It was a remarkable achievement in the isometric theory of Banach spaces of the years 1960's to characterize the contractive linear projections of Lebesgue $L_p$ spaces ($p \neq 2$). In the case of $L_p$ spaces of a probability space it was done by Douglas [4] in the case $p = 1$ and Andô [1] in the case $1 < p < \infty$, $p \neq 2$. They showed that the range of such a contractive projection is itself isometric to a $L_p$ space (for the same $p$, but a different measure space); if moreover the projection is positive then its range is a sublattice of the initial $L_p$ space and is lattice isomorphically isometric to a $L_p$ space. This was extended to the non-sigma-finite measure space setting by Tzafriri ([17]). In the case of a probability space, the structure of contractive projections is elucidated by Douglas-Andô works: a general contractive projection $P$ on $L_p(\Omega, \Sigma, \mu)$ has the form

$$P = M_\varepsilon \widehat{P} M_\varepsilon^{-1} + V \tag{1}$$

where $M_\varepsilon$ is the multiplication operator by a function $\varepsilon$ with $|\varepsilon| = \mathbf{1}$, $\widehat{P}$ is a positive contractive projection, and $V = 0$ if $p > 1$, while if $p = 1$, then $V$ is a contraction from $L_1$ into the range $R(P)$ of $P$ which vanishes on the band generated by $R(P)$. Moreover $\widehat{P}$ is a weighted conditional expectation, i.e. there exist a sub-sigma algebra $\mathcal{B}$, an element $B \in \mathcal{B}$ and a nonnegative function $w \in L^p$ such that $\mathbb{E}(w^p \mid \mathcal{B}) = \mathbf{1}$ and

$$\widehat{P} f = w \mathbb{E}(\mathbf{1}_B f \cdot w^{p-1} \mid \mathcal{B})$$

for every $f \in L_p$ (in particular if $P\mathbf{1} = \mathbf{1}$ then $P$ is a conditional expectation). This last formula can also be written

$$\widehat{P} f = w \mathbb{E}_\nu (\mathbf{1}_B f w^{-1} \mid \mathcal{B})$$

where $\mathbb{E}_\nu$ is the conditional expectation relative to the measure $\nu = w^p \cdot \mu$. If we denote by $S$ the isometric isomorphism $f \mapsto w \cdot f$ of $L_p(\Omega, \Sigma, \nu)$ onto $L_p(\Omega, \Sigma, \nu)$ and by $M_B$ the multiplication operator by the indicator function $\mathbf{1}_B$, we have:

$$\widehat{P} = S M_B \mathbb{E}_\nu( \;\mid \mathcal{B}) S^{-1}. \tag{2}$$

The structure of contractive projections in the non-sigma finite case was treated by Bernau and Lacey ([3]); their main result can be rephrased in saying that if we assume (as we may) that the measure space $(\Omega, \Sigma, \mu)$ is localizable ([7]) then formulas (1) and (2) are still valid; now $w$ is some $\Sigma$-measurable positive function, $\nu = w^p \cdot \mu$ and $\mathcal{B}$ is some semi-finite sigma-subalgebra of $\Sigma$.

The task of extending these results to various classical spaces was considered by numerous authors; see the recent survey paper [15] and the references inside. Here we are more specifically interested in the case of vector-valued Lebesgue $L_p$ spaces, in particular mixed norm spaces $L_p(L_q)$. Since the survey paper [5] on this specific subject, several partial results appeared. In particular B. Randrianantoanina ([14]) succeeded in solving thoroughly the complex sequential case $\ell_p(\ell_q)$ using hermitian operator techniques introduced in the subject by Kalton and Wood. More recently the case of finite dimensional real Banach spaces with $C^2$ norm was considered by the authors of [12]; under some additional conditions on the dual norm (in particular it is assumed to be $C^2$ on the complementary set of the coordinate hyperplanes associated to a distinguished basis) the contractively complemented subspaces are shown to be necessarily generated by a block-basis of the given basis. This can be applied in particular to the real spaces $\ell_p^n(\ell_q^m)$, when $2 < p, q < \infty$ (or by duality when $1 < p, q < 2$), obtaining the same description of their contractively complemented subspaces as in the complex case [16].

In the present paper we examine the case of Lebesgue spaces of Hilbert valued functions $L_p(H)$; this is done in the most general case (without any assumption of sigma-finiteness of $L_p$-space or separability of the Hilbert space; in fact we have in mind some applications to the ultrapowers of such spaces, which are neither separable nor sigma-finite). It turns out that the range of a contractive projection is a $\ell_p$-direct sum of spaces of the type $L_p(H)$. More precisely:

**Theorem 0.1.** *Let $1 \leq p < \infty$, $p \neq 2$; $H$ be a Hilbert space and $L_p = L_p(\Omega, \Sigma, \mu)$. The range of every contractive projection $P : L_p(H) \to L_p(H)$ is isometric to a $\ell_p$-direct sum of Hilbert-valued $L_p$-spaces, i.e.*

$$R(P) \approx_1 \Big(\bigoplus_{i\in I} L_p(\Omega_i, \mathcal{B}_i, \mu_i; H_i)\Big)_{\ell_p}$$

*where $(\Omega_i)_i$ is a family of pairwise almost disjoint members of $\Sigma$, each $\mathcal{B}_i$ is a sub-sigma-algebra of the trace $\Sigma_i$ of $\Sigma$ on $\Omega_i$; $\mu_i$ is the trace on $\Omega_i$ of the measure $\mu$; and the Hilbert spaces $H_i$ have Hilbertian dimension not greater than the Hilbertian dimension of $H$.*

Conversely a $\ell_p$-sum $\big(\bigoplus_{i\in I} L_p(\Omega_i, \Sigma_i, \mu_i; H_i)\big)_{\ell_p}$ embeds isometrically into $L_p(H)$, where $L_p = \big(\bigoplus_{i\in I} L_p(\Omega_i, \mathcal{B}_i, \mu_i)\big)_{\ell_p}$ and $H = \big(\bigoplus_{i\in I} H_i\big)_{\ell_2}$. Hence a contractively complemented subspace of a $\ell_p$-direct sum of Hilbert-valued $L_p$-spaces is still a $\ell_p$-direct sum of Hilbert-valued $L_p$-spaces. In other words:

**Corollary 0.2.** *The class $\mathcal{C}_p$ of $\ell_p$-direct sums of Hilbert-valued $L_p$-spaces is stable under contractive projections.*

The structure of the contractive projection $P$ can be easily explained in the case where the space $H$ is separable (the non-separable case is analogous and will be described in Section 5). Recall that given two Banach spaces $X$, $Y$, a family of operators $T_\omega : X \to Y$ is said to be strong-operator $\Sigma$-measurable if for every $x \in X$, the map $\omega \mapsto T_\omega x$ is $\Sigma$-measurable as a map $\Omega \to Y$. If moreover $\operatorname{Ess\,sup}_\omega \|T_\omega\| < \infty$, such a measurable family induces a bounded linear map $T$ from $L_p(\Omega, \Sigma, \mu; X)$ into $L_p(\Omega, \Sigma, \mu; Y)$ by the equation:

$$(Tf)(\omega) = T_\omega(f_\omega)$$

**Theorem 0.3.** *Under the conditions of Thm. 0.1, if moreover $H$ is separable, then*

$$P = \sum_{i\in I} S_i(\widetilde{P}_i \otimes \mathrm{Id}_{H_i})S_i^\sharp M_{\Omega_i} + V$$

*where $\widetilde{P}_i$ is a positive contractive projection in $L_p(\Omega_i, \Sigma_i, \mu_i)$; $S_i$ is an isometric embedding of $L_p(\Omega_i, \Sigma_i, \mu_i; H_i)$ into $L_p(\Omega_i, \Sigma_i, \mu_i; H)$ associated with a (strong-operator)-measurable family $(S_{i,\omega})_{\omega\in\Omega_i}$ of isometric embeddings $H_i \to H$, while $S_i^\sharp$ is associated with the adjoint family $(S_{i,\omega}^*)_{\omega\in\Omega_i}$ of projections $H \to H_i$; $M_{\Omega_i} : L_p(\Omega; H) \to L_p(\Omega_i; H_i)$ is the multiplication operator by the indicator function $\mathbf{1}_{\Omega_i}$; and $V = 0$ if $p > 1$, while if $p = 1$ then $V$ is a contraction of $L_1(\Omega, \Sigma, \mu; H)$ vanishing on every $L_1(\Omega_i, \Sigma_i, \mu_i; H)$ and taking values in the range of $P$.*

Let us present shortly an application of the Thm. 0.1 which was in fact our main motivation for starting this study. If $X, Y$ are Banach spaces, we say that $X$ is an *ultraroot* of $Y$ if $Y$ is isometric to some ultrapower of $X$. Recall that a Banach space $X$ embeds canonically isometrically in every of its ultrapowers $X_{\mathcal{U}}$, and that if $X$ is reflexive, then this canonical image is contractively complemented in $X_{\mathcal{U}}$. As a consequence of Thm. 0.1 we see that every ultraroot of a $L_p(H)$ space, $p > 1$ is a member of $\mathcal{C}_p$. By Cor. 0.2 the same is true for ultraroots of members of $\mathcal{C}_p$. On the other hand it was proved in [13] that every ultraproduct of $L_p(H)$ spaces is isometric to a $\ell_p$-direct sum of Hilbert-valued $L_p$-spaces. More generally every ultraproduct of members of $\mathcal{C}_p$ is itself isometric to a member of $\mathcal{C}_p$. Hence we obtain:

**Corollary 0.4.** *For every $1 < p < \infty$ the class $\mathcal{C}_p$ of $\ell_p$-direct sums of Hilbert-valued $L_p$-spaces is stable under ultraproducts and ultraroots.*

In other words the class $\mathcal{C}_p$ is *axiomatizable* in the sense of Henson-Iovino [9] in their language of normed spaces structures (see [9], Thm. 13.8).

The paper is organized as follows: after a section devoted to definitions, notations and a general result on orthogonally complemented subspaces of $L_p(H)$, we have two sections of preliminary results distinguishing the case $p = 1$ (Section 2) from the case $p > 1$ (Section 3). In these sections it is proved that if $f$ belongs to the range of a contractive projection $P$, then the whole subspace $Z_f := \overline{L_\infty(\Omega) \cdot f}$ is preserved by $P$ (i.e. $PZ_f \subset Z_f$) which suggests clearly a possible reduction to the scalar case. It is also proved that the "orthogonal projection" onto $Z_f$ preserves the range of $P$. This allows to find an "orthogonal system" in $R(P)$ which generates $Z_P := \overline{L_\infty(\Sigma) \cdot R(P)}$ over $L_\infty(\Sigma)$ which will furnish the orthogonal bases of the Hilbert spaces $H_i$ of Thm. 0.1. Section 4 is devoted to the proof of Thm. 0.1; a key point consists in proving that the different subalgebras of $\Sigma$ given by the scalar theorem (applied to each $Z_f$) are induced by the same sigma-subalgebra $\mathcal{F}$ of $\Sigma$. Finally Thm 0.3 is proved in Section 5 (in a more general version not requiring separability).

## 1. General preliminaries

### *1.1. Definitions and notations*

Let $1 \le p < \infty$, $H$ be an Hilbert space and $(\Omega, \Sigma, \mu)$ be a measure space. In the following we denote (when there is no ambiguity) by $L_p(H)$ the Lebesgue-Bochner space $L_p(\Omega, \Sigma, \mu; H)$ of classes of $H$-valued $p$-integrable functions (for $\mu$-a.e. equality). Similarly $L_\infty(H)$ will be the space of classes of Bochner measurable, essentially bounded $H$-valued functions. These spaces can be defined directly from the Banach lattices $L_p$ (resp. $L_\infty$) and the Hilbert space $H$, but we adopt the functional point of view for the simplicity of the exposition. In the case where $(\Omega, \Sigma, \mu)$ is not sigma-finite, it is preferable to suppose that this measure space is localizable: the measure $\mu$ is semifinite (every set in $\Sigma$ of positive measure contains a further one of positive and finite measure) and $L_\infty(\Omega,\Sigma,\mu)$ is order complete. In particular every family $(A_i)_{i\in I}$ in $\Sigma$ has a supremum $A$, denoted by $\bigvee_{i\in I} A_i$. The set $A$ is defined (up to a $\mu$-null set) by the conditions:

$$A \dot{\supset} A_i \text{ for every } i \in I,$$
$$\text{If } B\in\Sigma \text{ and } B\dot{\supset} A_i \text{ for every } i\in I \text{ then } B\dot{\supset} A,$$

where $B\dot{\supset}A$ means $\mu(A\setminus B)=0$ (define similarly $A\dot{\subset}B$ and $A\doteq B$). We say that $B,C$ are almost disjoint if $A\cap B\doteq\emptyset$.

To every $f\in L_p(H)$ we associate its "*random norm*" $N(f)\in L_p^+$ defined by $N(f)(\omega)=\|f(\omega)\|_H$, its *vectorial function support* $\mathbf{VS}(f)=\mathrm{Supp}(N(f))$ and its "random direction", i.e. the element $u_f$ of $L_\infty(H)$ defined by $u_f(\omega)=\frac{f(\omega)}{N(f)(\omega)}$ if $\omega\in\mathbf{VS}(f)$, $=0$ if $\omega\notin\mathbf{VS}(f)$. If $M\subset L_p(H)$ we set $\mathbf{VS}(M)=\bigvee\{\mathbf{VS}(f)\mid f\in M\}$. If $f\in L_p(H)$, $g\in L_q(H)$ we define their random scalar product $\langle\!\langle f,g\rangle\!\rangle\in L_r$ (where $\frac{1}{r}=\frac{1}{p}+\frac{1}{q}$) by $\langle\!\langle f|g\rangle\!\rangle(\omega)=\langle f(\omega)|g(\omega)\rangle_H$, where $\langle\cdot\,|\,\cdot\rangle_H$ denotes the scalar product in $H$ (which we suppose left linear, right antilinear in the complex case). When $p,q$ are conjugate ($\frac{1}{p}+\frac{1}{q}=1$), we obtain a sesquilinear pairing

$$\langle f,g\rangle=\int_\Omega \langle\!\langle f\mid g\rangle\!\rangle d\mu \tag{3}$$

which gives rise to a canonical antilinear identification of $L_q(H)$ with $L_p(H)^*$ (if $1<p,q<\infty$; the case $p=1$, $q=\infty$ is more delicate); it is the usual duality pairing in the real spaces case. We have also

$$\forall f\in L_p(H), \langle\!\langle f|u_f\rangle\!\rangle=N(f).$$

We say that two elements $f,g\in L_p(H)$ are *orthogonal*, and we write $f\perp g$ if $\langle\!\langle f\mid g\rangle\!\rangle=0$. A related notation is the following. We set

$$\{\,f\perp g\,\}=\{\,\omega\in\Omega\mid\langle\!\langle f\mid g\rangle\!\rangle(\omega)=0\,\}$$

We have then $f\perp g \iff \{\,f\perp g\,\}\doteq\Omega$.

Let $H$, $K$ two Hilbert spaces. We say that a linear operator $T: L_p(H)\to L_p(K)$ is $\Sigma$-*modular* iff $T(\varphi.f)=\varphi.Tf$ for every $f\in L_p(H)$ and $\varphi\in L_\infty(\Omega,\Sigma,\mu)$. It is *modularly contractive*, resp. *modularly isometric* iff $N(Tf)\le N(f)$, resp. $N(Tf)=N(f)$ for every $f\in L_p(H)$: it is then automatically $\Sigma$-modular (and, of course, contractive, resp. isometric). If $H$ is separable, then a modularly contractive, resp. modularly isometric operator $T$ is associated with a measurable family of contractions, resp. isometries $T_\omega: H\to K$.

Let $\mathcal{F}$ be a sub-sigma-algebra of $\Sigma$; a linear subspace $Z$ of $L_p(H)$ is a $L_\infty(\mathcal{F})$-*submodule* iff $\varphi\cdot f\in Z$ for every $f\in Z$ and $\varphi\in L_\infty(\Omega,\mathcal{F},\mu)$. To every $f\in L_p(H)$ we associate the bounded $\Sigma$-modular operator:

$$E_f: L_p(H)\to L_p(H), \qquad g\mapsto\langle\!\langle g|u_f\rangle\!\rangle\,u_f.$$

We have $N(E_f g) = |\langle\langle g|u_f\rangle\rangle| \mathbf{1}_{\mathbf{VS}(f)} \le N(g)$, hence $E_f$ is modularly contractive.

We have clearly $E_f(f) = N(f)u_f = f$. Consequently for every $\varphi \in L_\infty$, we have

$$E_f((\varphi N(f)) \cdot u_f) = E_f(\varphi f) = \varphi f = (\varphi N(f)) \cdot u_f$$

and by density we deduce that $E_f(\psi \cdot u_f) = \psi \cdot u_f$ for every $\psi \in L_p$. In particular $E_f(E_f g) = E_f g$, so $E_f$ is a projection (with range $R(E_f) = L_p(\Omega).u_f$). It is not hard to see that $R(E_f)$ is exactly the closed $L_\infty(\Sigma)$-submodule generated by $f$. Note also that if $f, g \in L_p(H)$,

$$f \perp g \iff E_f g = 0 \iff E_g f = 0.$$

### *1.2. Orthogonal projections*

We end this section by considering a special class of contractive projections, namely the orthogonal ones. A projection $Q$ in $L_p(H)$ is said to be *orthogonal* if $(f - Qf) \perp Qf$ for every $f \in L_p(H)$. Such a projection is trivially modularly contractive since

$$N(f)^2 = N(Qf)^2 + N((I-Q)f)^2 \ge N(Qf)^2.$$

Note that by polarization we have for every $f, g \in L_p(H)$:

$$\langle\langle f \mid g\rangle\rangle = \langle\langle Qf \mid Qg\rangle\rangle + \langle\langle (I-Q)f \mid (I-Q)g\rangle\rangle$$

Replacing $g$ by $Qg$, we have

$$\langle\langle f \mid Qg\rangle\rangle = \langle\langle Qf \mid Qg\rangle\rangle$$

that is $(I-Q)f \perp Qg$; hence $\ker Q = R(I-Q) \perp R(Q)$.

Conversely if $f \perp R(I-Q)$ then $f - Qf \perp R(I-Q)$ and in particular $f - Qf \perp f - Qf$, i.e. $f = Qf \in R(Q)$. Hence $R(Q) = \ker Q^\perp := \{f \in L_p(H) \mid f \perp \ker Q\}$ and similarly (exchanging the roles of $Q$ and $I-Q$) we have: $\ker Q = R(Q)^\perp$.

If $A$ is a subset of $L_p(H)$ then $A^\perp$ is a closed $L_\infty(\Sigma)$-submodule of $L_p(H)$. In particular the range of any orthogonal projection in $L_p(H)$ is a closed $L_\infty(\Sigma)$-submodule. The converse is true:

**Lemma 1.1.** *If $Z$ is a closed $L_\infty(\Sigma)$-submodule of $L_p(\Omega, \Sigma, \mu; H)$ there exists a unique orthogonal projection $Q_Z$ in $L_p(H)$ with range $Z$.*

*Proof.* Let $(f_\alpha)_{\alpha \in A}$ be a maximal family of pairwise orthogonal non zero elements of $Z$. For every family $(\varphi_\alpha)_\alpha$ in $L_p(\Omega)$ and every finite subset $B$ of $A$ we have

$$\Big\| \sum_{\alpha \in B} \varphi_\alpha u_{f_\alpha} \Big\|_{L_p(H)} = \Big\| N\Big(\sum_{\alpha \in B} \varphi_\alpha u_{f_\alpha}\Big)\Big\|_p = \Big\| \Big(\sum_{\alpha \in B} \mathbf{1}_{\mathbf{VS}(f_\alpha)} |\varphi_\alpha|^2\Big)^{1/2}\Big\|_p.$$

Hence, by Cauchy's criterion, $\sum_{\alpha\in A}\varphi_\alpha u_{f_\alpha}$ converges in $L_p(H)$ iff $(\sum_{\alpha\in A}\mathbf{1}_{\mathbf{VS}(f_\alpha)}|\varphi_\alpha|^2)^{1/2}$ exists in $L_p$ and

$$\Big\|\sum_{\alpha\in A}\varphi_\alpha u_{f_\alpha}\Big\|_{L_p(H)}=\Big\|\Big(\sum_{\alpha\in A}\mathbf{1}_{\mathbf{VS}(f_\alpha)}|\varphi_\alpha|^2\Big)^{1/2}\Big\|_p.$$

If now $f\in L_p(H)$ and $B$ is a finite subset of $A$ we have

$$\begin{aligned}N\Big(\sum_{\alpha\in B}\langle\!\langle f|u_{f_\alpha}\rangle\!\rangle u_{f_\alpha}\Big)^2&=\sum_{\alpha\in B}|\langle\!\langle f|u_{f_\alpha}\rangle\!\rangle|^2\\&=\Big\langle\!\!\Big\langle f,\sum_{\alpha\in B}\langle\!\langle f|u_{f_\alpha}\rangle\!\rangle u_{f_\alpha}\Big\rangle\!\!\Big\rangle\le N(f)N\Big(\sum_{\alpha\in B}\langle\!\langle f|u_{f_\alpha}\rangle\!\rangle u_{f_\alpha}\Big),\end{aligned}$$

whence

$$N\Big(\sum_{\alpha\in B}\langle\!\langle f|u_{f_\alpha}\rangle\!\rangle u_{f_\alpha}\Big)=\Big(\sum_{\alpha\in B}|\langle\!\langle f|u_{f_\alpha}\rangle\!\rangle|^2\Big)^{1/2}\le N(f),$$

so

$$\Big(\sum_{\alpha\in A}|\langle\!\langle f|u_{f_\alpha}\rangle\!\rangle|^2\Big)^{1/2}\le N(f).$$

Consequently $Qf:=\sum_{\alpha\in A}\langle\!\langle f|u_{f_\alpha}\rangle\!\rangle u_{f_\alpha}=\sum_{\alpha\in A}E_{f_\alpha}f$ converges in $L_p(H)$ (with $\|Qf\|\le\|f\|$). Since $R(E_{f_\alpha})$ is the closed $L_\infty(\Sigma)$-submodule generated by $f_\alpha$, we have $R(E_{f_\alpha})\subset Z$ for each $\alpha$ and consequently $Qf\in Z$ for every $f\in L_p(H)$. The map $Q$ is modular for the action of $L_\infty(\Omega)$, and clearly $Qf_\beta=f_\beta$ for every $\beta\in A$. It results easily that $Qf=f$ for every $f=\sum_{\alpha\in A}\varphi_\alpha u_{f_\alpha}$ (when this series converges), i.e. $Q$ is a contractive projection in $L_p(H)$ with range

$$\begin{aligned}R(Q)&=\Big\{\sum_{\alpha\in A}\varphi_\alpha u_{f_\alpha}\;\Big|\;\Big(\sum_\alpha|\varphi_\alpha|^2\Big)^{1/2}\in L_p(\Omega)\Big\}\\&=\Big\{\sum_{\alpha\in A}\psi_\alpha f_\alpha\;\Big|\;\Big(\sum_\alpha|\psi_\alpha|^2N(f_\alpha)^2\Big)^{1/2}\in L_p(\Omega)\Big\}.\end{aligned}$$

Since clearly $\langle\!\langle Qf|f_\alpha\rangle\!\rangle=\langle\!\langle f|f_\alpha\rangle\!\rangle$ for every $\alpha\in A$ we have $(f-Qf)\perp f_\alpha$ for every $\alpha\in A$. By maximality of the system $(f_\alpha)$ we deduce that

$$f=Qf\text{ for every }f\in Z$$

so $R(Q)$ contains $Z$, hence coincides with $Z$. Note also that $f-Qf\perp Z$ for all $f\in L_p(H)$, and so $Q$ is orthogonal.

The unicity of the orthogonal projection onto $Z$ is a consequence of the fact that its image and kernel are uniquely determined ($R(Q)=Z$ and $\ker Q=Z^\perp$). □

## 2. Preliminary results: the case $p = 1$

**Lemma 2.1.** *Let $P$ be a contractive projection in $L_1(H)$. Then for every $f \in R(P)$ we have*

$$PE_f = E_f PE_f$$

*Proof.* For every $\varphi \in L_1(\Omega)$ with $0 \le \varphi \le N(f)$ we have

$$\begin{aligned}
\|f\| - \|\varphi \cdot u_f\| &= \int N(f)\, d\mu - \int N(\varphi u_f)\, d\mu = \int (N(f) - \varphi)\, d\mu \\
&= \|(N(f) - \varphi) \cdot u_f\| = \|f - \varphi.u_f\| \\
&\ge \|P(f - \varphi \cdot u_f)\| = \|f - P(\varphi \cdot u_f)\| \\
&\ge \|f\| - \|P(\varphi.u_f)\| \\
&\ge \|f\| - \|\varphi \cdot u_f\|.
\end{aligned}$$

Hence all the inequalities are equalities, and in particular

$$\|f - P(\varphi \cdot u_f)\| = \|f\| - \|P(\varphi \cdot u_f)\|,$$

that is,

$$\int N(f - P(\varphi \cdot u_f))\, d\mu = \int [N(f) - N(P(\varphi \cdot u_f))]\, d\mu.$$

Note that the function in the left-hand integral is greater than the one in the right-hand integral. Thus,

$$N(f - P(\varphi \cdot u_f)) = N(f) - N(P(\varphi \cdot u_f))$$

(equality as elements of $L_1(\Omega)$). Since $H$ is strictly convex this implies that

$$P(\varphi \cdot u_f) = \alpha \cdot f$$

for some $\alpha \in L_\infty^+(\Omega)$. Hence

$$E_f P(\varphi \cdot u_f) = E_f(\alpha \cdot f) = \alpha \cdot f = P(\varphi \cdot u_f).$$

This property has been proved for $\varphi \in L_1(\Omega)$ with $0 \le \varphi \le N(f)$; it is extended by linearity and density to every $\varphi \in L_1(\Omega)$. In particular if we take $\varphi = \langle\langle h | u_f \rangle\rangle$, we obtain

$$\forall h \in L_1(H), \qquad E_f PE_f h = PE_f h,$$

that is, $E_f PE_f = PE_f$. □

**Lemma 2.2.** *Let $P$ be a contractive projection in $L_1(H)$. Then for every $f, g \in R(P)$ we have: $E_g f \in R(P)$. In other words $E_g P = PE_g P$.*

*Proof.* We have $(f - E_g f) \perp g$, while (by Lemma 2.1) $E_g f - PE_g f = E_g(f - PE_g f) \in L_1(\Omega) \cdot u_g$. Hence $(f - E_g f) \perp (E_g f - PE_g f)$. It results that

$$N(f - PE_g f) = [N(f - E_g f)^2 + N((E_g f - PE_g f)^2]^{1/2} \geq N(f - E_g f). \quad (4)$$

Hence:

$$\begin{aligned} \|f - PE_g f\| &\geq \|f - E_g f\| \\ &\geq \|P(f - E_g f)\| \\ &= \|f - PE_g f\| \end{aligned}$$

Hence the inequalities are equalities. In view of (4), the equality $\|f - PE_g f\| = \|f - E_g f\|$ implies

$$N(f - PE_g f) = [N(f - E_g f)^2 + N((E_g f - PE_g f)^2]^{1/2} = N(f - E_g f),$$

which implies in turn that $N(E_g f - PE_g f) = 0$, that is $E_g f = PE_g f$. So $E_g f \in R(P)$. □

## 3. Preliminary results: the case $p > 1$

**Notations.** Let $p_*$ be the conjugate exponent of $p$. If $T : L_p(H) \to L_p(H)$ is a bounded operator, we define its adjoint $T^* : L_{p_*}(H) \to L_{p_*}(H)$ by

$$\forall f \in L_{p_*}(H), \forall g \in L_p(H) \qquad \langle T^* f, g\rangle = \langle f, Tg\rangle,$$

where $\langle \cdot, \cdot \rangle$ denotes the sesquilinear pairing given by eq. (3).

If $f \in L_p(H)$, $f \neq 0$, let $Jf \in L_{p_*}(H)$ be the unique norm-one element such that $\langle f, Jf\rangle = \|f\|$. In fact it will be easier to consider the $(p-1)$-homogeneous functional $J_p(h) = \|h\|^{p-1} J(h)$. We have $J_p(h) = N(h)^{p-1} \cdot u_h = N(h)^{p-2} h$, hence $J_p$ is random direction preserving. Note that $pJ_p$ is the derivative of the $p^{\text{th}}$ power of the norm.

**Lemma 3.1.** *Let $1 < p < \infty$, $p \neq 2$, and $P$ be a contractive projection in $L_p(H)$. Then for every $f, g \in R(P)$ the function $F(f, g) := \operatorname{sgn}\langle\langle g \mid f\rangle\rangle f + \gamma_p \mathbf{1}_{\{f \perp g\}} N(f) u_g$ belongs to $R(P)$, where $\gamma_p$ is a positive constant depending only on $p$.*

*Proof.* a) Case $2 < p < \infty$.

Recall that since $L_p(H)$ is smooth the duality map $J$ maps $R(P)$ into $R(P^*)$ (see e.g. [6, Lemma 4.8]); hence $J_p(f + tg) \in R(P^*)$ for every $t \geq 0$. The derivative $\frac{\partial}{\partial t} J_p(f + tg)$ exists at $t = 0$ (since the norm to the power $p$ is twice differentiable) and it belongs to $R(P^*)$ too. We have

$$\frac{\partial}{\partial t} J_p(f + tg) = N(f + tg)^{p-2} g + \left(\frac{p-2}{2} \frac{\partial}{\partial t} N(f + tg)^2\right) N(f + tg)^{p-4}(f + tg).$$

Hence

$$A(f,g) := \frac{\partial}{\partial t} J_p(f+tg)\,|_{t=0} = N(f)^{p-2} g + (p-2)\operatorname{Re}(\langle\!\langle f \mid g\rangle\!\rangle) N(f)^{p-4} f$$
$$= N(f)^{p-2}[g + (p-2)\operatorname{Re}(\,\langle\!\langle u_f, g\rangle\!\rangle\,)\,u_f] \in R(P^*) \qquad (5)$$

In the complex case, replacing $f$ by $if$, we obtain

$$B(f,g) := N(f)^{p-2}[g - i(p-2)\operatorname{Im}(\,\langle\!\langle u_f, g\rangle\!\rangle\,)\,u_f] \in R(P^*) \qquad (5\text{bis})$$

adding

$$N(f)^{p-2}[2g + (p-2)\langle\!\langle g, u_f\rangle\!\rangle\,u_f] \in R(P^*)$$

With $E_f g = \langle\!\langle g, u_f\rangle\!\rangle\,u_f$ we obtain

$$N(f)^{p-2}[2(g - E_f g) + pE_f g] \in R(P^*).$$

In the case of a real space (5) is valid without the symbol Re and we obtain

$$N(f)^{p-2}[(g - E_f g) + (p-1)E_f g] \in R(P^*).$$

If $h \in R(P^*)$ then $J_{p_*} h = N(h)^{p_*-1} u_h \in R(P)$, hence if we set $Tg = \alpha_p(g - E_f g) + E_f g$, with $\alpha_p = \frac{2}{p}$ in the complex case, $\alpha_p = \frac{1}{p-1}$ in the real case, we obtain:

$$\Phi(g) := N(f)^{(p-2)(p_*-1)} N(Tg)^{(p_*-1)} u_{Tg} \in R(P).$$

Since $T$ is $\Sigma$-modular we have $u_{T(\varphi\cdot u_h)} = \mathbf{1}_{\operatorname{Supp}\varphi} \cdot u_{Th}$ for every $h \in L_p(H)$ and $\varphi \in L_p$, and more generally $u_{T^k(\varphi\cdot u_h)} = \mathbf{1}_{\operatorname{Supp}\varphi} \cdot u_{T^k h}$ for every $k \geq 1$. It is easily deduced that: $u_{T^k\Phi(g)} = \mathbf{1}_{\mathbf{VS}(f)} \cdot u_{T^{k+1}g}$ for every $k \geq 0$. Then

$$\begin{aligned} u_{\Phi^n(g)} &= u_{\Phi(\Phi^{n-1}(g))} = \mathbf{1}_{\mathbf{VS}(f)} \cdot u_{T\Phi^{n-1}(g)} \\ &= \mathbf{1}_{\mathbf{VS}(f)} \cdot u_{T\Phi(\Phi^{n-2}(g))} = \mathbf{1}_{\mathbf{VS}(f)} \cdot u_{T^2\Phi^{n-2}(g)} \cdots \\ &= \mathbf{1}_{\mathbf{VS}(f)} \cdot u_{T^n g} \end{aligned} \qquad (6)$$

for every $n \geq 1$. If $E_f g(\omega) \neq 0$ we have

$$u_{T^n g}(\omega) = \frac{\alpha_p^n (g - E_f g)(\omega) + E_f g(\omega)}{N(\alpha_p^n (g - E_f g) + E_f g)(\omega)} \longrightarrow \frac{E_f g(\omega)}{N(E_f g)(\omega)} = u_{E_f g}(\omega) \qquad (7)$$

(norm convergence in $H$) while if $E_f g(\omega) = 0$

$$u_{T^n g}(\omega) = \frac{(g - E_f g)(\omega)}{N((g - E_f g))(\omega)} = u_{(g - E_f g)}(\omega) = u_g(\omega). \qquad (7')$$

Since $g - E_f g \perp E_f g$ we have $N(Tg) \leq N(g)$. Hence

$$N(\Phi(g)) = N(f)^{2-p_*} N(Tg)^{p_*-1} \leq N(f)^{2-p_*} N(g)^{p_*-1}. \qquad (8)$$

In particular

$$N(\Phi(g)) \leq \max(N(f), N(g)). \tag{9}$$

Reiterating (8) we obtain for every $n \geq 1$

$$N(\Phi^n(g)) \leq N(f)^{(2-p_*)\Sigma_{k=0}^{n-1}(p_*-1)^k} N(g)^{(p_*-1)^n} = N(f)^{1-(p_*-1)^n} N(g)^{(p_*-1)^n}.$$

Since $0 < p_* - 1 < 1$ we obtain

$$\overline{\lim_{n\to\infty}} N(\Phi^n(g)) \leq \mathbf{1}_{\mathbf{VS}(g)} N(f). \tag{10}$$

We try now to be more precise. If $E_f g(\omega) = 0$ we have $N(Tg)(\omega) = \alpha_p N(g)(\omega)$. Hence

$$N(\Phi(g))(\omega) = N(f)(\omega)^{2-p_*}(\alpha_p N(g)(\omega))^{p_*-1}.$$

Moreover, since in this case $u_{\Phi^n(g)}(\omega) = u_g(\omega)$, we have $E_f \Phi^n(g)(\omega) = 0$ for every $n$, and we can reiterate. We obtain

$$N(\Phi^n(g))(\omega) = (\alpha_p^{p_*-1} N(f)(\omega)^{(2-p_*)})^{\Sigma_{k=0}^{n-1}(p_*-1)^k} N(g)(\omega)^{(p_*-1)^n}.$$

Hence

$$\begin{aligned}\lim_{n\to\infty} N(\Phi^n(g))(\omega) &= \alpha_p^{(p_*-1)/(2-p_*)} \mathbf{1}_{\mathbf{VS}(g)}(\omega) N(f)(\omega)\\ &= \alpha_p^{1/(p-2)} \mathbf{1}_{\mathbf{VS}(g)}(\omega) N(f)(\omega).\end{aligned} \tag{11}$$

If now $E_f(g)(\omega) \neq 0$, we have also $E_f(\Phi^n(g))(\omega) \neq 0$ for every $n \geq 0$. Set

$$\beta_n(\omega) = \frac{N(E_f \Phi^n(g))(\omega)}{N(\Phi^n(g))(\omega)}$$

We have then

$$N(T\Phi^n(g))(\omega) \geq \beta_n(\omega) N(\Phi^n(g))(\omega)$$

and consequently:

$$N(\Phi^{n+1}(g))(\omega) \geq N(f)^{2-p_*}(\beta_n(\omega) N(\Phi^n(g)(\omega))^{p_*-1}. \tag{12}$$

On the other hand

$$\beta_n(\omega) = |\langle\langle u_{\Phi^n(g)}, u_f \rangle\rangle(\omega)| = |\langle\langle u_{T^n(g)}, u_f \rangle\rangle(\omega)| = \frac{N(E_f T^n(g))(\omega)}{N(T^n(g))(\omega)} = \frac{N(E_f g)(\omega)}{N(T^n(g))(\omega)}$$

and since $N(T^n g) = (\alpha_p^{2n} N(g - E_f g)^2 + N(E_f g)^2)^{1/2} \searrow N(E_f g)$ pointwise (as $\alpha_p < 1$) we have $\beta_n(\omega) \nearrow 1$ on the set $\{\omega \mid E_f g(\omega) \neq 0\}$. Reiterating (12) from the step $n = n_0$ we obtain then

$$\underline{\lim_{n\to\infty}} N(\Phi^n(g))(\omega) \geq (\beta_{n_0}(\omega))^{1/(p-2)} \mathbf{1}_{\mathbf{VS}(\Phi_{n_0}(g))}(\omega) N(f)(\omega)$$

and letting $n_0 \to \infty$, we have, since $\mathbf{VS}(\Phi_n(g)) = \mathbf{VS}(g) \cap \mathbf{VS}(f)$ for every $n$,

$$\varliminf_{n\to\infty} N(\Phi^n(g))(\omega) \geq \mathbf{1}_{\mathbf{VS}(g)}(\omega) N(f)(\omega). \tag{13}$$

From (6), (7), (7'), and (11), (10), (13) we deduce that

$$\Phi^n(g) \to N(f)[u_{E_f(g)} + \alpha_p^{1/(p-2)} \mathbf{1}_{\{f\perp g\}} u_g\,] \tag{14}$$

almost everywhere in $H$-norm, hence in $L_p(H)$-norm by (9) and Lebesgue's Theorem. Hence the right-hand member of (14) belongs to $R(P)$. Since $u_{E_f g} = \operatorname{sgn}\langle\!\langle g \mid f\rangle\!\rangle u_f$ the right member of (14) can be written

$$\operatorname{sgn}\langle\!\langle g \mid f\rangle\!\rangle f + \gamma_p \mathbf{1}_{\{f\perp g\}} N(f) u_g = F_p(f, g) \tag{15}$$

where we have set $\gamma_p = \alpha_p^{1/(p-2)}$.

b) Case $1 < p < 2$.

This case is treated by duality. Set $\gamma_p = \gamma_{p_*}^{p_*-1}$ and define $F_p(f, g)$ by the formula (15). If $g = J_{p_*} g'$, $f = J_{p_*} h'$ with $f', g' \in L_{p_*}(H)$ we have

$$\operatorname{sgn}\langle\!\langle g \mid f\rangle\!\rangle = \operatorname{sgn}\langle\!\langle g' \mid f'\rangle\!\rangle.$$

Hence

$$\operatorname{sgn}\langle\!\langle g \mid f\rangle\!\rangle f = J_{p_*}(\operatorname{sgn}\langle\!\langle g' \mid f'\rangle\!\rangle f')$$

and similarly

$$N(f) = N(J_{p^*} f') = N(f')^{p_*-1}.$$

Hence

$$N(f) u_g = N(f')^{p_*-1} u_g = J_{p_*}(N(f') u_{g'}).$$

Finally, since $\{f \perp g\} = \{f' \perp g'\}$ and $J_{p_*}$ is additive on elements with disjoint functional supports, and positively homogeneous of degree $p_* - 1$,

$$F_p(f, g) = J_{p_*}(F_{p_*}(f', g')).$$

Then since $f' = J_p f, g' = J_p g$ belong to $R(P^*)$, the function $F_{p_*}(f', g'))$ belongs to $R(P^*)$ too by the case (a), and $F_p(f, g)$ belongs to $R(P)$. □

**Corollary 3.2.** *Let $p$ and $P$ be as in Lemma 3.1. Then for every $f, g \in R(P)$ the three elements $\operatorname{sgn}\langle\!\langle g \mid f\rangle\!\rangle f$, $\mathbf{1}_{\{f\perp g\}} f$ and $\mathbf{1}_{\{f\perp g\}} N(f) u_g$ belong to $R(P)$.*

*Proof.* The set $\Lambda$ of scalars $\lambda$ such that the set $\{\,\omega \in \mathbf{VS}(f) \mid \frac{\langle\!\langle g|f\rangle\!\rangle(\omega)}{\langle\!\langle f|f\rangle\!\rangle(\omega)} = -\lambda\,\}$ has positive measure is at most countable. This set is also the set of $\lambda$'s such that $\{(g + \lambda f) \perp f\} \cap \mathbf{VS}(f)$ has positive measure. Choose a sequence $(\varepsilon_n)$ of positive numbers not in $\Lambda \cup (-\Lambda)$ which converges to 0. Then by Lemma 3.1

$$\operatorname{sgn}\langle\!\langle g \pm \varepsilon_n f \mid f\rangle\!\rangle f \in R(P)$$

for every $n \geq 1$. Since

$$\operatorname{sgn}\langle\langle g \pm \varepsilon_n f \mid f\rangle\rangle(\omega) \to \begin{cases} \operatorname{sgn}\langle\langle g \mid f\rangle\rangle(\omega) & \text{if } \langle\langle g \mid f\rangle\rangle(\omega) \neq 0, \\ \pm 1 & \text{if } \langle\langle g \mid f\rangle\rangle(\omega) = 0 \quad \text{and } f(\omega) \neq 0, \end{cases}$$

we have

$$\operatorname{sgn}\langle\langle g \mid f\rangle\rangle f \pm \mathbf{1}_{\{f\perp g\}} f = \lim_n \operatorname{sgn}\langle\langle g \pm \varepsilon_n f \mid f\rangle\rangle f \in R(P)$$

and consequently $\operatorname{sgn}\langle\langle g \mid f\rangle\rangle f$ and $\mathbf{1}_{\{f\perp g\}} f$ belong to $R(P)$. Then $F_p(f,g) - \operatorname{sgn}\langle\langle g \mid f\rangle\rangle f = \gamma_p \mathbf{1}_{\{f\perp g\}} N(f) u_g$ belongs to $R(P)$ too. □

**Corollary 3.3.** *Let $p$ and $P$ be as in Lemma 3.1. Then for every $f, g \in R(P)$ we have $\mathbf{1}_{\mathbf{VS}(g)} f \in R(P)$.*

*Proof.* By Cor. 3.2, $h := G(f,g) := \mathbf{1}_{\{f\perp g\}} N(f) u_g$ belongs to $R(P)$. Then $G(h,f) = \mathbf{1}_{\{f\perp g\}} \mathbf{1}_{\{u_g \neq 0\}} N(f) u_f = \mathbf{1}_{\mathbf{VS}(g)\cap\{f\perp g\}} f$ belongs to $R(P)$ too. By Cor. 3.2, $f - \mathbf{1}_{\{f\perp g\}} f = \mathbf{1}_{\{f\not\perp g\}} f \in R(P)$, thus $\mathbf{1}_{\mathbf{VS}(g)} f = \mathbf{1}_{\{f\not\perp g\}} f + \mathbf{1}_{\mathbf{VS}(g)\cap\{f\perp g\}} f \in R(P)$. □

*Remark* 3.4. In the complex case, for every $f, g \in R(P)$ the elements $\operatorname{sgn}(\operatorname{Re}\langle\langle g \mid f\rangle\rangle) f$ and $\mathbf{1}_{\{\operatorname{sgn}(\operatorname{Re}\langle\langle g|f\rangle\rangle)=0\}} f$ belong to $R(P)$ too. Indeed, $L_p(H)$ is a real Hilbert-valued $L_p(K)$ space, where $K$ is the real vector space $H$ equipped with the scalar product $(x,y)_K = \operatorname{Re}(x \mid y)_H$. As a consequence, the element $\mathbf{1}_{\{\operatorname{Re}\langle\langle g|f\rangle\rangle>0\}} f = \frac{1}{2}(\operatorname{sgn}\operatorname{Re}\langle\langle g \mid f\rangle\rangle + \mathbf{1}_{\{\operatorname{sgn}(\operatorname{Re}\langle\langle g|f\rangle\rangle)\neq 0\}}) f$ belongs to $R(P)$.

**Lemma 3.5.** *Let $p$ and $P$ be as in Lemma 3.1. For every $f, g \in R(P)$ denote by $\Sigma_{f,g}$ the $\sigma$-field generated by the element $\frac{\langle\langle g|f\rangle\rangle}{\langle\langle f|f\rangle\rangle}$. Then for every $\Sigma_{f,g}$-measurable function $\varphi$ such that $\varphi \cdot N(f) \in L_p(\Omega, \Sigma, \mu)$, the element $\varphi \cdot f$ belongs to $R(P)$.*

*Proof.* Since $R(P)$ is a closed linear subspace, it is sufficient to prove this for indicator functions of $\Sigma_{f,g}$-measurable sets. The sigma-algebra $\Sigma_{f,g}$ is generated by the sets $\big\{\frac{\operatorname{Re}\langle\langle g|f\rangle\rangle}{\langle\langle f|f\rangle\rangle} > \lambda\big\}$, $\big\{-\frac{\operatorname{Re}\langle\langle g|f\rangle\rangle}{\langle\langle f|f\rangle\rangle} > \lambda\big\}$, $\big\{\frac{\operatorname{Im}\langle\langle g|f\rangle\rangle}{\langle\langle f|f\rangle\rangle} > \lambda\big\}$, and $\big\{-\frac{\operatorname{Im}\langle\langle g|f\rangle\rangle}{\langle\langle f|f\rangle\rangle} > \lambda\big\}$, $\lambda \in \mathbb{R}_+$. If $A_{f,g,\lambda} = \big\{\frac{\operatorname{Re}\langle\langle g|f\rangle\rangle}{\langle\langle f|f\rangle\rangle} > \lambda\big\}$ we have $A_{f,g,\lambda} = \{\operatorname{Re}\langle\langle g - \lambda f \mid f\rangle\rangle > 0\}$, hence $\mathbf{1}_{A_{f,g,\lambda}} f \in R(P)$ by Rem. 3.4. The conclusion is the same for the three others kinds of sets (replacing $g$ by $-g$ or $\pm ig$). Now if $\mathbf{1}_B f \in R(P)$ then $A_{f,g,\lambda} \cap B = A_{f',g,\lambda}$ with $f' = \mathbf{1}_B f$, hence $\mathbf{1}_{A_{f,g,\lambda}\cap B} f = \mathbf{1}_{A_{f',g,\lambda}} f' \in R(P)$. It results that the class $\mathcal{C}$ of the sets $A \in \Sigma$ such that $\mathbf{1}_A f \in R(P)$ contains finite intersections of sets of the four preceding types. Since $\mathcal{C}$ is closed by complementation and monotone limits, it contains the sigma-algebra $\Sigma_{f,g}$. □

**Corollary 3.6.** *Let $p$ and $P$ be as in Lemma 3.1. For every $f \in R(P)$ we have $E_f P = P E_f$.*

*Proof.* Let $g \in R(P)$. Applying Lemma 3.5 to the function $\varphi = \frac{\langle\langle g|f\rangle\rangle}{\langle\langle f|f\rangle\rangle}$ we obtain that $E_f g \in R(P)$. Hence for every $h \in L_p(H)$, we have $E_f Ph \in R(P)$, i.e. $E_f Ph = PE_f Ph$; thus $E_f P = PE_f P$. Similarly, reasoning with the contractive projection $P^*$ in $L_{p_*}(H)$, and the element $J_p f$ of $R(P^*)$, we have $E_{J_p f} P^* = P^* E_{J_p f} P^*$. Dualizing we obtain $PE^*_{J_p f} = PE^*_{J_p f} P$. We claim that $E^*_f = E_{J_p f}$. This will show that $PE_f = PE_f P = E_f P$. Let us show this claim. Since $u_{J_p f} = u_f$, we have for every $g \in L_p(H)$ and $h' \in L_{p_*}(H)$

$$\begin{aligned}\langle E_f g, h'\rangle &= \int \langle\langle E_f g, h'\rangle\rangle\, d\mu = \int \langle\langle \langle\langle g, u_f\rangle\rangle u_f, h'\rangle\rangle\, d\mu \\ &= \int \langle\langle g, u_f\rangle\rangle \langle\langle u_f, h'\rangle\rangle\, d\mu \\ &= \int \langle\langle g, \langle\langle h', u_f\rangle\rangle u_f\rangle\rangle\, d\mu \\ &= \int \langle\langle g, E_{J_p f} h'\rangle\rangle\, d\mu = \langle g, E_{J_p f} h'\rangle \qquad \square\end{aligned}$$

*Remark.* The preceding proof of Cor. 3.6 is essentially a real one. In the complex case it can be replaced by a shorter one, of more algebraic nature, due to Arazy and Friedman in the context of spaces $C_p$ (see [2]). It seemed interesting to us to reproduce this proof in the Annex (see §6), after simplifying it considerably by eliminating the unnecessary non-commutative apparatus.

## 4. The range of a contractive projection

This section is devoted to the proof of Thm. 0.1, which consists in four lemmas.

**Lemma 4.1.** *The closed $L_\infty(\Sigma)$-module $Z$ generated by $R(P)$ in $L_p(H)$ is generated (as $L_\infty$-module) by a family $(f_\alpha)_{\alpha\in A}$ of pairwise orthogonal elements of $R(P)$. We have in fact a Schauder (orthogonal) decomposition*

$$Z = \bigoplus_{\alpha\in A} L_p(\Omega)\cdot u_{f_\alpha}$$

*Proof.* Let $(f_\alpha)_{\alpha\in A}$ be a maximal family of pairwise orthogonal non zero elements of $R(P)$ and $Z_0$ be the closed $L_\infty(\Sigma)$-submodule generated by the family $(f_\alpha)_{\alpha\in A}$. Let $Q_{Z_0}$ be the orthogonal projection onto $Z_0$. By the proof of Lemma 1.1 we know that $Q_{Z_0} = \sum_{\alpha\in A} E_{f_\alpha}$ (convergence in strong operator topology). Hence, by Lemma 2.2 if $p = 1$, resp. Cor. 3.6 if $p > 1$, $Q_{Z_0} f \in R(P)$ for every $f \in R(P)$. Since $Q_{Z_0}$ is orthogonal and $f_\alpha \in R(Q_{Z_0})$ we have $(f - Q_{Z_0} f) \perp f_\alpha$ for every $\alpha \in A$. By maximality of the system $(f_\alpha)$ we deduce that

$$f = Q_{Z_0} f \text{ for every } f \in R(P)$$

i.e. $Q_{Z_0}P = P$. Then $Z_0 = R(Q_{Z_0})$ is a closed $L_\infty$-module containing $R(P)$ and generated by a subset of $R(P)$; hence it coincides with the closed $L_\infty$-module generated by $R(P)$. □

**Lemma 4.2.** *There exists a sub-$\sigma$-algebra $\mathcal{F}$ of $\Sigma$ containing the vectorial function supports of all elements of $R(P)$ such that for every $f \in R(P)$ and $\varphi \in L_p(\Omega, \Sigma, \mu)$, the product $\varphi \cdot u_f$ belongs to $R(P)$ iff $\mathbf{1}_{\mathbf{VS}(f)} N(f)^{-1}\varphi$ is $\mathcal{F}$-measurable. In particular $R(P)$ is a $L_\infty(\Omega, \mathcal{F}, \mu)$-submodule.*

*Proof.* Since $PE_f = E_f PE_f$ by Lemma 2.1 (if $p = 1$) or by Cor. 3.6 (if $p > 1$), we have $P(\varphi \cdot u_f) \in L_p(\Omega) \cdot u_f$ for every $f \in R(P)$ and $\varphi \in L_p(\Omega, \Sigma, \mu)$. We may write $P(\varphi \cdot u_f) = (\widetilde{P}_f \varphi) \cdot u_f$, with $\mathrm{Supp}(\widetilde{P}_f \varphi) \subset \mathbf{VS}(f)$. Clearly $\widetilde{P}_f$ is linear, $\widetilde{P}_f^2 = \widetilde{P}_f$ and

$$\|\widetilde{P}_f \varphi\|_p = \|P(\varphi \cdot u_f)\| \le \|\varphi \cdot u_f\| \le \|\varphi\|_p,$$

hence $\widetilde{P}_f$ is a contractive projection in $L_p(\Omega, \Sigma, \mu)$. Moreover $\widetilde{P}_f(N(f)) = N(f)$ and $\widetilde{P}_f \psi = 0$ for every $\psi \in L_p(\Omega, \Sigma, \mu)$ disjoint from $N(f)$.

It results from Douglas' theorem (in case $p = 1$) or Andô's theorem (in case $p > 1$) that $\widetilde{P}_f$ is positive and

$$\widetilde{P}_f(\varphi) = N(f)\mathbb{E}_{\nu_f}^{\mathcal{F}_f}\left(\frac{\mathbf{1}_{\mathrm{Supp}(N(f))}\varphi}{N(f)}\right)$$

where $\mathbb{E}_{\nu_f}^{\mathcal{F}_f}$ is the conditional expectation with respect to some subalgebra $\mathcal{F}_f$ of $\Sigma$ containing $\mathbf{VS}(f)$ and to the measure $\nu_f = N(f)^p \, d\mu$. (We may assume that $\Omega \setminus \mathbf{VS}(f)$ is an atom of $\mathcal{F}_f$). In particular $L_p(\Omega, \Sigma, \mu) \cdot u_f \cap R(P) = L_p(\Omega, \mathcal{F}_f, \nu_f) \cdot f$ is a $L_\infty(\Omega, \mathcal{F}_f, \mu)$-module.

Let us denote $\mathbb{E}^f \psi = \mathbb{E}_{\nu_f}^{\mathcal{F}_f}(\mathbf{1}_{\mathbf{VS}(f)}\psi)$, we have then $P(\psi \cdot f) = \mathbb{E}^f(\psi) \cdot f$ for every $\psi \in L_\infty(\Omega, \Sigma, \mu)$. Let now $f, g \in R(P)$. If $g = h \cdot u_f$ with $h \in L^p(\Omega)$ then $\frac{h}{N(f)}$ is $\mathcal{F}_f$-measurable and for every $\varphi \in L_\infty(\Omega, \Sigma, \mu)$ we have

$$\mathbb{E}^g(\varphi) \cdot g = P(\varphi h \cdot u_f) = N(f)\mathbb{E}^f\Big(\frac{\varphi \cdot h}{N(f)}\Big) \cdot u_f = h\mathbb{E}^f(\varphi) \cdot u_f = \mathbb{E}^f(\varphi) \cdot g,$$

Hence

$$\mathbb{E}^g(\varphi) = \mathbf{1}_{\mathbf{VS}(g)}.\mathbb{E}^f(\varphi) = \mathbf{1}_{\mathrm{Supp}\, h}\mathbb{E}^f(\varphi). \tag{16}$$

Let now $g$ be a general element of $R(P)$. For every $\varphi \in L_\infty(\Omega)$ the equation

$$P(\varphi \cdot (f+g)) = P(\varphi \cdot f) + P(\varphi \cdot g)$$

is equivalent to

$$\mathbb{E}^{f+g}(\varphi) \cdot (f+g) = \mathbb{E}^f(\varphi) \cdot f + \mathbb{E}^g(\varphi) \cdot g. \tag{17}$$

Let $g = h \cdot u_f + g'$ be the orthogonal decomposition, i.e. $h = \langle\langle g \mid u_f \rangle\rangle$ and $g' \perp f$. Note that $h \cdot u_f = E_f g \in R(P)$. Set $A = \mathbf{VS}(f)$, $B = \mathbf{VS}(g)$ and $B' = \mathbf{VS}(g')$. Taking the images of both sides of (17) by the orthogonal projection $I - E_f$ we obtain

$$\mathbb{E}^{f+g}(\varphi) \cdot g' = \mathbb{E}^g(\varphi) \cdot g',$$

hence $\mathbf{1}_{B'}\mathbb{E}^{f+g}(\varphi) = \mathbf{1}_{B'}\mathbb{E}^g(\varphi)$. Then by (17) again, $\mathbf{1}_{B'}\mathbb{E}^{f+g}(\varphi)f = \mathbf{1}_{B'}\mathbb{E}^f(\varphi)f$ and finally

$$\mathbf{1}_{A\cap B'}\mathbb{E}^{f+g}(\varphi) = \mathbf{1}_{A\cap B'}\mathbb{E}^f(\varphi) = \mathbf{1}_{A\cap B'}\mathbb{E}^g(\varphi). \tag{18}$$

On the other hand similarly to (17) we have

$$\mathbb{E}^{h\cdot u_f - g}(\varphi) \cdot (h \cdot u_f - g) = \mathbb{E}^{hu_f}(\varphi) \cdot hu_f - \mathbb{E}^g(\varphi) \cdot g.$$

Since $h \cdot u_f - g = -g'$ we deduce that

$$\mathbf{1}_{\Omega\setminus B'}\mathbb{E}^{hu_f}(\varphi) \cdot hu_f = \mathbf{1}_{\Omega\setminus B'}\mathbb{E}^g(\varphi) \cdot g,$$

hence

$$\mathbf{1}_{B\setminus B'}\mathbb{E}^{hu_f}(\varphi) = \mathbf{1}_{B\setminus B'}\mathbb{E}^g(\varphi). \tag{19}$$

We have $\mathbb{E}^{hu_f}(\varphi) = \mathbf{1}_{\operatorname{Supp} h}\mathbb{E}^f(\varphi)$ by eq. (16). Hence since $B \setminus B' \subset \operatorname{Supp} h$, eq. (19) gives

$$\mathbf{1}_{B\setminus B'}\mathbb{E}^f(\varphi) = \mathbf{1}_{B\setminus B'}\mathbb{E}^g(\varphi)$$

which together with eq. (18) gives

$$\mathbf{1}_{A\cap B}\mathbb{E}^f(\varphi) = \mathbf{1}_{A\cap B}\mathbb{E}^g(\varphi)$$

for every $\varphi \in L_\infty(\Omega, \Sigma, \mu)$. In particular

$$\begin{aligned}
\mathbf{1}_{\mathbf{VS}(f)\cap\mathbf{VS}(g)} &= \mathbf{1}_{\mathbf{VS}(f)\cap\mathbf{VS}(g)}\mathbb{E}^g(\mathbf{1}_{\mathbf{VS}(g)}) \\
&= \mathbf{1}_{\mathbf{VS}(f)\cap\mathbf{VS}(g)}\mathbb{E}^f(\mathbf{1}_{\mathbf{VS}(g)}) \\
&= \mathbf{1}_{\mathbf{VS}(f)\cap\mathbf{VS}(g)}\mathbb{E}^f(\mathbf{1}_{\mathbf{VS}(f)\cap\mathbf{VS}(g)}),
\end{aligned}$$

hence

$$\mathbb{E}^f(\mathbf{1}_{\mathbf{VS}(f)\cap\mathbf{VS}(g)}) \geq \mathbf{1}_{\mathbf{VS}(f)\cap\mathbf{VS}(g)}$$

and since $\mathbb{E}^f$ is a contraction in $L_p(\Omega, \Sigma, N(f)^p \cdot \mu)$ we have in fact

$$\mathbb{E}^f(\mathbf{1}_{\mathbf{VS}(f)\cap\mathbf{VS}(g)}) = \mathbf{1}_{\mathbf{VS}(f)\cap\mathbf{VS}(g)},$$

that is, $\mathbf{VS}(f) \cap \mathbf{VS}(g) \in \mathcal{F}_f$. In particular $\mathbf{1}_{\mathbf{VS}(g)} \cdot f = \mathbf{1}_{\mathbf{VS}(f)\cap\mathbf{VS}(g)} \cdot f \in R(P)$. More generally for every $A \in \mathcal{F}_g$ its trace $\mathbf{VS}(f) \cap A$ belongs to $\mathcal{F}_f$ (as is easily seen by treating separately the cases $A \subset \mathbf{VS}(g)$ and $A = \Omega \setminus \mathbf{VS}(g)$). Let $\mathcal{F}$ be the $\sigma$-algebra consisting of sets $A \in \Sigma$ such that $A \cap \mathbf{VS}(f)$ belongs to $\mathcal{F}_f$ for every $f \in R(P)$. Then for every $f \in R(P)$ and $\varphi \in L_0(\Omega, \Sigma, \mu)$ the function $\mathbf{1}_{\mathbf{VS}(f)}\varphi$ is $\mathcal{F}$ measurable iff it is $\mathcal{F}_f$-measurable, and the Lemma follows. □

**Lemma 4.3.** *There is a weight $w \in L_0(\Omega, \Sigma, \mu)$ with support $\mathbf{VS}(R(P))$ such that for every $f \in R(P)$, $w^{-1}N(f)$ is $\mathcal{F}$-measurable.*

*Proof.* a) First we claim that for every $f, g \in R(P)$ then $\mathbf{1}_{\mathbf{VS}(f)}\frac{N(g)}{N(f)}$ is $\mathcal{F}$-measurable. Since $E_f g = \langle\!\langle g \mid u_f \rangle\!\rangle u_f \in R(P)$ by Lemma 2.2, it results from Lemma 4.2 that $N(f)^{-1}\langle\!\langle g \mid u_f\rangle\!\rangle = N(f)^{-2}\langle\!\langle g \mid f \rangle\!\rangle$ is $\mathcal{F}$-measurable; hence its absolute value $N(f)^{-2}$ $|\langle\!\langle g \mid f\rangle\!\rangle|$ is $\mathcal{F}$-measurable, and similarly $N(g)^{-2}|\langle\!\langle f \mid g \rangle\!\rangle|$ is $\mathcal{F}$-measurable too. Then the ratio of these functions, that is $\mathbf{1}_{\mathrm{Supp}\langle\!\langle g|f\rangle\!\rangle}N(g)^2N(f)^{-2}$ is $\mathcal{F}$-measurable, and so is its square root $\mathbf{1}_{\mathrm{Supp}\langle\!\langle g|f\rangle\!\rangle}N(g)N(f)^{-1}$. Replacing $g$ by $g_\varepsilon = g + \varepsilon f$, $\varepsilon > 0$ we obtain that $\mathbf{1}_{\mathrm{Supp}\langle\!\langle g_\varepsilon|f\rangle\!\rangle}N(g_\varepsilon)N(f)^{-1}$ is $\mathcal{F}$ measurable. When $\varepsilon \to 0$ we have $g_\varepsilon \to g$, $N(g_\varepsilon) \to N(g)$ (in $L_p$-norm) and $\mathrm{Supp}\langle\!\langle g_\varepsilon \mid f\rangle\!\rangle \to \mathrm{Supp}\, N(f) = \mathbf{VS}(f)$ (in probability). At the limit $\mathbf{1}_{\mathbf{VS}(f)}\frac{N(g)}{N(f)}$ is $\mathcal{F}$-measurable.

b) Let $(f_i)_{i\in I}$ be a maximal family of non zero elements in $R(P)$ with pairwise almost disjoint functional supports $\mathbf{VS}(f_i)$. Then $\mathbf{VS}(R(P)) = \bigvee_{i\in I}\mathbf{VS}(f_i)$: if $f \in R(P)$ then, since $S = \bigvee_{i\in I}\mathbf{VS}(f_i)$ belongs to $\mathcal{F}$, so does its complementary set $S^c$, and thus $\mathbf{1}_{S^c}f \in R(P)$; then, by maximality of the family $(f_i)$, we have $\mathbf{1}_{S^c}.f = 0$, that is, $f = \mathbf{1}_S \cdot f$. We set $w = \sum_{i\in I}N(f_i)$ (which converges in $L_0(\Omega, \Sigma, \mu)$): this is a $\Sigma$-measurable weight with support $\mathbf{VS}(R(P))$. For every $f \in R(P)$ and every $i \in I$, $\mathbf{1}_{\mathbf{VS}(f_i)}w^{-1}N(f) = \mathbf{1}_{\mathbf{VS}(f_i)}N(f_i)^{-1}N(f)$ is $\mathcal{F}$-measurable; hence $w^{-1}N(f) = \sum_{i\in I}\mathbf{1}_{\mathbf{VS}(f_i)}w^{-1}N(f)$ is $\mathcal{F}$-measurable. □

We can now give the

*Proof of the Thm. 0.1.* Consider the new measure $\nu = w^p \cdot \mu$, which has support $\Omega_P = \mathbf{VS}(R(P))$ and set $T : L_p(\Omega_P, \Sigma_P, \mu) \to L_p(\Omega_P, \Sigma_P, \nu)$, defined by $Tf = w^{-1}f$ (we denote by $\Sigma_P$ the trace of $\Sigma$ on $\Omega_P$). Then $T$ is an isometry; $Y := (T\otimes \mathrm{Id_H})(R(P))$ is a $L_\infty(\mathcal{F}_P)$-module isometric to $R(P)$ and for every $f \in Y$ its new random norm $\widetilde{N}(f) = w^{-1}N(f)$ belongs to $L_p(\Omega_P, \mathcal{F}_P, \nu)$. It results from an argument in [13] that $Y$ is isometric to $\left(\bigoplus_{i\in I}L_p(\Omega_i, \mathcal{F}_{|\Omega_i}, \nu_{|\Omega_i}; H_i)\right)_{\ell_p}$, for some families $(\Omega_i)$ of pairwise almost disjoint sets in $\mathcal{F}$ and $(H_i)$ of Hilbert spaces. Set then $\widehat{w}_i = (\mathbb{E}(\mathbf{1}_{\Omega_i}\cdot w^p \mid \mathcal{F}))^{1/p}$, and define an isometry $S_i : L_p(\Omega_i, \mathcal{F}_{|\Omega_i}, \nu_{|\Omega_i}) \to L_p(\Omega_i, \mathcal{F}_{|\Omega_i}, \mu_{|\Omega_i})$ by $S_i f = \widehat{w}_i \cdot f$. Then each $S_i\otimes \mathrm{Id_{H_i}}$ is an onto isometry $L_p(\Omega_i, \mathcal{F}_{|\Omega_i}, \nu_{|\Omega_i}; H_i) \to L_p(\Omega_i, \mathcal{F}_{|\Omega_i}, \mu_{|\Omega_i}; H_i)$; the collection of these isometries induces an isometry of the corresponding $\ell_p$-direct sums. The proof of Thm. 0.1 is complete. □

Let us finally adapt to the present situation the argument of [13] for the commodity of the reader (and for further reference in Section 5).

**Lemma 4.4.** *Let $(\Omega, \Sigma, \nu)$ be a localizable measure space, $\mathcal{F}$ be a sub-sigma algebra such that $(\Omega, \mathcal{F}, \nu)$ is still localizable and $H$ be a Hilbert space. Let $Y$ be a closed $L_\infty(\mathcal{F})$-submodule of $L_p(\Omega, \Sigma, \nu; H)$ such that for every $f \in Y$ its random norm $N(f)$ is $\mathcal{F}$-measurable. Then there exist a family $(\Omega_i)_{i\in I}$ of pairwise almost disjoint members of $\mathcal{F}$, a family $(\mathcal{H}_i)$ of Hilbert spaces (of lower Hilbertian dimension than $H$) and a random norm preserving isometry from $Y$ onto $\left(\bigoplus_{i\in I}L_p(\Omega_i, \mathcal{F}_{|\Omega_i}, \nu_{|\Omega_i}; \mathcal{H}_i)\right)_{\ell_p}$.*

*Proof.* Note that by an elementary polarization argument all the scalar products $\langle\!\langle f \mid g\rangle\!\rangle$, $f, g \in Y$ are $\mathcal{F}$-measurable. Hence for every $f \in Y$, the projection $E_f$ restricts to a projection from $Y$ onto $L_p(\Omega, \mathcal{F}) \cdot u_f$. It results that for every closed $L_\infty(\mathcal{F})$-submodule $Z$ of $Y$ there is an orthogonal projection from $Y$ onto $Z$ (which is the restriction of the orthogonal projection from $L_p(\Omega, \Sigma; H)$ onto the closed $L_\infty(\Sigma)$-submodule generated by $Z$). In particular $Y = Z \oplus (Z^\perp \cap Y)$.

Remark that if $A \in \mathcal{F}$ is $\nu$-sigma-finite and $M \subset Y$ is a closed $L_\infty(\mathcal{F})$-submodule such that $\mathbf{VS}(M) \supset A$ then there exists $g \in M$ such that $\mathbf{VS}(g) = A$: take a maximal family $(g_n)$ in $M$ of norm-one elements with almost disjoint functional supports included in $A$; this family is necessarily at most countable and $\bigvee_n \mathbf{VS}(g_n) = A$; then set $g = \sum_n 2^{-n} g_n$.

Now we claim that for every $A \in \mathcal{F}$, $A \subset \mathbf{VS}(Y)$ with positive measure, there exists a $\mathcal{F}$-measurable subset $B$ of $A$ of positive measure and a family of pairwise orthogonal element $(f_\gamma)_{\alpha \in \Gamma_B}$, such that $\mathbf{VS}(f_\gamma) = B$ for every $\gamma \in \Gamma_B$, which generates $\mathbf{1}_B \cdot Y$ as closed $L_\infty(\mathcal{F})$-submodule. For, let $A' \subset A$ be a sigma-finite $\mathcal{F}$-measurable subset with positive measure, and $(g_\gamma)_{\gamma \in \Gamma}$ be a maximal family of pairwise orthogonal elements of $Y$ with $\mathbf{VS}(g_\gamma) = A'$. If this family generates $\mathbf{1}_{A'}.Y$ as closed $L_\infty(\mathcal{F})$-submodule we can take $B = A'$. If not, consider the set $M = \{ f \in Y \mid f \perp g_\gamma, \forall \gamma \in \Gamma \}$. Then $M$ is a closed $L_\infty(\mathcal{F})$-submodule of $Y$, and $\mathbf{VS}(M) \not\supset A'$ by the maximality of $(g_\gamma)_{\gamma \in \Gamma}$ (and the preceding remark). Let $B = A' \setminus \mathbf{VS}(M)$, then $(\mathbf{1}_B g_\gamma)_{\gamma \in \Gamma}$ is a maximal family in $\mathbf{1}_B \cdot Y$ of nonzero, pairwise orthogonal elements of $\mathbf{1}_B \cdot Y$. Consequently it generates $\mathbf{1}_B \cdot Y$ as $L_\infty(\mathcal{F})$-submodule, and moreover $\mathbf{VS}(\mathbf{1}_B g_\gamma) = B$ for every $\gamma \in \Gamma$.

Let now $(\Omega_i)_{i \in I}$ be a maximal family of $\mathcal{F}$-measurable almost disjoint subsets of $\mathbf{VS}(Y)$ of positive measure, such that there exists for each $i \in I$ a family $(f^i_\gamma)_{\gamma \in \Gamma_i}$ of pairwise orthogonal elements with $\mathbf{VS}(f^i_\gamma) = \Omega_i$ for every $\gamma \in \Gamma_i$, which generates $\mathbf{1}_{\Omega_i} \cdot Y$ as closed $L_\infty(\mathcal{F})$-submodule. By the claim, we have $\bigvee_{i \in I} \Omega_i = \mathbf{VS}(Y)$. Every $f \in \mathbf{1}_{\Omega_i} \cdot Y$ can be written $f = \sum_{\gamma \in \Gamma_i} \varphi_\gamma f^i_\gamma$ with $\varphi_\gamma \in L_0((\Omega_i, \mathcal{F}_{|\Omega_i}, \nu)$; then $N(f) = (\sum_{\gamma \in \Gamma_i} |\varphi_\gamma|^2 N(f^i_\gamma)^2)^{1/2} \in L_p(\Omega_i, \mathcal{F}_{|\Omega_i}, \nu)$.

Note that, by refining if necessary the "partition" $(\Omega_i)$ we may suppose that each $\Omega_i$ has finite $\nu$-measure. Then, replacing each $f^i_\gamma$ by $u_{f^i_\gamma} = N(f^i_\gamma)^{-1} f^i_\gamma$, we may assume that $N(f^i_\gamma) = \mathbf{1}_{\Omega_i}$. We have then $N(f) = (\sum_{\gamma \in \Gamma_i} |\varphi_\gamma|^2)^{1/2}$ for each $f = \sum_{\gamma \in \Gamma_i} \varphi_\gamma f^i_\gamma$ in $\mathbf{1}_{\Omega_i} \cdot Y$. Let $\mathcal{H}_i = \ell^2(\Gamma_i)$. Then $T_i : L_p(\Omega_i, \mathcal{F}_{|\Omega_i}, \nu; \mathcal{H}_i) \to \mathbf{1}_{\Omega_i} \cdot Y$, $\sum_{\gamma \in \Gamma_i} \varphi_\gamma e_\gamma \mapsto \sum_{\gamma \in \Gamma_i} \varphi_\gamma u_{f^i_\gamma}$ is an (onto) isometry (preserving the random norm), and finally $Y$ is isometric to $\big(\bigoplus_{i \in I} L_p(\Omega_i; \mathcal{H}_i)\big)_{\ell_p}$.

For proving the assertion about the Hilbertian dimension of $\mathcal{H}_i$, suppose that for some $i \in I$, the Hilbertian dimension $d_H$ of $H$ is strictly smaller than that of $\mathcal{H}_i$, $d_{\mathcal{H}_i}$. We distinguish two cases:

(i) if $H$ is finite dimensional: select a finite subset $\Gamma'_i$ of $\Gamma_i$ with cardinality $d_H + 1$; since $\langle\!\langle f^i_\gamma \mid f^i_\delta \rangle\!\rangle = 0$ for every $\gamma \neq \delta \in \Gamma'_i$, there exists $\omega \in \Omega$ such that $\langle\!\langle f^i_\gamma \mid f^i_\delta \rangle\!\rangle(\omega) = 0$, i.e. the vectors $f^i_\gamma(\omega)$, $\gamma \in \Gamma'_i$ of $H$ are pairwise orthogonal: a contradiction.

(ii) if $H$ is infinite dimensional: for every $x \in H$ the set $\{\gamma \in \Gamma_i \mid \langle\langle f^i_\gamma \mid \mathbf{1}_{\Omega_i} x\rangle\rangle \neq 0\}$ is at most countable (since $\sum_\gamma |\langle\langle f^i_\gamma \mid \mathbf{1}_{\Omega_i} x\rangle\rangle|^2 \leq N(\mathbf{1}_{\Omega_i} \cdot x)^2 = \mathbf{1}_{\Omega_i} \|x\|^2$), hence if $D$ is a dense set in $H$ of cardinality $d_H$, the set $\{\, \gamma \in \Gamma_i \mid \exists x \in D, \langle\langle f^i_\gamma \mid \ \mathbf{1}_{\Omega_i} x\rangle\rangle \neq 0 \,\}$ has cardinality $d_H < d_{\mathcal{H}_i} = \#\Gamma_i$. Hence there exists some $\gamma \in \Gamma_i$ such that $f^i_\gamma \perp \mathbf{1}_{\Omega_i} x$ for every $x \in D$, and thus for every $x \in H$, which means $f^i_\gamma = 0$, a contradiction. □

*Remark* 4.5. The final argument in the proof of Lemma 4.4 shows indeed that if $L_p(\Omega, \Sigma, \nu; \mathcal{H})$ embeds in $L_p(\Omega, \Sigma, \nu; H)$ by a modularly isometric map then $\dim \mathcal{H} \leq \dim H$.

*Remark* 4.6. In a forthcoming paper ([10]) it will be proved that contractively complemented sublattices of $L_p(L_q)$ are isometric to "abstract $L_p(L_q)$ spaces", i.e. bands in $L_p(L_q)$ spaces. Let us show how this permits to deduce shortly the essence of Thm. 0.1 from Lemma 4.1.

As in the proof of Lemma 4.1 let $(f_\alpha)_{\alpha\in\mathcal{A}}$ be a maximal family of non zero, pairwise orthogonal elements of $R(P)$ and $Z = \bigoplus_\alpha L_p(\Omega, \Sigma, \mu) \cdot u_{f_\alpha}$ be the closed $L_\infty(\Sigma)$-submodule generated by $R(P)$. There is clearly a $\Sigma$-modular isometry $U$ from the closed submodule Z onto a band $Y$ of the Banach lattice $L_p(\Omega, \Sigma, \mu; \mathcal{H})$ where $\mathcal{H}$ is the discrete Banach lattice $\ell_2(A)$, such that $Ue_\alpha = N(f_\alpha)e_\alpha$, where $(e_\alpha)_{\alpha\in A}$ is a Hilbertian basis of $\mathcal{H}$. Then $P\,|_Z$ is similar by $U$ to a contractive projection $\widehat{P}$ of $Y$ which preserves the spaces $Y_\alpha = L_p(A_\alpha) \cdot e_\alpha$ (where $A_\alpha = \operatorname{Supp} N(f_\alpha)$) by Lemma 2.1 if $p = 1$ and Cor. 3.6 if $p > 1$, as well as the elements $N(f_\alpha) \cdot e_\alpha$. By the classical (scalar) theorem of Douglas if $p = 1$, Andô if $p > 1$, $\widehat{P}\,|_{Y_\alpha}$ is positive and its image is a sublattice of $Y_\alpha$. Since $Y = \bigoplus_\alpha Y_\alpha$ is a decomposition in disjoint subbands, $\widehat{P}$ is itself positive and its range is a sublattice of $Y$, hence of $L_p(\mathcal{H})$. By the analysis of contractive projections on sublattices in $L_p(L_q)$-spaces developed in [10], the range $R(\widehat{P})$ is an abstract $L_p(L_2)$-space, hence by [13] it is Banach-isometric to a $\ell_p$-direct sum $\bigoplus_{i\in I} L_p(\Omega_i, H_i)$, where the $H_i$ are Hilbert spaces. □

## 5. Structure of the contractive projections

**Theorem 5.1.** *Let $1 \leq p < \infty$, $p \neq 2$. For every contractive projection $P$ of $L_p(H)$ there exist a family $(u_\gamma)_{\gamma\in\Gamma}$ of pairwise orthogonal elements of $L_\infty(H)$, a positive contractive projection $\widetilde{P}$ of $L_p(\Omega)$ and, if $p = 1$, a contractive linear operator $V : L_1(H) \to L_1(H)$ verifying $\ker V \supset \mathbf{1}_A L_1(H)$ where $A = \bigvee_{\gamma\in\Gamma} \mathbf{VS}(u_\gamma)$, and $R(V) \subset \sum_\gamma R(\widetilde{P}) \cdot u_\gamma$, such that:*

$$Pf = \begin{cases} \sum_\gamma \widetilde{P}(\langle\langle f \mid u_\gamma\rangle\rangle) u_\gamma & \text{if } p \neq 1, \\ \sum_\gamma \widetilde{P}(\langle\langle f \mid u_\gamma\rangle\rangle) u_\gamma + V(f) & \text{if } p = 1, \end{cases} \tag{20}$$

*for every $f \in L_p(H)$.*

*Conversely for every family $(u_\gamma)_{\gamma\in\Gamma}$ of pairwise orthogonal elements of $L_\infty(H)$, every positive contractive projection $\widetilde{P}$ of $L_p(\Omega)$ [and every linear contraction $V$ of $L_1(H)$ satisfying the previous conditions of kernel and range in the case $p = 1$], the formula (20) defines a contractive projection $P$ of $L_p(H)$.*

*Moreover if $p \neq 1$ the inequality $N(Pf) \leq \widetilde{P}(N(f))$ holds for every $f \in L_p(H)$ [this happens also for a contractive projection of $L_1(H)$ for which the operator $V$ of formula (20) is zero]. ($\widetilde{P}$ is a "majorizing $L_p$-contraction" for $P$ in the terminology of [8]).*

The proof of Thm. 5.1 will require the two following Lemmas, the first of which is specific to the $p = 1$ case:

**Lemma 5.2.** *Let $P$ be a contractive projection in $L_1(H)$. Then $Pf = 0$ for every $f \in L_1(H)$ with $\mathbf{VS}(f) \subset \mathbf{VS}(R(P))$ and $f \perp R(P)$.*

*Proof.* Assume that $f \perp R(P)$ and $\mathbf{VS}(f) \subset \mathbf{VS}(h)$ for some $h \in R(P)$. Then $g := Pf + \mathbf{1}_{(\mathbf{VS}(Pf))^c} h$ belongs to $R(P)$ and $\mathbf{VS}(g) \supset \mathbf{VS}(f) \cup \mathbf{VS}(Pf)$. We have for every $t > 0$:

$$\begin{aligned}\int (N(g)^2 + t^2 N(f)^2)^{1/2}\, d\mu &= \|g + tf\| \\ &\geq \|P(g + tf)\| = \|g + tPf\| \\ &= (1 + t)\|Pf\| + \|\mathbf{1}_{(\mathbf{VS}(Pf))^c} \cdot h\| \\ &= \|g\| + t\|Pf\|.\end{aligned}$$

Hence:

$$\|Pf\| \leq \lim_{t\to 0} \left(\frac{\|g + tf\| - \|g\|}{t}\right) = \lim_{t\to 0} \int \frac{(N(g)^2 + t^2 N(f)^2)^{1/2} - N(g)}{t}\, d\mu = 0. \quad \square$$

**Lemma 5.3.** *Let $P$ be a contractive projection in $L_p(H)$. There exists a positive contractive projection $\widetilde{P}$ on $L_p(\Omega, \Sigma, \mu)$ such that $P(\varphi \cdot u_f) = (\widetilde{P}\varphi) \cdot u_f$ for every $f \in R(P)$ and $\varphi \in L_p(\Omega, \Sigma, \mu)$.*

*Proof.* Let $\mathcal{F}$ be the $\sigma$-algebra of Lemma 4.2 and $w$ be the weight of Lemma 4.3. Define $\widetilde{P}_f(\varphi)$ as in the proof of Lemma 4.2. Recall that for every $f \in R(P)$ the function $w^{-1}N(f)$ is $\mathcal{F}$-measurable. We have then for every $h \in L_\infty(\Omega, \mathcal{F}, \mu)$:

$$\begin{aligned}\int \widetilde{P}_f(\varphi) h N(f)^{p-1}\, d\mu &= \int \mathbb{E}^{\mathcal{F}}_{N(f)^p\cdot\mu}(N(f)^{-1}\mathbf{1}_{\mathbf{VS}(f)}\varphi) h\, N(f)^p \cdot d\mu \\ &= \int \mathbf{1}_{\mathbf{VS}(f)}\varphi \cdot h N(f)^{p-1} \cdot d\mu\end{aligned}$$

$$= \int (\mathbf{1}_{\mathbf{VS}(R(P))} w^{-1}\varphi) \cdot (\mathbf{1}_{\mathbf{VS}(f)} h (w^{-1}N(f))^{p-1}) w^p \cdot d\mu$$
$$= \int \mathbb{E}^{\mathcal{F}}_{w^p\mu}(\mathbf{1}_{\mathbf{VS}(R(P))} w^{-1}\varphi) \mathbf{1}_{\mathbf{VS}(f)} h (w^{-1}N(f))^{p-1} w^p \cdot d\mu$$
$$= \int \mathbf{1}_{\mathbf{VS}(f)} w \mathbb{E}^{\mathcal{F}}_{w^p\mu}(\mathbf{1}_{\mathbf{VS}(R(P))} w^{-1}\varphi) h N(f)^{p-1} d\mu.$$

Hence $\widetilde{P}_f\varphi = \mathbf{1}_{\mathbf{VS}(f)}\widetilde{P}\varphi$ if we set $\widetilde{P}\varphi = w\mathbb{E}^{\mathcal{F}}_{w^p\cdot\mu}(\mathbf{1}_{\mathbf{VS}(R(P))}w^{-1}\varphi)$ for every $\varphi \in L_p(\Omega,\Sigma,\mu)$. Then $\widetilde{P}$ is a positive contractive projection in $L_p(\Omega,\Sigma,\mu)$ and $P(\varphi.u_f) = \widetilde{P}(\varphi)\cdot u_f$ for every $f \in R(P)$ and $\varphi \in L_p(\Omega,\Sigma,\mu)$. □

*Proof of Thm. 5.1.* Let $Q$ be the orthogonal projection from $L_p(H)$ onto the closed submodule generated by $R(P)$. It results from Lemma 4.1 that if $(f_\gamma)_{\gamma\in\Gamma}$ is a maximal family of pairwise orthogonal elements of $R(P)$ then $Q = \sum_{\gamma\in\Gamma} E_{f_\gamma}$ (convergence for s.o.t.), hence $PQ = \sum_{\gamma\in\Gamma} PE_{f_\gamma}$. If $p > 1$ we know by Cor. 3.6 that $E_{f_\gamma}P = PE_{f_\gamma}$ for every $\gamma$, hence $P = QP = PQ$. If $p = 1$ let $\Pi : L_1(H) \to L_1(H)$ the projection defined by $\Pi f = \mathbf{1}_{\mathbf{VS}(R(P))}\cdot f$, then $\Pi$ and $I-\Pi$ are contractive. We have $Q\Pi = \Pi Q = Q$ and it results from the preceding Lemma 5.2 that $P(I-Q)\Pi = 0$. Hence $P = PQ + V$, where $V = P(I-\Pi)$.

Let us express now $PE_f$ when $f \in R(P)$. If $\widetilde{P}$ is the positive projection in $L_p(\Omega,\Sigma,\mu)$ defined in Lemma 5.3 we have for every $g \in L_p(\Omega,\Sigma,\mu;H)$

$$PE_f g = P(\langle\langle g \mid u_f\rangle\rangle \cdot u_f) = \widetilde{P}(\langle\langle g \mid u_f\rangle\rangle)\cdot u_f$$

The formula (20) in Thm. 5.1 is now clear if we set $u_\gamma = u_{f_\gamma}$.

Conversely given $(u_\gamma)$, $\widetilde{P}$ and $V$, let us prove first that $P$ is a contraction. We have for every finite subset $G$ of $\Gamma$ (using the positivity of $\widetilde{P}$):

$$\begin{aligned} N\Big(\sum_{\gamma\in G}\widetilde{P}(\langle\langle f \mid u_\gamma\rangle\rangle)u_\gamma\Big) &= \Big(\sum_{\gamma\in G}|\widetilde{P}(\langle\langle f \mid u_\gamma\rangle\rangle)|^2\Big)^{1/2} \\ &= \bigvee\Big\{\Big|\sum_{\gamma\in G} a_\gamma\widetilde{P}\langle\langle f \mid u_\gamma\rangle\rangle\Big| \Big| a_\gamma\in\mathbb{C}, \sum_{\gamma\in G}|a_\gamma|^2 \le 1\Big\} \\ &\le \widetilde{P}\Big(\bigvee\Big\{\Big|\sum_{\gamma\in G} a_\gamma\langle\langle f \mid u_\gamma\rangle\rangle\Big| \Big| a_\gamma\in\mathbb{C}, \sum_{\gamma\in G}|a_\gamma|^2 \le 1\Big\}\Big) \\ &= \widetilde{P}\Big(\Big(\sum_{\gamma\in G}|\langle\langle f \mid u_\gamma\rangle\rangle|^2\Big)^{1/2}\Big). \end{aligned}$$

Hence $\|\sum_{\gamma\in G}\widetilde{P}(\langle\langle f \mid u_\gamma\rangle\rangle)u_\gamma\|^p \le \int(\sum_{\gamma\in G}|\langle\langle f \mid u_\gamma\rangle\rangle|^2)^{p/2}\,d\mu$ and the sum $P_0 f := \sum_{\gamma\in\Gamma}\widetilde{P}(\langle\langle f \mid u_\gamma\rangle\rangle)u_\gamma$ converges in $L_p(H)$. Moreover

$$N(P_0 f) \le \widetilde{P}\Big(\Big(\sum_{\gamma\in\Gamma}|\langle\langle f \mid u_\gamma\rangle\rangle|^2\Big)^{1/2}\Big) \le \widetilde{P}(N(\mathbf{1}_A\cdot f))$$

(see section 1.2 and the proof of Lemma 1.1) and

$$\|P_0 f\| \leq \|\mathbf{1}_A \cdot f\|$$

where $A = \bigvee_\gamma \mathbf{VS}(u_\gamma)$. That $P_0$ is a projection follows immediately from the fact that $\widetilde{P}$ is. If $p = 1$ we have to care with the contraction $V$. Since $\|Vf\| \leq \|\mathbf{1}_{A^c} \cdot f\|$ we obtain $\|Pf\| \leq \|\mathbf{1}_A \cdot f\| + \|\mathbf{1}_{A^c} \cdot f\| = \|f\|$. Then since $VP_0 = 0$, $P_0 V = V$, it follows clearly that $P = P_0 + V$ is a projection. □

We can now give the structure theorem for contractive projections:

**Theorem 5.4.** *For every contractive projection $P$ of $L_p(\Omega, \Sigma, \mu; H)$ ($1 \leq p < \infty$, $p \neq 2$) there exist:*

- *a modularly isometric automorphism $W$ of $L_p(H)$;*
- *a family $(\Omega_i)_{i\in I}$ of pairwise almost disjoint $\Sigma$-measurable subsets of $\Omega$ of positive measure;*
- *a family $(\mathcal{H}_i)_{i\in I}$ of Hilbert spaces;*
- *for every $i \in I$ a (strong operator) measurable family $(U_{i,\omega})_{\omega\in\Omega}$ of isometric embeddings of $\mathcal{H}_i$ into $H$;*
- *a positive contractive projection $\widetilde{P}$ of $L_p(\Omega, \Sigma, \mu)$ commuting with the band projections associated with the sets $\Omega_i$;*
- *and if $p = 1$ a contraction $V$ from $L_1(S, \Sigma_{|S}, \mu_{|S}; H)$ into $R(P)$, where $S = \Omega \setminus \bigvee_i \Omega_i$*

*such that (setting $V = 0$ if $p > 1$):*

$$P = WU\Big(\sum_i \widetilde{P} M_{\Omega_i} \otimes \mathrm{Id}_{\mathcal{H}_i}\Big) U^\sharp W^{-1} + V$$

*where $M_{\Omega_i}$ denotes the multiplication operator by the characteristic function $\mathbf{1}_{\Omega_i}$; $U$ is the modularly isometric embedding of $\bigoplus L_p(\Omega_i, \Sigma_{|\Omega_i}, \mu_{|\Omega_i}; \mathcal{H}_i)$ into $L_p(H)$ naturally associated with the family $(U_{i,\omega})_{i\in I,\omega\in\Omega}$ by mean of the formula:*

$$(Uf)(\omega) = U_{i,\omega}(f(\omega)) \quad \text{when } \omega \in \Omega_i$$

*and similarly $U^\sharp : L_p(H) \to \bigoplus L_p(\Omega_i, \Sigma_{|\Omega_i}, \mu_{|\Omega_i}; \mathcal{H}_i)$ is the modularly contractive map associated with the family $(U^*_{i,\omega})_{i\in I,\omega\in\Omega}$.*

*Remark* 5.5. In fact the families $(U_{i,\omega})_{\omega\in\Omega}$ may be chosen locally constant, i.e. there is a partition of $\Omega_i$ in $\Sigma$-measurable subsets of positive measure on which $U_{i,\omega}$ is constant.

*Remark* 5.6. In the case where $H$ is separable, it is a standard (and easy) fact that every modularly isometric automorphism $W$ of $L_p(H)$ is associated with a measurable family $(W_\omega)_{\omega\in\Omega}$ of unitary operators on $H$; so we recover the Theorem 0.3 of the Introduction.

*Proof.* By the proof of Thm. 0.1 in Section 4, there are a sub-$\sigma$algebra $\mathcal{F}$ of $\Sigma$, a family $(\Omega_i)_{i\in I}$ of pairwise almost disjoint elements of $\mathcal{F}$, a positive weight $w$ on $\Omega$ with support $\bigvee_{i\in I}\Omega_i$, a family $(\mathcal{H}_i)_{i\in I}$ of Hilbert spaces and for every $i\in I$ an isometry $T_i$ from $L_p(\Omega_i,\mathcal{F}_{|\Omega_i},\nu\mid_{\Omega_i};\mathcal{H}_i)$ into $L_p(\Omega_i,\mathcal{F}_{|\Omega_i},\nu_{|\Omega_i};H)$ such that $R(P)=\bigoplus_{i\in I}w\cdot R(T_i)$ and moreover $N(T_if)=N(f)$ for all $f\in L_p(\Omega_i,\mathcal{F}_{|\Omega_i},\nu_{|\Omega_i};\mathcal{H}_i)$ (recall that $\nu=w^p\cdot\mu$). Moreover $P$ commutes with the action of $L_\infty(\mathcal{F})$, in particular with the multiplication operators $M_{\Omega_i}$.

Each $T_i$ extends uniquely to a modularly isometric map $\widetilde{T}_i$ from $L_p(\Omega_i,\Sigma_{|\Omega_i},\nu_{|\Omega_i};\mathcal{H}_i)$ onto the closed $L_\infty(\Sigma)$-submodule generated by $R(T_i)$ in $L_p(\Omega,\Sigma,\nu;H)$: set simply $\widetilde{T}_i(\sum_k\varphi_kf_k)=\sum_k\varphi_kT_i(f_k)$ when $\varphi_1,\dots,\varphi_n\in L_\infty(\Omega_i,\Sigma_{|\Omega_i})$ and $f_1,\dots,f_k\in L_p(\Omega_i,\mathcal{F}_{|\Omega_i},\nu_{|\Omega_i};\mathcal{H}_i)$ and verify that $N(\sum_k\varphi_kT_i(f_k))=N(\sum_k\varphi_kf_k)$ (since $T_i$ preserves the random scalar products).

Now define $S_i:L_p(\Omega_i,\Sigma_{|\Omega_i},\mu_{|\Omega_i};\mathcal{H}_i)\to L_p(\Omega,\Sigma,\mu;H)$ by $S_if=w\widetilde{T}_i(w^{-1}f)$: the range $R(S_i)=wR(\widetilde{T}_i)$ is exactly $\mathbf{1}_{\Omega_i}\cdot Z$, where $Z$ is the closed $L_\infty(\Sigma)$-submodule generated by $R(P)$. We can glue up the maps $S_i$ and obtain a modularly isometric embedding $S$ from $\bigoplus_{i\in I}L_p(\Omega_i,\Sigma_{|\Omega_i},\mu_{|\Omega_i};\mathcal{H}_i)$ into $L_p(\Omega,\Sigma,\mu;H)$, with range $R(S)=Z$.

By Lemma 5.3 there exists a positive projection $\widetilde{P}$ on $L_p(\Omega,\Sigma,\mu)$ such that $P(\varphi.u_f)=(\widetilde{P}\varphi).u_f$ for every $f\in R(P)$ and $\varphi\in L_p(\Omega,\Sigma,\mu)$. Note that $\widetilde{P}$ is $\mathcal{F}$-modular, in particular it commutes with every multiplication operator $M_{\Omega_i}$, $i\in I$.

If $A\in\mathcal{F}$ is a $\nu$-integrable subset of $\Omega_i$ and $e\in H_i$ we have $S_i(\mathbf{1}_Aw\otimes e)=wT_i(\mathbf{1}_A\otimes e)\in R(P)$, and $N(S_i(\mathbf{1}_A\cdot w\otimes e))=N(\mathbf{1}_A\cdot w\otimes e)=\mathbf{1}_A\cdot w$, and consequently for $f=S_i(\mathbf{1}_Aw\otimes e)$ we have $f=w\cdot u_f$. Thus for every $\psi\in L_\infty(\Omega,\Sigma,\mu)\cap L_p(\Omega,\Sigma,\mu)$ we have

$$\begin{aligned}PS_i(\psi\mathbf{1}_Aw\otimes e)&=P(\psi\cdot wu_f)=\widetilde{P}(\psi\cdot w\mathbf{1}_{\Omega_i})\cdot u_f=\widetilde{P}(\psi\cdot w\mathbf{1}_{\Omega_i})\cdot w^{-1}S_i(\mathbf{1}_Aw\otimes e)\\&=S_i(\widetilde{P}(\psi\cdot w\mathbf{1}_{\Omega_i})\cdot w^{-1}\cdot\mathbf{1}_Aw\otimes e)=S_i(\widetilde{P}(\psi\cdot w\mathbf{1}_{\Omega_i})\mathbf{1}_A\otimes e),\end{aligned}$$

hence by linearity and density we have for every $\varphi\in L_p(\Omega_i,\Sigma_{|\Omega_i},\mu_{|\Omega_i})$ and $e\in\mathcal{H}_i$:

$$PS_i(\varphi\otimes e)=S_i(\widetilde{P}(\varphi)\otimes e),$$

that is, the restriction of $P$ to $\mathbf{1}_{\Omega_i}\cdot Z$ is similar by $S_i$ to the projection $\widetilde{P}\otimes\mathrm{id}_{\mathcal{H}_i}$; consequently the restriction of $P$ to $Z$ is similar by $S$ to the projection $\sum_{i\in I}\widetilde{P}M_{\Omega_i}\otimes\mathrm{id}_{\mathcal{H}_i}$.

In the case where $Z=L_p(\Omega,\Sigma,\mu;H)$ we have necessarily $\dim H=\dim\mathcal{H}_i$ for every $i\in I$ since $S_i$ is a modularly isometric map from $L_p(\Omega_i;\mathcal{H}_i)$ onto $\mathbf{1}_{\Omega_i}.Z=$

$L_p(\Omega_i; H)$ (see Remark 4.5). Thus we may assume that $\mathcal{H}_i = H$ and the conclusion of Thm. 5.4 is obtained with $W = S$ and $U = \mathrm{Id}$.

In the general case we apply Lemma 4.4 to the $L_\infty(\Sigma)$ submodule $Z^\perp$. We find a family $(\Omega'_j)_{j\in J}$ of pairwise almost disjoint members of $\Sigma$, a family $(\mathcal{K}_j)_{j\in J}$ of Hilbert spaces and a modularly isometric map $S'$ from $(\bigoplus_{j\in J} L_p(\Omega'_j, \Sigma_{|\Omega'_j}, \mu_{|\Omega'_j}; \mathcal{K}_j))_{\ell_p}$ onto $Z^\perp$. Note that now the sets $\Omega'_j$ have no reason to belong to the smaller $\sigma$-algebra $\mathcal{F}$. We have $\bigvee_j \Omega'_j = \mathbf{VS}(Z^\perp)$. For the commodity of the notation we may assume $\bigvee_j \Omega'_j = \Omega$, adding if necessary one extra set $\Omega'_0 = \Omega \setminus \bigvee_j \Omega'_j$ for which we set $\mathcal{K}_0 = \{0\}$, the 0-dimensional Hilbert space. Similarly, up to the cost of adding one extra set $\Omega_0 = \Omega \setminus \Omega_P$ and setting $\mathcal{H}_0 = \{0\}$, we may assume that $\bigvee_i \Omega_i = \Omega$. We may also refine the partition $(\Omega'_j)$ by setting $\Omega'_{ij} = \Omega_i \cap \Omega'_j$ and removing the $\Omega'_{ij}$ which are almost void. This operation gives a doubly indexed family $(\Omega'_{ij})_{i\in I; j\in J_i}$.

For every $i \in I, j \in J_i$, set $L_{ij} = \mathcal{H}_i \oplus \mathcal{K}_j$ (direct Hilbertian sum). Then $L_p(\Omega'_{ij}; \mathcal{H}_i)$ and $L_p(\Omega'_{ij}; \mathcal{K}_j)$ identify naturally to a pair of mutually orthogonal $L_\infty(\Sigma)$-submodules of $L_p(\Omega'_{ij}; L_{ij})$: if $u^0_{ij}$ and $u'^0_{ij}$ are the inclusion maps of $\mathcal{H}_i$, resp. $\mathcal{K}_j$ into $L_{ij}$ then $U^0_{ij} = \mathrm{id} \otimes u^0_{ij}$ and $U'^0_{ij} = \mathrm{id} \otimes u'^0_{ij}$ are the corresponding embeddings of $L_p(\Omega'_{ij}; \mathcal{H}_i)$ and $L_p(\Omega'_{ij}; \mathcal{K}_j)$ into $L_p(\Omega'_{ij}; L_{ij})$. Since $u^{0*}_{ij}$ and $u'^{0*}_{ij}$ are the orthogonal projections $L_{ij} \to \mathcal{H}_i$, resp. $L_{ij} \to \mathcal{K}_j$, we see that $U^{0\sharp}_{ij}$ and $U'^{0\sharp}_{ij}$ are the orthogonal projections (in the sense given in Section 1.2) onto $L_p(\Omega'_{ij}; \mathcal{H}_i)$, resp. $L_p(\Omega'_{ij}; \mathcal{K}_j)$.

Now define $W^0_{ij} : L_p(\Omega'_{ij}; L_{ij}) \to L_p(\Omega'_{ij}; H)$ by $W^0_{ij} f = S_i(U^{0\sharp}_{ij} f) + S'_j(U'^{0\sharp}_{ij} f)$: we have

$$N(W^0_{ij} f)^2 = N(S_i(U^{0\sharp}_{ij} f))^2 + N(S'_i(U^{0\sharp}_{ij} f))^2 = N(U^{0\sharp}_{ij} f)^2 + N(U'^{0\sharp}_{ij} f)^2 = N(f)^2$$

since $S_i$ and $S'_i$ are modularly isometric and have values in orthogonal subspaces $Z$, resp. $Z^\perp$. Hence $W^0_{ij}$ is modularly isometric and $R(W^0_{ij}) = \mathbf{1}_{\Omega'_{ij}} Z + \mathbf{1}_{\Omega'_{ij}} Z^\perp = L_p(\Omega'_{ij}; H)$.

We know by the proof of Thm. 5.1 that $P = PQ + V$, where $Q$ is the orthogonal projection onto $Z$. Since $V$ satisfies the requirements of the theorem, we look only for a representation of $P_0 = PQ$. From the first part of the proof we know that $P_0 S_i = S_i(\widetilde{P} \otimes \mathrm{id}_{\mathcal{H}_i})$; on the other hand $P_0 S'_j = 0$ since $R(S'_j) \subset Z^\perp = \ker Q$. Hence, for every $f \in L_p(\Omega_{ij}; L_{ij})$,

$$P_0 W^0_{ij} f = P_0 S_i U^{0\sharp}_{ij} f + P_0 S'_j U'^{0\sharp}_{ij} f = S_i(\widetilde{P} \otimes \mathrm{id}_{\mathcal{H}_i}) U^{0\sharp}_{ij} f = W^0_{ij} U^0_{ij} (\widetilde{P} \otimes \mathrm{id}_{\mathcal{H}_i}) U^{0\sharp}_{ij} f$$

i.e. $P_0$ is similar by $W^0_{ij}$ to $U^0_{ij}(\widetilde{P} M_{\Omega'_{ij}} \otimes \mathrm{id}_{\mathcal{H}_i}) U^{0\sharp}_{ij}$.

Since $L_p(\Omega_{ij}; L_{ij})$ is modularly isometric to $L_p(\Omega_{ij}; H)$ (by $W^0_{ij}$), we have $\dim L_{ij} = \dim H$ by Rem. 4.5, so we may identify $L_{ij}$ with $H$ by an isomorphism $\theta_{ij}$. This isomorphism induces in turn a modular isometry $\Theta_{ij} = \mathrm{Id} \otimes \theta_{ij}$ from $L_p(\Omega'_{ij}; L_{ij})$ onto $L_p(\Omega'_{ij}; H)$. Set $W_{ij} = W^0_{ij} \Theta^{-1}_{ij}$: then $W_{ij}$ is a modular automorphism of $L_p(\Omega'_{ij}; H)$. Let also $u_{ij} = \theta_{ij} \circ u^0_{ij}$ be the embedding of $\mathcal{H}_i$ into $H$ resulting from this identification and $U_{ij} = \mathrm{id}_{L_p(\Omega_{ij})} \otimes u_{ij} = \Theta_{ij} U^0_{ij}$ be the associated embedding of $L_p(\Omega_{ij}; \mathcal{H}_i)$ into $L_p(\Omega_{ij}; \mathcal{H})$. Since $\Theta_{ij}^{-1} = \Theta_{ij}^\sharp$ we see that $P_0$ is similar by $W_{ij}$ to $U_{ij}(\widetilde{P} M_{\Omega'_{ij}} \otimes \mathrm{id}_{\mathcal{H}_i}) U_{ij}^\sharp$.

Finally we glue up the automorphisms $W_{ij}$ to an automorphism $W$ of $L_p(\Omega; \mu; H)$ by setting

$$Wf = \sum_{i \in I} \sum_{j \in J_i} W_{ij} M_{\Omega'_{ij}} f$$

and similarly we glue up the embeddings $U_{ij}$ to an embedding $U$ of $\bigoplus_{i \in I} L_p(\Omega_i; \mathcal{H}_i)$ into $L_p(\Omega; H)$. The maps $W$ and $U$ are still modularly isometric and $P_0$ is similar by $W$ to $U(\sum_{i \in I} \widetilde{P} M_{\Omega_i} \otimes \mathrm{id}_{\mathcal{H}_i}) U^\sharp$. □

## 6. Annex: a proof of Corollary 3.6 specific to the complex case

The following proof is an adaptation of that of Thm. 4.1 in [2]. We assume that $2 < p < \infty$ (the case $1 < p < 2$ follows by duality).

If $f \in R(P)$ we introduce besides the projection $E_f$ (defined in §1) the operators $F_f$ and $G_f$ defined by

$$F_f g = \mathbf{1}_{\mathbf{VS}(f)^c} g \ ; \ G_f g = \mathbf{1}_{\mathbf{VS}(f)} g - E_f g.$$

Then $E_f$, $G_f$ and $F_f$ are commuting modularly contractive projections in $L_p(H)$ with $E_f + F_f + G_f = I$.

Let $f, g \in R(P)$, then the elements $A(f,g)$ and $B(f,g)$ defined in §3 (eqs. (5) and (5bis) belong to the range of $P^*$; so do the sum and difference: $M_f(g) := A(f,g) + B(f,g)$ and $\Gamma_f(g) := \frac{p}{p-2}[A(f,g) - B(f,g)]$ belong to $R(P^*)$. Set

$$Q_f g = \langle\!\langle u_f, g \rangle\!\rangle \, u_f.$$

We have then

$$M_f(g) = N(f)^{p-2}[2g + (p-2)E_f g]$$
$$\Gamma_f(g) = pN(f)^{p-2} Q_f g$$

Then $M_f$, resp. $\Gamma_f$ are bounded linear, resp. antilinear operators from $L_p(H)$ into $L_{p_*}(H)$, and $Q_f$ is a contractive antilinear endomorphism of $L_p(H)$ such that $Q_f^2 = E_f$; moreover:

$$M_f P = P^* M_f P, \qquad \Gamma_f P = P^* \Gamma_f P. \tag{21}$$

Consider the positive symmetric bounded bilinear form defined on $L_p(H)$ by

$$(g,h)_f := \langle M_f(g), h \rangle = \int N(f)^{p-2} \langle\!\langle (2I + (p-2)E_f) g \mid h \rangle\!\rangle \, d\mu$$

Note that $\Gamma_f = M_f Q_f = Q_{J_p f} M_f$ and $Q_f^* = Q_{J_p f}$; then $Q_f$ is hermitian for $(\cdot,\cdot)_f$ since

$$\begin{aligned}(Q_f g, h)_f &= \langle M_f Q_f g, h\rangle = \langle Q_f g, M_f h\rangle \\ &= \langle Q_{J_p f} M_f h, g\rangle = \langle M_f Q_f h, g\rangle = (Q_f h, g)_f\end{aligned}$$

On the other hand $P$ is hermitian for $(\cdot,\cdot)_f$ since (using (21))

$$\begin{aligned}(Pg, h)_f &= \langle M_f Pg, h\rangle = \langle P^* M_f Pg, h\rangle = \langle M_f Pg, Ph\rangle = (Pg, Ph)_f \\ &= \overline{(Ph, Pg)_f} = \overline{(Ph, g)_f} = (g, Ph)_f\end{aligned}$$

Let $N_f$ be the kernel of the form $(\cdot,\cdot)_f$: we have $g \in N_f$ iff $(g,g)_f = 0$ iff $(g,h)_f = 0$ for all $h \in L_p(H)$ (by Cauchy-Schwartz inequality). Then $PN_f \subset N_f$ since

$$(Pg, Pg)_f = (g, Pg)_f = 0 \quad \text{if } g \in N_f$$

On the other hand the operator $2\cdot \mathbf{1}_{\mathbf{VS}(f)} + (p-2)E_f$ maps $L_p(H)$ onto $\mathbf{1}_{\mathbf{VS}(f)} L_p(H)$; hence $g \in N_f$ iff $\langle N(f)^{p-2} g, h\rangle = 0$ for every $h \in 1_{\mathbf{VS}(f)} L_p(H)$ iff $N(f)^{p-2} g = 0$ iff $\mathbf{1}_{\mathbf{VS}(f)} g = 0$.

We have thus $R(F_f) = N_f$ and consequently

$$PF_f = F_f P F_f$$

Since $L_p(H)$ is a strictly convex Banach space as well as its dual, we have by the auxiliary Lemma 6.1 below:

$$PF_f = F_f P$$

Let us show that $Q_f P$ is hermitian for $(\cdot,\cdot)_f$, using eq. (21) again:

$$\begin{aligned}(Q_f Pg, h)_f &= \langle M_f Q_f Pg, h\rangle = \langle \Gamma_f Pg, h\rangle \\ &= \langle P^* \Gamma_f Pg, h\rangle = \langle \Gamma_f Pg, Ph\rangle \\ &= (Q_f Pg, Ph)_f = (Q_f Ph, Pg)_f \\ &= (Q_f Ph, g)_f\end{aligned}$$

Since $Q_f$ and $P$ are separately hermitian for $(\cdot,\cdot)_f$ we have

$$(Q_f Pg, h)_f = (PQ_f h, g)_f,$$

hence $(PQ_f - Q_f P)h \in N_f$, i.e. $(I - F_f)PQ_f = (I - F_f)Q_f P$. Composing on the left by $G_f$ and on the right by $Q_f$, or conversely, we obtain

$$G_f P E_f = 0 = E_f P G_f.$$

Since, on the other hand,

$$F_f P E_f = P F_f E_f = 0 = E_f F_f P = E_f P F_f,$$

we obtain

$$PE_f = E_f P E_f = E_f P. \qquad \square$$

We state now and give a proof of the announced auxiliary Lemma.

**Lemma 6.1.** *Let $X$ be a strictly convex Banach space with strictly convex dual, and $P, Q$ two contractive projections on $X$. The following conditions are equivalent:*

(i) *$PQ$ is a projection.*

(ii) *$PQ = QPQ$.*

(iii) *$PQ = PQP$.*

*If moreover the complementary projection $Q^\perp$ is contractive too then $PQ = QP$.*

*Proof.* If (ii) is verified then $(PQ)^2 = PQPQ = P \cdot PQ = PQ$; while if (iii) is verified then $(PQ)^2 = PQPQ = PQ \cdot Q = PQ$. Hence both (ii) and (iii) imply (i) (without any contractiveness assumption). Conversely if (i) is verified then for every $x \in R(PQ)$ we have $x = Qx = PQx$ (by [2, Prop. 1.1 (iii)]; only the strict convexity of $X$ is needed) so $x \in R(P) \cap R(Q)$. Since the converse is trivial, we see that $R(PQ) = R(P) \cap R(Q)$; in particular $QPQ = PQ$ and (ii) is verified. Dualizing we have that $P^*$, $Q^*$ and $Q^*P^*$ are contractive projections in $X^*$; hence $Q^*P^* = P^*Q^*P^*$, so $PQ = PQP$ and (iii) is verified. Now (iii) implies $PQ^\perp = PQ^\perp P$, and if $Q^\perp$ is contractive this implies $PQ^\perp = Q^\perp PQ^\perp$ by the preceding. Then

$$Q = PQ + PQ^\perp = QPQ + Q^\perp PQ^\perp$$

which in turn implies $QP = PQ = QPQ$. $\square$

*Remark.* The final assertion $PQ = QP$ of Lemma 6.1 is stated in [2] (for $X = C_p$) as Cor. 1.7 without the assumption that the complementary projection $Q^\perp$ is contractive. This statement is not correct: if $p \neq 2$ it is easy to construct rank 1 contractive projections $P$, $Q$ in $X = \ell_p$ or $C_p$ such that $PQ = 0 \neq QP$: choose non zero elements $a, b \in X$ such that their norming functionals $Ja$, $Jb$ verify $\langle Ja, b\rangle = 0$ and $\langle Jb, a\rangle \neq 0$ and set $P = a \otimes Ja$, $Q = b \otimes Jb$.

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