Entropy Solution for Anisotropic Reaction-Diffusion-Advection Systems with L^1 Data

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ABSTRACT

In this paper, we study the question of existence and uniqueness of entropy solutions for a system of nonlinear partial differential equations with general anisotropic diffusivity and transport effects, supplemented with no-flux boundary conditions, modeling the spread of an epidemic disease through a heterogeneous habitat.

Key words: entropy solutions, uniqueness, anisotropic parabolic equations.

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1. Introduction

Let us first consider $p = u+v$ a simple population, where $u = u(t)$ and $v = v(t)$ are the respective densities of susceptible and infected individuals at time t. When no spatial consideration is involved the dynamics of the propagation of Feline Immunodeficiency Virus (F.I.V.) within a population of cats is governed by the following system of ordinary differential equations

$$\begin{cases} u' = -\sigma(u,v) + b(u+v) - (m + k(u+v))u, & u(0) > 0, \\ v' = \sigma(u,v) - \alpha v - (m + k(u+v))v, & v(0) > 0, \end{cases}$$

where b is the (linear) natural birth rate, m is natural death rate and $k > 0$ is a positive constant yielding a density dependent death rate $\delta(p) = m + kp$. For $b - m > 0$, $K_p = \frac{b-m}{k}$ is the carrying capacity and $p(t) \to K_p$ as $t \to +\infty$. Let $\sigma(u, v)$ be the incidence function, i.e. the recruitment of newly infected cats and α the disease induced death rate in the infected class. The incidence term is a mathematical expression describing the loss of individuals from the susceptible class and their entry into the latently infected class. Two commons of the incidence terms are proportionate mixing and mass action. In the case of mass action we have a term of the form $\sigma(u, v) = \sigma_1 uv$ with $\sigma_1 > 0$, while a proportionate mixing term has the form $\sigma(u, v) = \sigma_2 \frac{uv}{u+v}$ with $\sigma_2 > 0$. More details concerning the propagation of F.I.V. may be found in [15] and the references therein.

Actually we shall be concerned with spatial densities and the total population of our subclass will be given by

$$U(t) = \int_\Omega u(t, x)\, dx \quad \text{and} \quad V(t) = \int_\Omega v(t, x)\, dx$$

where Ω in $\mathbb{R}^N$ ($N \geq 1$) is a bounded domain representing the habitat under consideration. With this in mind the total population density $p(t, x)$ is given by

$$p(t, x) = u(t, x) + v(t, x),$$

with total population

$$P(t) = \int_\Omega p(t, x)\, dx.$$

Here, $u(t, x)$ and $v(t, x)$ represent the spatial densities at time t and location $x \in \Omega$ of susceptible and infectious individuals. We are led to consider spatially dependent birth rate $b(t, x)$, death rate $m(t, x)$ and the additional disease induced death rate in the infected class $\alpha(t, x)$. We denote by $\mathbf{A}_i$, $\mathbf{K}_i$ and r_i, for $i = 1, 2$ respectively the diffusivity terms, the transport field and the density dependent mortality rates terms (we see later the dependence of these functions on solutions).

A prototype of a nonlinear system that governs the spreading of F.I.V. through a cat population in a heterogeneous spatial domain with seasonal variations and external supply is given by the following reaction-diffusion-advection system

$$\begin{cases} \partial_t u(t,x) - \operatorname{div}(\mathbf{A}_1(t,x,\nabla u(t,x)) + u(t,x)\mathbf{K}_1(t,x)) + r_1(t,x,u,v) = \\ \qquad -\sigma(t,x,u,v) + b(t,x)(u(t,x) + v(t,x)) - m(t,x)u(t,x) + f(t,x), \\ \partial_t v(t,x) - \operatorname{div}(\mathbf{A}_2(t,x,\nabla v(t,x)) + v(t,x)\mathbf{K}_2(t,x)) + r_2(t,x,u,v) = \\ \qquad \sigma(t,x,u,v) - (m(t,x) + \alpha(t,x))v(t,x) + g(t,x); \end{cases} \tag{1}$$

in $Q_T = (0, T) \times \Omega$, together with no-flux boundary conditions on $(0, T) \times \partial\Omega$

$$\begin{cases} (\mathbf{A}_1(t,x,\nabla u(t,x)) + u(t,x)\mathbf{K}_1(t,x)) \cdot \eta(x) = 0, \\ (\mathbf{A}_2(t,x,\nabla v(t,x)) + v(t,x)\mathbf{K}_2(t,x)) \cdot \eta(x) = 0, \\ \mathbf{K}_i(t,x) \cdot \eta(x) \geq 0 \text{ for } i = 1, 2, \end{cases} \tag{2}$$

and initial distributions in Ω

$$u(0,x) = u_0(x) \text{ and } v(0,x) = v_0(x), \tag{3}$$

where η denote the outward normal to Ω on $\partial\Omega$.

Now, we precise the assumptions on the given functions which appears in the system (1). We confine ourselves to a model where the diffusivity $\mathbf{A}_i$, $i = 1, 2$, is a Carathéodory function in $(0,T) \times \Omega \times \mathbb{R}^N$ whose components are $a_{l,i}$ for $l = 1, \dots, N$ and $i = 1, 2$, satisfying for $\xi \in \mathbb{R}^N$:

$$\text{There exists } p_l > 1, \quad a_{l,i}(t,x,\xi) = \beta_{l,i}(t,x)|\xi_l|^{p_l-2}\xi_l. \tag{4}$$

Herein the nonnegative function $\beta_{l,i}$ is bounded on Q_T. We assume there exists a real positive constant $\underline{\text{a}}$, such that for $i = 1, 2$ and for any $\xi \in \mathbb{R}^N$:

$$\mathbf{A}_i(t,x,\xi) \cdot \xi \geq \underline{\text{a}} \sum_{l=1}^{N} |\xi|^{p_l}, \text{ a.e. } (t,x) \in Q_T.$$

The transport vector $\mathbf{K}_i$ $(i = 1, 2)$ is bounded on Q_T and satisfies

$$\mathbf{K}_i \in (L^\infty(Q_T))^N \text{ and } \operatorname{div}(\mathbf{K}_i) \in L^\infty(Q_T) \text{ for } i = 1, 2.$$

The functions m, b, and α are defined on Q_T with values in $\mathbb{R}_+$ and satisfy

$$m, b, \alpha \in L^\infty(Q_T).$$

The density dependent mortality rates have the following form:

$$\begin{cases} r_1(t,x,u,v) = k_1(t,x)\, u|u+v|^{p_u-1}, \\ r_2(t,x,u,v) = k_2(t,x)\, v|u+v|^{p_v-1} \end{cases}$$

where p_θ, for $\theta = u, v$, satisfies

$$p_\theta \geq \max_{1 \leq l \leq N} (\frac{p_l}{p_l - 1}, p_l) > 1,$$

and the functions k_i, $i = 1, 2$, defined on Q_T with values in $\mathbb{R}_+$ satisfy

$$k_i \in L^\infty(Q_T) \text{ and } k_i(t,x) \geq k_0 > 0 \text{ a.e.}(t,x) \in Q_T \text{ for } i = 1, 2, 3.$$

Last, $\sigma : Q_T \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}_+$ is measurable on Q_T, continuous with respect to u and v, a.e. in Q_T and satisfies a *growth condition*:

$$\begin{cases} \text{there exists two bounded functions } L, M : (0,\infty) \times \mathbb{R}^N \to (0,\infty), \\ \text{and } s',\ s, \in \mathbb{R}_+ \text{ such that} \\ 1 \leq s < \max_{1 \leq l \leq N} \left(\frac{p_l}{\overline{p}} \left(\overline{p} - \frac{N}{N+1} \right), p_u, p_v \right) \text{ and} \\ |\sigma(t,x,u,v)| \leq L(t,x)\, (|u|^{s'}\ |v|^s) + M(t,x) \text{ a.e. } (t,x) \in Q_T, \end{cases} \tag{5}$$

where $\frac{1}{\overline{p}} = \frac{1}{N}\sum_{l=1}^{N}\frac{1}{p_l}$, and a *nonnegativity condition*

$$\begin{cases} \sigma(t,x,0,v) = 0 & \text{if} \quad v \geq 0 \\ \sigma(t,x,u,0) \geq 0 & \text{if} \quad u \geq 0 \\ \sigma(t,x,u,v) \geq 0 & \text{if} \quad u \geq 0 \text{ and } v \geq 0. \end{cases} \tag{6}$$

Before we discuss the concept of solution, we need to go into the functional setting especially the anisotropic Sobolev space. Set $W^{1,p,l}(\Omega) = \big\{ u \in W^{1,1}(\Omega) \mid \frac{\partial u}{\partial x_l} \in L^p(\Omega) \big\}$ the anisotropic Sobolev space, with

$$\|u\|_{W^{1,p,l}(\Omega)} = \|u\|_{W^{1,1}(\Omega)} + \left\|\frac{\partial u}{\partial x_l}\right\|_{L^p(\Omega)}.$$

We denote

$$L^1_+(\Omega) = \{u \in L^1(\Omega), u \geq 0 \text{ a.e. in } \Omega\},$$

and $C^1_c([0,T)\times\Omega)$ the set of all C^1-functions with compact support in $[0,T)\times\Omega$. For given constant $\gamma > 0$ we define the cut function T_γ as the real-valued Lipschitz function

$$T_\gamma(z) = \min(\gamma, \max(z, -\gamma)).$$

By the Stampacchia Theorem ([10]), if $u \in W^{1,q}(\Omega)$ $(q \geq 1)$, we have $\nabla T_\gamma(u) = \mathbf{1}_{\{|u|<\gamma\}}\nabla u$, where $1_{\{|u|<\gamma\}}$ denotes the characteristic function of a measurable set $\{|u| < \gamma\} \subset (0,T)\times\Omega$. We denote $S_\gamma(z) = \int_0^z T_\gamma(\tau)\, d\tau$, $\phi_\gamma = T_{\gamma+1} - T_\gamma$ and we set $\Psi_\gamma(z) = \int_0^z \phi_\gamma(\tau)\, d\tau$.

We note that T_γ and ϕ_γ are continuous Lipschitz functions, satisfying $0 \leq |\phi_\gamma(z)| \leq 1$ and $|\Psi_\gamma(z)| \leq |z|$ for $\gamma > 0$ and $z \in \mathbb{R}$.

The usual weak formulation of parabolic problems, in the case where the data are in L^1, does not ensure the uniqueness of the solution, some counterexamples are given in [17, 20]. Then, for an isotropic parabolic equation with L^1 data, without nonlinear, reaction and advection terms, in [16] the author introduced a entropy formulation which allows to achieve existence and uniqueness. For the corresponding anisotropic parabolic equations with measure data or elliptic equations with L^1 data, existence of weak solutions is proved in [8] and [7] respectively. In [12], an existence result of entropy solutions to some parabolic equations is established. The data is considered in L^1 and no growth assumption is made on the lower-order term in divergence form. Another concept in terms of renormalized solutions permitting to ensure the uniqueness of the solution, can be found in [6, 18, 19]. Existence and uniqueness of renormalized solutions for a linear parabolic equation involving a first order term with free divergence coefficient is discussed in [14]. Also, in [9], existence and uniqueness of entropy solutions for linear parabolic equations involving 0^{th} and 1^{st} order terms with L^1 data are proved.

In this paper, we extend the results of [8, 16] to anisotropic reaction-diffusion-advection systems arising in population dynamics and modeling the propagation of Feline Immunodeficiency Virus.

2. Main results

Definition 2.1. Let $1 \leq q_l < \frac{p_l}{\overline{p}}\big(\overline{p} - \frac{N}{N+1}\big)$, $l = 1, \dots, N$. An entropy solution of (1)–(3) is a couple (u, v) of nonnegative functions, with $\theta = u, v$ belonging to $\bigcap_{l=1}^N L^{q_l}(0, T; W^{1,q_l,l}(\Omega)) \cap L^{p_\theta}(0, T; L^{p_\theta}(\Omega)) \cap C([0, T]; L^1(\Omega))$, such that $\sigma(\cdot, \cdot, u, v)$ and $r_i(\cdot, \cdot, u, v)$, for $i = 1, 2$, belong to $L^1(Q_T)$, $T_\gamma(u)$ and $T_\gamma(v)$ belong to $\bigcap_{l=1}^N L^{p_l}(0, T; W^{1,p_l,l}(\Omega))$, and satisfying

$$\begin{aligned}
\int_\Omega S_\gamma(u-\varphi)(T,x)\,dx &- \int_\Omega S_\gamma(u_0(x)-\varphi(0,x))\,dx + \int_0^T \langle \partial_t\varphi, T_\gamma(u-\varphi)\rangle\,dt \\
&+ \int_0^T\int_\Omega (\mathbf{A}_1(t,x,\nabla u) + u\mathbf{K}_1(t,x))\cdot\nabla T_\gamma(u-\varphi)\,dx\,dt \\
&+ \int_0^T\int_\Omega r_1(t,x,u,v)\,T_\gamma(u-\varphi)\,dx\,dt + \int_0^T\int_\Omega \sigma(t,x,u,v)T_\gamma(u-\varphi)\,dx\,dt \\
&- \int_0^T\int_\Omega b\;(u+v)\;T_\gamma(u-\varphi)\,dx\,dt + \int_0^T\int_\Omega m\;u\;T_\gamma(u-\varphi)\,dx\,dt \\
&\leq \int_0^T\int_\Omega fT_\gamma(u-\varphi)\,dx\,dt, \qquad (7)
\end{aligned}$$

$$\begin{aligned}
\int_\Omega S_\gamma(v-\psi)(T,x)\,dx &- \int_\Omega S_\gamma(v_0(x)-\psi(0,x))\,dx + \int_0^T \langle \partial_t\psi, T_\gamma(v-\psi)\rangle\,dt \\
&+ \int_0^T\int_\Omega (\mathbf{A}_2(t,x,\nabla v) + v\;\mathbf{K}_2(t,x))\cdot\nabla T_\gamma(v-\psi) \\
&+ \int_0^T\int_\Omega r_2(t,x,u,v)\;T_\gamma(v-\psi)\,dx\,dt - \int_0^T\int_\Omega \sigma(t,x,u,v)T_\gamma(v-\psi)\,dx\,dt \\
&+ \int_0^T\int_\Omega (m+\alpha)\;v\;T_\gamma(v-\psi)\,dx\,dt \leq \int_0^T\int_\Omega gT_\gamma(v-\psi)\,dx\,dt, \qquad (8)
\end{aligned}$$

for all $\gamma > 0$ and $\varphi,\ \psi \in \bigcap_{l=1}^N L^{p_l}(0, T; W^{1,p_l,l}(\Omega)) \cap L^\infty(Q_T) \cap C([0, T]; L^1(\Omega))$ such that $\partial_t\phi, \partial_t\psi \in \sum_{l=1}^N L^{\frac{p_l}{p_l-1}}(0, T; (W^{1,p_l,l}(\Omega))')$.

The results proved here are summarized in two theorems, the first one concerns the existence of entropy solutions and the second one establishes the uniqueness of these solutions.

Theorem 2.2. *Assume that* (4)–(6) *hold. Let* $u_0,\ v_0 \in L^1_+(\Omega)$ *and* $f,\ g \in L^1_+(Q_T)$, *then the system* (1)–(3) *has an entropy solution.*

The uniqueness of entropy solution is established in the case where the incidence terms and the density dependent mortality rates are Lipschitz functions with respect to u and v.

Theorem 2.3. *Assuming that*

$$\begin{cases} \exists\beta(t,x) > 0, \ \text{a.e. } (t,x) \in Q_T, \beta \in L^\infty(Q_T), \ \text{such that} \\ |\sigma(t,x,u_1,v_1) - \sigma(t,x,u_2,v_2)| \le \beta(t,x)(|u_1-u_2| + |v_1-v_2|) \\ |r_i(t,x,u_1,v_1) - r_i(t,x,u_2,v_2)| \le \beta(t,x)(|u_1-u_2| + |v_1-v_2|) \ \text{for } i=1,2, \end{cases} \tag{9}$$

then the entropy solution defined by (7)–(8) *is unique.*

Remark 2.4. Under Dirichlet boundary conditions, Theorem 2.2 and Theorem 2.3 remain valid by considering an adequate functional space. Precisely, the space $W^{1,q_l,l}(\Omega)$ in Definition 2.1 is replaced by $W_0^{1,q_l,l}(\Omega) = \{u \in W^{1,q_l,l}(\Omega), u|_{\partial\Omega} = 0\}$.

The plan of the paper is as follows. Section 3 is devoted to explain how the solution of system (1) is obtained and to precise its regularity. In the last two sections, we show existence and uniqueness of the entropy solution.

3. Approximate Solutions

The method used in [3] for showing the existence of a solution for $u_0,\ v_0 \in L^1_+(\Omega)$ and $f,\ g \in L^1_+(Q_T)$ consists in :

- introducing a measurable may on Q_T, $\hat{\sigma}$ continuous with respect to u and v, a.e. in Q_T,
- regularizing the following data $f,\ g,\ u_0$ and v_0 with nonnegative smooth sequences $(f_\varepsilon)_\varepsilon$, $(g_\varepsilon)_\varepsilon$ $(u_{0,\varepsilon})_\varepsilon$, and $(v_{0,\varepsilon})_\varepsilon$.

Then, classical results (see e.g. [13], [11]) provide the existence of a sequence $u_\varepsilon,\ v_\varepsilon \in \bigcap_{l=1}^N L^{p_l}(0,T;W^{1,p_l,l}(\Omega)) \cap L^{p_{max}}(Q_T) \cap C([0,T];L^2(\Omega))$, $p_{max} = \max(p_u,p_v)$, with $\partial_t u_\varepsilon,\ \partial_t v_\varepsilon \in \sum_{l=1}^N L^{p'_l}(0,T;(W^{1,p_l,l}(\Omega))')$, of solutions of (1)–(3) where $u_0,\ v_0$ and $f,\ g$ are replaced by $u_{0,\varepsilon},\ v_{0,\varepsilon}$ and $f_\varepsilon,\ g_\varepsilon$ respectively, and σ is replaced by $\hat{\sigma}$, $\sum_{l=1}^N L^{p'_l}(0,T;(W^{1,p_l,l}(\Omega))')$ denotes the dual space of $\bigcap_{l=1}^N L^{p_l}(0,T;W^{1,p_l,l}(\Omega))$ with $p'_l = \frac{p_l}{p_l-1}$.

In order to obtain estimates on solutions independent of the parameter ε, the authors in [3] introduced a new parameter $\lambda > 0$ satisfying

$$\lambda \ge \max(b(t,x), b(t,x) - m(t,x) + \operatorname{div}(\mathbf{K}_1(t,x)), \operatorname{div}(\mathbf{K}_2(t,x))) \text{ a.e. } (t,x) \in Q_T. \tag{10}$$

We often omit the dependence of functions in (t,x) when no confusion is possible. We set $u_\varepsilon = e^{\lambda t}\tilde{u}_\varepsilon$ and $v_\varepsilon = e^{\lambda t}\tilde{v}_\varepsilon$; then $\tilde{u}_\varepsilon$ and $\tilde{v}_\varepsilon$ satisfy

$$\begin{aligned}\int_0^T \langle \partial_t \tilde{u}_\varepsilon, \varphi\rangle\,dt + \int_0^T\int_\Omega \tilde{\mathbf{A}}_1(t,x,\nabla\tilde{u}_\varepsilon)\cdot\nabla\varphi\,dx\,dt + \int_0^T\int_\Omega \tilde{u}_\varepsilon \mathbf{K}_1\cdot\nabla\varphi\,dx\,dt \\ + \int_0^T\int_\Omega r_{1,\lambda}(\tilde{u}_\varepsilon,\tilde{v}_\varepsilon)\varphi\,dx\,dt + \int_0^T\int_\Omega e^{-\lambda t}\hat{\sigma}(e^{\lambda t}\tilde{u}_\varepsilon, e^{\lambda t}\tilde{v}_\varepsilon)\varphi\,dx\,dt \\ + \int_0^T\int_\Omega (\lambda+m-b)\tilde{u}_\varepsilon\varphi\,dx\,dt - \int_0^T\int_\Omega b\tilde{v}_\varepsilon\varphi\,dx\,dt = \int_0^T\int_\Omega e^{-\lambda t} f_\varepsilon\varphi\,dx\,dt, \quad (11)\end{aligned}$$

$$\begin{aligned}\int_0^T \langle \partial_t \tilde{v}_\varepsilon, \psi\rangle\,dt + \int_0^T\int_\Omega \tilde{\mathbf{A}}_2(t,x,\nabla\tilde{v}_\varepsilon)\cdot\nabla\psi\,dx\,dt + \int_0^T\int_\Omega \tilde{v}_\varepsilon \mathbf{K}_2\cdot\nabla\psi\,dx\,dt \\ + \int_0^T\int_\Omega r_{2,\lambda}(\tilde{u}_\varepsilon,\tilde{v}_\varepsilon)\psi dx\,dt - \int_0^T\int_\Omega e^{-\lambda t}\hat{\sigma}(e^{\lambda t}\tilde{u}_\varepsilon, e^{\lambda t}\tilde{v}_\varepsilon)\psi\,dx\,dt \\ + \int_0^T\int_\Omega (\lambda+m+\alpha)\tilde{v}_\varepsilon\psi\,dx\,dt = \int_0^T\int_\Omega e^{-\lambda t} g_\varepsilon\psi\,dx\,dt, \quad (12)\end{aligned}$$

for all $\varphi,\ \psi \in \bigcap_{l=1}^N L^{p_l}(0,T;W^{1,p_l,l}(\Omega)) \cap L^\infty(Q_T)$.

Herein

$$\tilde{\mathbf{A}}_i(t,x,\xi) = e^{-\lambda t}\mathbf{A}_i(t,x,e^{\lambda t}\xi),\ \text{for } \xi\in\mathbb{R}^N,$$

and

$$r_{i,\lambda}(t,x,\tilde{u}_\varepsilon,\tilde{v}_\varepsilon) = e^{-\lambda t} r_i(t,x,e^{\lambda t}\tilde{u}_\varepsilon, e^{\lambda t}\tilde{v}_\varepsilon).$$

It is shown in [3] that the solutions satisfy

(i) $\tilde{u}_\varepsilon \geq 0$, $\tilde{v}_\varepsilon \geq 0$ a.e. $(t,x)\in Q_T$,

(ii) $\exists\ c_1 > 0$, $c_2 > 0$ not depending on ε such that

$$\|\tilde{u}_\varepsilon + \tilde{v}_\varepsilon\|_{L^\infty(0,T;L^1(\Omega))} \leq c_1, \quad (13)$$

$$\|r_{i,\lambda}(\tilde{u}_\varepsilon,\tilde{v}_\varepsilon)\|_{L^1(Q_T)} + \|\sigma(e^{\lambda t}\tilde{u}_\varepsilon, e^{\lambda t}\tilde{v}_\varepsilon)\|_{L^1(Q_T)} \leq c_2 \text{ for } i=1,2. \quad (14)$$

(iii) Let $\overline{p} \leq N + \frac{N}{N+1}$, then for every $1 \leq q_l < \frac{p_l}{\overline{p}}\left(\overline{p} - \frac{N}{N+1}\right)$, $l=1,\dots,N$, $\exists c_3 > 0$ such that

$$\left\|\frac{\partial \tilde{z}_\varepsilon}{\partial x_l}\right\|_{L^{q_l}(Q_T)} + \|\tilde{z}_\varepsilon\|_{L^{\overline{q}}(Q_T)} \leq c_3,$$

where $\tilde{z}_\varepsilon = \tilde{u}_\varepsilon, \tilde{v}_\varepsilon$ and $\overline{p}$ satisfy $\frac{1}{\overline{p}} = \frac{1}{N}\sum_{l=1}^N \frac{1}{p_l}$.

Let $q_0 = \min_{1\leq l\leq N}\{q_l\}$, then $(\tilde{u}_\varepsilon)_\varepsilon$ and $(\tilde{v}_\varepsilon)_\varepsilon$ are bounded in $L^{q_0}(0,T;W^{1,q_0}(\Omega))$, this implies that $(\partial_t\tilde{u}_\varepsilon)_\varepsilon, (\partial_t\tilde{v}_\varepsilon)_\varepsilon$ are bounded in $L^1(0,T;(W^{1,q_0}(\Omega))')+L^1(Q_T)$; therefore, possibly at the cost of extracting subsequences still denoted $(\tilde{u}_\varepsilon)_\varepsilon$ and $(\tilde{v}_\varepsilon)_\varepsilon$ (see Corollary 4 in [21]) we can assume that

$$\begin{cases} \tilde{u}_\varepsilon \longrightarrow \tilde{u} & \text{strongly in } L^{q_0}(Q_T) \text{ and a.e. in } Q_T, \\ \tilde{v}_\varepsilon \longrightarrow \tilde{v} & \text{strongly in } L^{q_0}(Q_T) \text{ and a.e. in } Q_T. \end{cases} \tag{15}$$

Finally, to treat the nonlinear terms and to prove the continuity in time of the solution , it is shown in [3] that

$$\sigma(e^{\lambda t}\tilde{u}_\varepsilon, e^{\lambda t}\tilde{v}_\varepsilon) \longrightarrow \sigma(e^{\lambda t}\tilde{u}, e^{\lambda t}\tilde{v}) \text{ a.e. in } Q_T \text{ and strongly in } L^1(Q_T), \tag{16}$$

$$r_{i,\lambda}(\tilde{u}_\varepsilon,\tilde{v}_\varepsilon) \longrightarrow r_{i,\lambda}(\tilde{u},\tilde{v}), \text{ a.e. in } Q_T \text{ and strongly in } L^1(Q_T), \text{ for } i=1,2, \tag{17}$$

$$(\nabla\tilde{u}_\varepsilon(t,x), \nabla\tilde{v}_\varepsilon(t,x)) \longrightarrow (\nabla\tilde{u}(t,x), \nabla\tilde{v}(t,x)) \text{ a.e. in } (t,x)\in(0,T)\times\Omega, \tag{18}$$

$$(\tilde{u}_\varepsilon,\tilde{v}_\varepsilon) \longrightarrow (\tilde{u},\tilde{v}) \text{ strongly in } (C(0,T;L^1(\Omega)))^2. \tag{19}$$

Now, to complete the above estimates and the convergence results, we are concerned with the sequences $(T_\gamma(\tilde{u}_\varepsilon))_\varepsilon$ and $(T_\gamma(\tilde{v}_\varepsilon))_\varepsilon$.

Proposition 3.1. *Let* (4)–(6) *hold. Then the sequences*

$$(T_\gamma(\tilde{u}_\varepsilon))_\varepsilon \textit{ and } (T_\gamma(\tilde{v}_\varepsilon))_\varepsilon \textit{ are uniformly bounded in } \bigcap_{l=1}^N L^{p_l}(0,T;W^{1,p_l,l}(\Omega)). \tag{20}$$

Proof. Let us choose $\varphi = T_\gamma(\tilde{u}_\varepsilon)$ as test function in (11), we have

$$\begin{aligned} &\int_\Omega S_\gamma(\tilde{u}_\varepsilon)(T,x)\,dx - \int_\Omega S_\gamma(\tilde{u}_\varepsilon)(0,x)\,dx \\ &\quad + \int_{Q_T} \tilde{\mathbf{A}}_1(t,x,\nabla\tilde{u}_\varepsilon)\cdot\nabla T_\gamma(\tilde{u}_\varepsilon)\,dx\,dt + \int_{Q_T} \tilde{u}_\varepsilon\mathbf{K}_1\cdot\nabla T_\gamma(\tilde{u}_\varepsilon)\,dx\,dt \\ &\quad + \int_{Q_T} e^{-\lambda t}\sigma(e^{\lambda t}\tilde{u}_\varepsilon, e^{\lambda t}\tilde{v}_\varepsilon)T_\gamma(\tilde{u}_\varepsilon)\,dx\,dt + \int_{Q_T}(\lambda+m-b)\tilde{u}_\varepsilon T_\gamma(\tilde{u}_\varepsilon)\,dx\,dt \\ &\quad - \int_{Q_T} b\tilde{v}_\varepsilon T_\gamma(\tilde{u}_\varepsilon)\,dx\,dt + \int_{Q_T} r_{1,\lambda}(\tilde{u}_\varepsilon,\tilde{v}_\varepsilon)T_\gamma(\tilde{u}_\varepsilon)\,dx\,dt \\ &\quad = \int_{Q_T} e^{-\lambda t} f_\varepsilon T_\gamma(\tilde{u}_\varepsilon)\,dx\,dt, \end{aligned} \tag{21}$$

Using the positivity of solutions, the estimates (13), (14), the fact that $|T_\gamma(\tilde{u}_\varepsilon)|\leq\gamma$ and $|S_\gamma(\tilde{u}_\varepsilon)|\leq\gamma|\tilde{u}_\varepsilon|$, we deduce from (21)

$$\begin{aligned} &\int_{Q_T} \tilde{\mathbf{A}}_1(t,x,\nabla\tilde{u}_\varepsilon)\cdot\nabla T_\gamma(\tilde{u}_\varepsilon)\,dx\,dt + \int_{Q_T} \tilde{u}_\varepsilon\mathbf{K}_1\cdot\nabla T_\gamma(\tilde{u}_\varepsilon)\,dx\,dt \\ &\quad + \int_{Q_T}(\lambda+m-b)\tilde{u}_\varepsilon T_\gamma(\tilde{u}_\varepsilon)\,dx\,dt \leq C\gamma \end{aligned} \tag{22}$$

where C is a positive constant independent of ε. Define $F(z) = \int_0^z yT'_\gamma(y)dy$, then one gets $F(z) \le zT_\gamma(z)$. Using now the following equality,

$$\operatorname{div}(F(\tilde u_\varepsilon)\mathbf{K}_1) = \operatorname{div}(\mathbf{K}_1)F(\tilde u_\varepsilon) + \mathbf{K}_1 \cdot \nabla F(\tilde u_\varepsilon),$$

from (2) we have $\mathbf{K}_1 \cdot \eta \ge 0$ on $(0,T) \times \partial\Omega$, thus

$$\int_{Q_T} \tilde u_\varepsilon \mathbf{K}_1 \cdot \nabla T_\gamma(\tilde u_\varepsilon)\,dx\,dt + \int_{Q_T} (\lambda + m - b)\tilde u_\varepsilon T_\gamma(\tilde u_\varepsilon)\,dx\,dt \\ \ge \int_{Q_T} (\lambda + m - b - \operatorname{div}(\mathbf{K}_1))F(\tilde u_\varepsilon)\,dx\,dt \ge 0.$$

Consequently, from (22) we have

$$\int_{Q_T} \tilde{\mathbf{A}}_1(t,x,\nabla \tilde u_\varepsilon) \cdot \nabla T_\gamma(\tilde u_\varepsilon) dx\,dt \le C\gamma, \tag{23}$$

which yields

$$\int_{Q_T} \sum_{l=1}^N e^{-\lambda t}\beta_{l,1}(t,x) \left|\frac{\partial T_\gamma(\tilde u_\varepsilon)}{\partial x_l}\right|^{p_l} dx\,dt \le C\gamma. \tag{24}$$

It follows that $(T_\gamma(\tilde u_\varepsilon))_\varepsilon$ is bounded sequence in $\bigcap_{l=1}^N L^{p_l}(0,T;W^{1,p_l,l}(\Omega))$. In the same way, one gets that the above estimate remains valid on solution $\tilde v_\varepsilon$ and consequently the proof of estimate (20) is complete. □

4. Existence of entropy solution

Let us derive the entropy formulation for the regularized sequence $(\tilde u_\varepsilon)_\varepsilon$. Let $\varphi \in \bigcap_{l=1}^N L^{p_l}(0,T;W^{1,p_l,l}(\Omega)) \cap L^\infty(Q_T) \cap C([0,T];L^1(\Omega))$ such that

$$\partial_t \varphi \in \sum_{l=1}^N L^{\frac{p_l}{p_l-1}}(0,T;(W^{1,p_l,l}(\Omega))').$$

Consider now $\phi = T_\gamma(u_\varepsilon - \varphi)$ as test function in (11). Since $u_\varepsilon = e^{\lambda t}\tilde u_\varepsilon$ and $v_\varepsilon = e^{\lambda t}\tilde v_\varepsilon$, we have

$$\int_\Omega S_\gamma(u_\varepsilon - \varphi)(T,x)\,dx - \int_\Omega S_\gamma(u_\varepsilon - \varphi)(0,x)\,dx + \int_0^T \langle \partial_t\varphi, T_\gamma(u_\varepsilon - \varphi)\rangle\,dt \\ + \int_{Q_T} \mathbf{A}_1(t,x,\nabla u_\varepsilon) \cdot \nabla T_\gamma(u_\varepsilon - \varphi)\,dx\,dt + \int_{Q_T} u_\varepsilon \mathbf{K}_1 \cdot \nabla T_\gamma(u_\varepsilon - \varphi)\,dx\,dt \\ + \int_{Q_T} \sigma(u_\varepsilon, v_\varepsilon) T_\gamma(u_\varepsilon - \varphi)\,dx\,dt + \int_{Q_T} m u_\varepsilon T_\gamma(u_\varepsilon - \varphi)\,dx\,dt$$

$$- \int_{Q_T} b(u_\varepsilon + v_\varepsilon) T_\gamma(u_\varepsilon - \varphi)\, dx\, dt + \int_{Q_T} r_1(u_\varepsilon, v_\varepsilon) T_\gamma(u_\varepsilon - \varphi)\, dx\, dt = \int_{Q_T} f_\varepsilon T_\gamma(u_\varepsilon - \varphi)\, dx\, dt, \quad (25)$$

Thus, using the fact that $(u_\varepsilon)_\varepsilon$ converges to u almost everywhere in Q_T, it follows from Proposition 3.1 that

$$T_\gamma(u_\varepsilon) \longrightarrow T_\gamma(u) \text{ weakly in } \bigcap_{l=1}^{N} L^{p_l}(0,T; W^{1,p_l,l}(\Omega)), \quad (26)$$

as ε goes to zero. Let us now study the limit for ε goes to 0 of each term of equality (25).

Since S_γ is γ-Lipschitz continuous and that u_ε converges to u in $C(0,T,L^1(\Omega))$ (see (19)), when ε goes to 0, one gets

$$\int_\Omega S_\gamma(u_\varepsilon - \varphi)(T,x)\, dx \longrightarrow \int_\Omega S_\gamma(u - \varphi)(T,x)\, dx$$

and

$$\int_\Omega S_\gamma(u_\varepsilon - \varphi)(0,x)\, dx \longrightarrow \int_\Omega S_\gamma(u - \varphi)(0,x)\, dx.$$

We now pass to the limit in $\int_0^T \langle \partial_t \varphi, T_\gamma(u_\varepsilon - \varphi)\rangle\, dt$. Notice now that, setting $k = \|\varphi\|_{L^\infty(Q_T)}$, one has

$$T_\gamma(u_\varepsilon - \varphi) = T_\gamma(T_{\gamma+k}(u_\varepsilon) - \varphi) \text{ and } T_\gamma(u - \varphi) = T_\gamma(T_{\gamma+k}(u) - \varphi).$$

Since for $l = 1, \dots, N$

$$\frac{\partial}{\partial x_l}(T_\gamma(T_{\gamma+k}(u_\varepsilon) - \varphi) = \mathbf{1}_{\{|T_{\gamma+k}(u_\varepsilon) - \varphi| \le \gamma\}} \frac{\partial}{\partial x_l}(T_{\gamma+k}(u_\varepsilon) - \varphi),$$

the weak convergence of $T_{\gamma+k}(u_\varepsilon)$ from (26) and $T_{\gamma+k}(u_\varepsilon)$ converges almost everywhere, moreover $\partial_t \varphi$ in $\sum_{l=1}^N L^{p_l'}(0,T; (W^{1,p_l,l}(\Omega))')$. Hence

$$\int_0^T \langle \partial_t \varphi, T_\gamma(T_{\gamma+k}(u_\varepsilon) - \varphi)\rangle\, dt \longrightarrow \int_0^T \langle \partial_t \varphi, T_\gamma(T_{\gamma+k}(u) - \varphi)\rangle\, dt,$$

which is equivalent to

$$\int_0^T \langle \partial_t \varphi, T_\gamma(u_\varepsilon - \varphi)\rangle\, dt \longrightarrow \int_0^T \langle \partial_t \varphi, T_\gamma(u - \varphi)\rangle\, dt.$$

One observes that

$$\begin{aligned}\int_{Q_T} \mathbf{A}_1(t,x,\nabla u_\varepsilon)\cdot\nabla T_\gamma(u_\varepsilon-\varphi)\,dx\,dt \\ = \int_{Q_T} \mathbf{A}_1(t,x,\nabla T_{\gamma+k}(u_\varepsilon))\cdot\nabla T_{\gamma+k}(u_\varepsilon)\mathbf{1}_{\{\,|T_{\gamma+k}(u)-\varphi|\le\gamma\,\}}\,dx\,dt \\ - \int_{Q_T} \mathbf{A}_1(t,x,\nabla T_{\gamma+k}(u_\varepsilon))\cdot\nabla\varphi\mathbf{1}_{\{\,|T_{\gamma+k}(u)-\varphi|\le\gamma\,\}}\,dx\,dt.\end{aligned}$$

Indeed, from the Proposition 3.1, the convergence (18), and the assumption (4), we deduce, when ε goes to 0, that $\mathbf{a}_{l,1}(t,x,\nabla T_\gamma(u_\varepsilon)) \longrightarrow \mathbf{a}_{l,1}(t,x,\nabla T_\gamma(u))$ weakly in $L^{\frac{p_l}{p_l-1}}(Q_T)$, for $l=1,\dots,N$, where $\mathbf{a}_{l,1}$ are components of $\mathbf{A}_1$. Using the dominated convergence theorem and the convergence of $\frac{\partial\varphi}{\partial x_l}\mathbf{1}_{\{|T_{\gamma+k}(u_\varepsilon)-\varphi|\le\gamma\}}$ in $L^{p_l}(Q_T)$, for each $l=1...N$, we deduce

$$\begin{aligned}\int_{Q_T} \mathbf{A}_1(t,x,\nabla T_{\gamma+k}(u_\varepsilon))\cdot\nabla\varphi\mathbf{1}_{\{\,|T_{\gamma+k}(u_\varepsilon)-\varphi|\le\gamma\,\}} \longrightarrow \\ \int_{Q_T} \mathbf{A}_1(t,x,\nabla T_{\gamma+k}(u))\cdot\nabla\varphi\mathbf{1}_{\{\,|T_{\gamma+k}(u)-\varphi|\le\gamma\,\}}.\end{aligned}$$

Thus from Fatou's Lemma, we have

$$\begin{aligned}\int_{Q_T} \mathbf{A}_1(t,x,\nabla T_{\gamma+k}(u))\cdot\nabla T_{\gamma+k}(u)\mathbf{1}_{\{\,|T_{\gamma+k}(u)-\varphi|\le\gamma\,\}}\,dx\,dt \\ \le \liminf_{\varepsilon\to0}\int_{Q_T} \mathbf{A}_1(t,x,\nabla T_{\gamma+k}(u_\varepsilon)\cdot\nabla T_{\gamma+k}(u_\varepsilon)\mathbf{1}_{\{\,|T_{\gamma+k}(u_\varepsilon)-\varphi|\le\gamma\,\}}\,dx\,dt.\end{aligned}$$

Next,

$$\int_{Q_T} u_\varepsilon\mathbf{K}_1\cdot\nabla T_\gamma(u_\varepsilon-\varphi)dx\,dt = \int_{Q_T} T_{\gamma+k}(u_\varepsilon)\mathbf{K}_1\cdot\nabla(T_{\gamma+k}(u_\varepsilon)-\varphi)\mathbf{1}_{\{|T_{\gamma+k}(u_\varepsilon)-\varphi|\le\gamma\}}dx\,dt.$$

Since $(u_\varepsilon)_\varepsilon$ converges to u almost everywhere in Q_T and convergence result (26), one gets

$$\begin{aligned}\int_{Q_T} T_{\gamma+k}(u_\varepsilon)\mathbf{K}_1\cdot\nabla(T_{\gamma+k}(u_\varepsilon)-\varphi)\mathbf{1}_{\{\,|T_{\gamma+k}(u_\varepsilon)-\varphi|\le\gamma\,\}}\,dx\,dt \\ \longrightarrow \int_{Q_T} T_{\gamma+k}(u)\mathbf{K}_1\cdot\nabla(T_{\gamma+k}(u)-\varphi)\mathbf{1}_{\{\,|T_{\gamma+k}(u)-\varphi|\le\gamma\,\}}\,dx\,dt \\ = \int_{Q_T} u\mathbf{K}_1\cdot\nabla T_\gamma(u-\varphi)\,dx\,dt,\end{aligned}$$

when ε goes to zero.

We complete the existence part of Theorem 2.2 by using the dominated convergence theorem to obtain that $T_\gamma(u_\varepsilon - \varphi)$ converges to $T_\gamma(u-\varphi)$ weak $\star$ in $L^\infty(Q_T)$ and from the strong convergence results (15), (16), (17) in $L^1(Q_T)$, hence, when ε goes to 0,

$$\int_{Q_T} \sigma(u_\varepsilon, v_\varepsilon) T_\gamma(u_\varepsilon - \varphi)\,dx\,dt \longrightarrow \int_{Q_T} \sigma(u,v) T_\gamma(u-\varphi)\,dx\,dt,$$
$$\int_{Q_T} ((m-b)u_\varepsilon - bv_\varepsilon) T_\gamma(u_\varepsilon - \varphi)\,dx\,dt \longrightarrow \int_{Q_T} ((m-b)u - bv) T_\gamma(u-\varphi)\,dx\,dt,$$
$$\int_{Q_T} r_1(u_\varepsilon, v_\varepsilon) T_\gamma(u_\varepsilon - \varphi)\,dx\,dt \longrightarrow \int_{Q_T} r_1(u,v) T_\gamma(u_\varepsilon-\varphi)\,dx\,dt,$$
$$\int_{Q_T} f_\varepsilon T_\gamma(u_\varepsilon - \varphi)\,dx\,dt \longrightarrow \int_{Q_T} f T_\gamma(u-\varphi)\,dx\,dt.$$

Now passing to the limit as ε goes to zero on the formulation (25) we obtain that the limit u satisfies (7). In the same way, one gets v entropy solution.

Remark 4.1. Note that, one can prove, when ε tends to 0,

$$\int_0^T \int_\Omega \mathbf{A}_1(t,x,\nabla T_\gamma(u_\varepsilon)) \cdot \nabla T_\gamma(u_\varepsilon)\,dx\,dt \longrightarrow \int_0^T \int_\Omega \mathbf{A}_1(t,x,\nabla T_\gamma(u)) \cdot \nabla T_\gamma(u)\,dx\,dt,$$
$$\int_0^T \int_\Omega \mathbf{A}_2(t,x,\nabla T_\gamma(v_\varepsilon)) \cdot \nabla T_\gamma(v_\varepsilon)\,dx\,dt \longrightarrow \int_0^T \int_\Omega \mathbf{A}_2(t,x,\nabla T_\gamma(v)) \cdot \nabla T_\gamma(v)\,dx\,dt,$$

and

$$T_\gamma(z_\varepsilon) \longrightarrow T_\gamma(z) \text{ strongly in } L^{p_l}(0,T;W^{1,p_l,l}(\Omega)) \text{ for } l=1,\dots,N,$$

where $z_\varepsilon = u_\varepsilon,\ v_\varepsilon$ and $z = u, v$, (see [1]). Then, the inequalities in (7) and (8) are equalities, however the inequality in the entropy formulation is sufficient to have uniqueness.

5. Uniqueness of entropy solution

In this section, we study the uniqueness question of the entropy solutions constructed in this paper. The method we adopt, to prove Theorem 2.3, is rather close to those introduced in [4, 9, 16]. However, new difficulties arise essentially related to the influence of the transport terms and nonlinear terms. For that, we prove the uniqueness of the solutions $(\tilde{u}, \tilde{v})$ defined by $\tilde{u} = e^{-\lambda t} u$ and $\tilde{v} = e^{-\lambda t} v$, where (u, v) is the entropy solution defined by (7)-(8) and λ is defined by (10).

We often write

$$F(u, v) = e^{-\lambda t}(f - \sigma(e^{\lambda t} u, e^{\lambda t} v) - r_1(e^{\lambda t} u, e^{\lambda t} v)) - b(u + v) + mu,$$

and

$$G(u, v) = e^{-\lambda t}(g + \sigma(e^{\lambda t} u, e^{\lambda t} v) - r_2(e^{\lambda t} u, e^{\lambda t} v)) - (m + \alpha)v.$$

Recall that $(\tilde{u}, \tilde{v})$ satisfies the following relations

$$\begin{aligned} &\int_\Omega S_\gamma(\tilde{u} - \varphi)(T, x)\,dx - \int_\Omega S_\gamma(u_0(x) - \varphi(0, x))\,dx \\ &+ \int_0^T \langle \partial_t \varphi, T_\gamma(\tilde{u} - \varphi)\rangle\,dt + \int_0^T \int_\Omega (\tilde{\mathbf{A}}_1(t, x, \nabla \tilde{u}) + \tilde{u}\mathbf{K}_1(t, x)) \cdot \nabla T_\gamma(\tilde{u} - \varphi)\ dx\,dt \\ &\qquad + \int_0^T \int_\Omega \lambda \tilde{u} T_\gamma(\tilde{u} - \varphi)\,dx\,dt \le \int_0^T \int_\Omega F(\tilde{u}, \tilde{v}) T_\gamma(\tilde{u} - \varphi)\,dx\,dt, \end{aligned} \tag{27}$$

$$\begin{aligned} &\int_\Omega S_\gamma(\tilde{v} - \psi)(T, x)\,dx - \int_\Omega S_\gamma(v_0(x) - \psi(0, x))\,dx \\ &+ \int_0^T \langle \partial_t \psi, T_\gamma(\tilde{v} - \psi)\rangle\,dt + \int_0^T \int_\Omega (\tilde{\mathbf{A}}_2(t, x, \nabla \tilde{v}) + \tilde{v}\ \mathbf{K}_2(t, x)) \cdot \nabla T_\gamma(\tilde{v} - \psi)\,dx\,dt \\ &\qquad + \int_0^T \int_\Omega \lambda \tilde{v} T_\gamma(\tilde{v} - \psi)\,dx\,dt \le \int_0^T \int_\Omega G(\tilde{u}, \tilde{v}) T_\gamma(\tilde{v} - \psi)\,dx\,dt, \end{aligned} \tag{28}$$

for all $\gamma > 0$ and the test functions φ and ψ are defined in the definition 2.1.

The proof of Theorem 2.3 is divided in two steps. The first step consists in establishing Lemma 5.1. This lemma proves that entropy solutions are limit of solutions obtained by approximation. And the second step is devoted to prove the uniqueness of entropy solutions in general. This result is a consequence of this lemma.

Lemma 5.1. *Let $(\tilde{u}_2, \tilde{v}_2)$ be the entropy solution solution of system defined by* (27)–(28). *Let $(\tilde{u}_1, \tilde{v}_1)$ be the limit of solution obtained by approximation of $(\tilde{u}_{1,\varepsilon})_\varepsilon$, $(\tilde{v}_{1,\varepsilon})_\varepsilon$ solution of equations* (11) *and* (12). *Then $(\tilde{u}_2, \tilde{v}_2) = (\tilde{u}_1, \tilde{v}_1)$ almost everywhere in $(0, T) \times \Omega$.*

Proof. According to [17], we introduce the functions T_h^n and R_h^n in $C^2(\mathbb{R}, \mathbb{R})$, such that for $h > \frac{1}{n}$,

$$\begin{gathered} (T_h^n)'(s) = 0 \text{ if } |s| \ge h, \\ (T_h^n)'(s) = 1 \text{ if } |s| \le h - \frac{1}{n}, \\ 0 \le (T_h^n)'(s) \le 1 \text{ for all } s, \\ R_h^n(0) = 0,\ (R_h^n)'(s) = 1 - (T_h^n)' \text{ and } R_h^n(-s) = R_h^n(s), \text{for all } s \ge 0. \end{gathered}$$

Note that $T_h^n(\tilde v_{1,\varepsilon})$ and $(T_h^n)'(\tilde v_{1,\varepsilon})$ are in $\bigcap_{l=1}^N L^{p_l}(0,T;W^{1,p_l,l}(\Omega))\cap L^\infty(Q_T)$.

Let us substitute $\psi = T_h^n(\tilde v_{1,\varepsilon})$ in equation (28) where $\tilde u$ and $\tilde v$ are replaced respectively by $\tilde u_2$ and $\tilde v_2$. We have

$$\begin{aligned}
&\int_\Omega S_\gamma(\tilde v_2 - T_h^n(\tilde v_{1,\varepsilon}))(t,x)\,dx \\
&\quad - \int_\Omega S_\gamma(v_0(x) - T_h^n(v_{0,\varepsilon}(x)))\,dx + \int_0^t \langle \partial_t \tilde v_{1,\varepsilon}, (T_h^n)'(\tilde v_{1,\varepsilon})T_\gamma(\tilde v_2 - T_h^n(\tilde v_{1,\varepsilon}))\rangle\,d\tau \\
&\qquad + \int_0^t\int_\Omega \tilde{\mathbf{A}}_2(t,x,\nabla\tilde v_2)\cdot\nabla T_\gamma(\tilde v_2 - T_h^n(\tilde v_{1,\varepsilon}))\,dx\,d\tau \\
&\quad + \int_0^t\int_\Omega \tilde v_2\ \mathbf{K}_2\cdot\nabla T_\gamma(\tilde v_2 - T_h^n(\tilde v_{1,\varepsilon}))\,dx\,d\tau + \int_0^t\int_\Omega \lambda\ \tilde v_2 T_\gamma(\tilde v_2 - T_h^n(\tilde v_{1,\varepsilon}))\,dx\,d\tau \\
&\qquad\qquad \le \int_0^t\int_\Omega G(\tilde u_2,\tilde v_2)T_\gamma(\tilde v_2 - T_h^n(\tilde v_{1,\varepsilon}))\,dx\,d\tau. \qquad (29)
\end{aligned}$$

Hence, let $\psi\in\bigcap_{l=1}^N L^{p_l}(0,T;W^{1,p_l,l}(\Omega))\cap L^\infty(Q_T)$, and take $(T_h^n)'(\tilde v_{1,\varepsilon})\psi$ as test function in (12), where $\tilde u_\varepsilon$ and $\tilde v_\varepsilon$ are replaced respectively by $\tilde u_{1,\varepsilon}$ and $\tilde v_{1,\varepsilon}$. we get

$$\begin{aligned}
&\int_0^t \langle \partial_t\tilde v_{1,\varepsilon}, (T_h^n)'(\tilde v_{1,\varepsilon})\psi\rangle\,d\tau + \int_0^t\int_\Omega (T_h^n)''(\tilde v_{1,\varepsilon})\psi\tilde{\mathbf{A}}_2(t,x,\nabla\tilde v_{1,\varepsilon})\cdot\nabla\tilde v_{1,\varepsilon}dx\,d\tau \\
&+\int_0^t\int_\Omega (T_h^n)'(\tilde v_{1,\varepsilon})\mathbf{A}_2(t,x,\nabla\tilde v_{1,\varepsilon})\cdot\nabla\psi\,dx\,d\tau + \int_0^t\int_\Omega (T_h^n)''(\tilde v_{1,\varepsilon})\psi\ \tilde v_{1,\varepsilon}\ \mathbf{K}_2\cdot\nabla\tilde v_{1,\varepsilon}\,dx\,d\tau \\
&\quad + \int_0^t\int_\Omega (T_h^n)'(\tilde v_{1,\varepsilon})\ \tilde v_{1,\varepsilon}\ \mathbf{K}_2\cdot\nabla\psi\,dxd\tau + \int_0^t\int_\Omega \lambda\ \tilde v_{1,\varepsilon}(T_h^n)'(\tilde v_{1,\varepsilon})\psi\,dx\,d\tau \\
&\qquad\qquad = \int_0^t\int_\Omega G(\tilde u_{1,\varepsilon},\tilde v_{1,\varepsilon})(T_h^n)'(\tilde v_{1,\varepsilon})\psi\,dx\,d\tau.
\end{aligned}$$

Now, choosing $\psi = T_\gamma(\tilde v_2 - T_h^n(\tilde v_{1,\varepsilon}))$ in the above inequality, the entropy formulation (29) is equivalent to

$$\begin{aligned}
&\int_\Omega S_\gamma(\tilde v_2 - T_h^n(\tilde v_{1,\varepsilon}))(t,x)\,dx - \int_\Omega S_\gamma(v_0(x) - T_h^n(v_{0,\varepsilon}(x)))\,dx \\
&\quad - \int_0^t\int_\Omega (T_h^n)''(\tilde v_{1,\varepsilon}) T_\gamma(\tilde v_2 - T_h^n(\tilde v_{1,\varepsilon}))\big(\tilde{\mathbf{A}}_2(t,x,\nabla\tilde v_{1,\varepsilon}) + \tilde v_{1,\varepsilon}\ \mathbf{K}_2\big)\cdot\nabla\tilde v_{1,\varepsilon}\,dx\,d\tau \\
&\quad - \int_0^t\int_\Omega (T_h^n)'(\tilde v_{1,\varepsilon})\big(\tilde{\mathbf{A}}_2(t,x,\nabla\tilde v_{1,\varepsilon}) + \tilde v_{1,\varepsilon}\ \mathbf{K}_2\big)\cdot\nabla T_\gamma(\tilde v_2 - T_h^n(\tilde v_{1,\varepsilon}))\,dx\,d\tau \\
&\qquad + \int_0^t\int_\Omega \big(\tilde{\mathbf{A}}_2(t,x,\nabla\tilde v_2) + \tilde v_2\ \mathbf{K}_2\big)\cdot\nabla T_\gamma(\tilde v_2 - T_h^n(\tilde v_{1,\varepsilon}))\,dx\,d\tau \\
&\qquad + \int_0^t\int_\Omega \lambda\big(\tilde v_2 - \tilde v_{1,\varepsilon}(T_h^n)'(\tilde v_{1,\varepsilon})\big)T_\gamma(\tilde v_2 - T_h^n(\tilde v_{1,\varepsilon}))\,dx\,d\tau \\
&\quad \le \int_0^t\int_\Omega \big(G(\tilde u_2,\tilde v_2) - G(\tilde u_{1,\varepsilon},\tilde v_{1,\varepsilon})(T_h^n)'(\tilde v_{1,\varepsilon})\big)T_\gamma(\tilde v_2 - T_h^n(\tilde v_{1,\varepsilon}))\,dx\,d\tau. \qquad (30)
\end{aligned}$$

Let us denote the six integrals of the left hand side as L_1 to L_6 and the integral of the right hand side as L_7. In order to obtain an estimate on L_3, we take $\psi = (R_h^n)'(\tilde v_{1,\varepsilon})$ in equation (12). We have

$$\begin{aligned}
&\int_\Omega R_h^n(\tilde v_{1,\varepsilon}(t,x))\,dx - \int_\Omega R_h^n(v_{0,\varepsilon}(x))\,dx \\
&\qquad + \int_0^t\int_\Omega (R_h^n)''(\tilde v_{1,\varepsilon})\big(\tilde{\mathbf{A}}_2(t,x,\nabla\tilde v_{1,\varepsilon}) + \tilde v_{1,\varepsilon}\ \mathbf{K}_2\big)\cdot\nabla\tilde v_{1,\varepsilon}\,dx\,d\tau \\
&\quad + \int_0^t\int_\Omega \lambda\tilde v_{1,\varepsilon}(R_h^n)'(\tilde v_{1,\varepsilon})dx\,d\tau = \int_0^t\int_\Omega G(\tilde u_{1,\varepsilon},\tilde v_{1,\varepsilon})(R_h^n)'(\tilde v_{1,\varepsilon})\,dx\,d\tau. \qquad (31)
\end{aligned}$$

Note that from the choice of λ in (10) and the fact that $\mathbf{K}_2(t,x)\cdot\eta(x) \ge 0$ on $(0,T)\times\partial\Omega$, we have

$$\int_0^t\int_\Omega \lambda\tilde v_{1,\varepsilon}(R_h^n)'(\tilde v_{1,\varepsilon})\,dx\,d\tau + \int_0^t\int_\Omega \tilde v_{1,\varepsilon}\ \mathbf{K}_2\cdot\nabla(R_h^n)'(\tilde v_{1,\varepsilon})\,dx\,d\tau \ge 0. \qquad (32)$$

We deduce from the definition of the function R_h^n and estimate (32) that

$$\begin{aligned}
|L_3| &\le \gamma\int_0^t\int_\Omega |(T_h^n)''(\tilde v_{1,\varepsilon})\tilde{\mathbf{A}}_2(t,x,\nabla\tilde v_{1,\varepsilon})\cdot\nabla\tilde v_{1,\varepsilon} + (T_h^n)''(\tilde v_{1,\varepsilon})\tilde v_{1,\varepsilon}\ \mathbf{K}_2\cdot\nabla\tilde v_{1,\varepsilon}|\,dx\,d\tau \\
&= \gamma\int_0^T\int_\Omega |(R_h^n)''(\tilde v_{1,\varepsilon})\tilde{\mathbf{A}}_2(t,x,\nabla\tilde v_{1,\varepsilon})\cdot\nabla\tilde v_{1,\varepsilon} + (R_h^n)''(\tilde v_{1,\varepsilon})\tilde v_{1,\varepsilon}\mathbf{K}_2\cdot\nabla\tilde v_{1,\varepsilon}|\,dx\,d\tau \\
&\le \gamma\int_0^T\int_\Omega |(R_h^n)''(\tilde v_{1,\varepsilon})\tilde{\mathbf{A}}_2(t,x,\nabla\tilde v_{1,\varepsilon})\cdot\nabla\tilde v_{1,\varepsilon} + \lambda\tilde v_{1,\varepsilon}(R_h^n)'(\tilde v_{1,\varepsilon}) \\
&\qquad + (R_h^n)''(\tilde v_{1,\varepsilon})\tilde v_{1,\varepsilon}\mathbf{K}_2\cdot\nabla\tilde v_{1,\varepsilon}|\,dx\,d\tau + \gamma\lambda\int_0^T\int_\Omega |\tilde v_{1,\varepsilon}(R_h^n)'(\tilde v_{1,\varepsilon})|\,dx\,d\tau,
\end{aligned}$$

and using the equation (31) one gets

$$|L_3| \le C\gamma\Big(\int_0^T \int_\Omega |G(\tilde u_{1,\varepsilon},\tilde v_{1,\varepsilon})|\mathbf{1}_{\{\,|\tilde v_{1,\varepsilon}|\ge h-\frac1n\,\}}\,dx\,d\tau + \int_\Omega |v_{0,\varepsilon}|\mathbf{1}_{\{\,|v_{0,\varepsilon}|\ge h-\frac1n\,\}}\,dx\Big), \\ + C\gamma\int_0^T\int_\Omega |\tilde v_{1,\varepsilon}|\mathbf{1}_{\{\,|\tilde v_{1,\varepsilon}|\ge h-\frac1n\,\}}\,dx\,d\tau, \quad (33)$$

where C denotes a positive constant independent of ε. Using the Dominated Convergence Theorem and estimate (33), it follows after letting n go to $+\infty$ in estimate (30) that

$$\begin{aligned}
&\int_\Omega S_\gamma(\tilde v_2 - T_h(\tilde v_{1,\varepsilon}(t,x)))\,dx - \int_\Omega S_\gamma(v_0 - T_h(v_{0,\varepsilon}(x)))\,dx \\
&\quad + \int_0^t\int_\Omega \big(\tilde{\mathbf{A}}_2(t,x,\nabla\tilde v_2) - \tilde{\mathbf{A}}_2(t,x,\nabla T_h(\tilde v_{1,\varepsilon}))\big)\cdot\nabla T_\gamma(\tilde v_2 - T_h(\tilde v_{1,\varepsilon}))\,dx\,d\tau \\
&\quad + \int_0^t\int_\Omega (\tilde v_2 - \tilde v_{1,\varepsilon}(T_h)'(\tilde v_{1,\varepsilon}))\mathbf{K}_2\cdot\nabla T_\gamma(\tilde v_2 - T_h(\tilde v_{1,\varepsilon}))\,dx\,d\tau \\
&\quad + \int_0^t\int_\Omega \lambda(\tilde v_2 - \tilde v_{1,\varepsilon}(T_h^n)'(\tilde v_{1,\varepsilon}))T_\gamma(\tilde v_2 - T_h(\tilde v_{1,\varepsilon}))\,dx\,d\tau \\
&\le \int_0^t\int_\Omega (G(\tilde u_2,\tilde v_2) - G(\tilde u_{1,\varepsilon},\tilde v_{1,\varepsilon})(T_h)'(\tilde v_{1,\varepsilon}))T_\gamma(\tilde v_2 - T_h(\tilde v_{1,\varepsilon}))\,dx\,d\tau \\
&\quad + C\gamma\Big(\int_0^t\int_\Omega |G(\tilde u_{1,\varepsilon},\tilde v_{1\varepsilon})|\mathbf{1}_{\{\,|\tilde v_{1,\varepsilon}|\ge h\,\}}\,dx\,d\tau + \int_\Omega |v_{0,\varepsilon}|\mathbf{1}_{\{\,|v_{0,\varepsilon}|\ge h\,\}}\,dx\Big) \\
&\quad + C\gamma\int_0^T\int_\Omega |\tilde v_{1,\varepsilon}|\mathbf{1}_{\{\,|\tilde v_{1,\varepsilon}|\ge h\,\}}\,dx.
\end{aligned}$$

Thanks to the monotonicity of $\tilde{\mathbf{A}}_2$, we have by Fatou's lemma

$$\begin{aligned}
&\int_0^t\int_\Omega \big(\tilde{\mathbf{A}}_2(t,x,\nabla\tilde v_2) - \tilde{\mathbf{A}}_2(t,x,\nabla T_h(\tilde v_1))\big)\cdot\nabla T_\gamma(\tilde v_2 - T_h(\tilde v_1))\,dx\,d\tau \\
&\le \liminf_{\varepsilon\to0}\int_0^T\int_\Omega \big(\tilde{\mathbf{A}}_2(t,x,\nabla\tilde v_2) - \mathbf{A}_1(t,x,\nabla T_h(\tilde v_{1,\varepsilon}))\big)\cdot\nabla T_\gamma(\tilde v_2 - T_h(\tilde v_{1,\varepsilon}))\,dxd\tau,
\end{aligned}$$

and the properties of the sequences $\tilde v_{1,\varepsilon}$ and $\tilde u_{1,\varepsilon}$ allow to pass to the limit as $\varepsilon\to0$. It remains to consider $h\to\infty$ in the following relation:

$$\begin{aligned}
&\int_\Omega S_\gamma(\tilde v_2 - T_h(\tilde v_1(t,x)))\,dx - \int_\Omega S_\gamma(v_0 - T_h(v_0(x)))\,dx \\
&\quad + \int_0^t\int_\Omega \big(\tilde{\mathbf{A}}_2(t,x,\nabla\tilde v_2) - \tilde{\mathbf{A}}_2(t,x,\nabla T_h(\tilde v_1))\big)\cdot\nabla T_\gamma(\tilde v_2 - T_h(\tilde v_1))\,dx\,d\tau \\
&\quad + \int_0^t\int_\Omega (\tilde v_2 - \tilde v_1(T_h)'(\tilde v_1))\,\mathbf{K}_2\cdot\nabla T_\gamma(\tilde v_2 - T_h(\tilde v_1))\,dx\,d\tau
\end{aligned}$$

$$+\int_0^t \int_\Omega \lambda\,(\tilde{v}_2 - \tilde{v}_1 (T_h)'(\tilde{v}_1))\, T_\gamma(\tilde{v}_2 - T_h(\tilde{v}_1))\, dx\, d\tau$$
$$\leq \int_0^t \int_\Omega (G(\tilde{u}_2, \tilde{v}_2) - G(\tilde{u}_1, \tilde{v}_1)(T_h)'(\tilde{v}_1)) T_\gamma(\tilde{v}_2 - T_h(\tilde{v}_1))\, dx\, d\tau + \gamma R(h), \quad (34)$$

where $R(h)$ stands for

$$R(h) = \int_0^t \int_\Omega |G(u_1, v_1)| \mathbf{1}_{\{\,|u_1| \geq h\,\}}\, dx\, d\tau + \int_\Omega |v_0| \mathbf{1}_{\{\,|v_0| \geq h\,\}}\, dx + \int_0^T \int_\Omega |v_1| \mathbf{1}_{\{\,|v_1| \geq h\,\}}\, dx,$$

which goes to 0 as $h \to \infty$.

Now, let us write

$$G(\tilde{u}_2, \tilde{v}_2) - G(\tilde{u}_1, \tilde{v}_1)(T_h)'(\tilde{v}_1) = G(\tilde{u}_1, \tilde{v}_1)(1 - (T_h)'(\tilde{v}_1)) + (G(\tilde{u}_2, \tilde{v}_2) - G(\tilde{u}_1, \tilde{v}_1)).$$

We have $(1 - (T_h)'(\tilde{v}_1)) \to 0$ as $h \to \infty$, then the term

$$\int_0^t \int_\Omega G(\tilde{u}_1, \tilde{v}_1)(1 - (T_h)'(\tilde{v}_1)) T_\gamma(\tilde{v}_2 - T_h(\tilde{v}_1))\, dx\, d\tau$$

can be included in the general expression $\gamma R(h)$ which tends to zero as $h \to \infty$, and using the fact that $|T_\gamma(.)| \leq \gamma$ and the function G is Lipschitz, we have

$$\int_0^t \int_\Omega (G(\tilde{u}_2, \tilde{v}_2) - G(\tilde{u}_1, \tilde{v}_1)) T_\gamma(\tilde{v}_2 - T_h(\tilde{v}_1))\, dx\, d\tau \leq C\gamma \int_0^t \int_\Omega (|\tilde{u}_2 - \tilde{u}_1| + |\tilde{v}_2 - \tilde{v}_1|)\, dx\, d\tau.$$

In the classical way, by Lebesgue's theorem and Fatou's lemma, letting h going to ∞, we deduce from (34)

$$\int_\Omega S_\gamma(\tilde{v}_2 - \tilde{v}_1)(t,x) dx + \int_0^t \int_\Omega \big(\tilde{\mathbf{A}}_2(t,x,\nabla \tilde{v}_2) - \tilde{\mathbf{A}}_2(t,x,\nabla \tilde{v}_1)\big) \cdot \nabla T_\gamma(\tilde{v}_2 - \tilde{v}_1)\, dx\, d\tau$$
$$+ \int_0^t \int_\Omega (\tilde{v}_2 - \tilde{v}_1) \mathbf{K}_2 \cdot \nabla T_\gamma(\tilde{v}_2 - \tilde{v}_1)\, dx\, d\tau + \int_0^t \int_\Omega \lambda(\tilde{v}_2 - \tilde{v}_1) T_\gamma(\tilde{v}_2 - \tilde{v}_1)\, dx\, d\tau$$
$$\leq C\gamma \int_0^t \int_\Omega (|\tilde{u}_2 - \tilde{u}_1| + |\tilde{v}_2 - \tilde{v}_1|)\, dx\, d\tau.$$

Then by the choice of λ in (10), the coercivity of $\tilde{\mathbf{A}}_2$ and letting γ going to 0, we have

$$\int_\Omega |\tilde{v}_2 - \tilde{v}_1|(t,x)\, dx \leq C \int_0^t \int_\Omega (|\tilde{u}_2 - \tilde{u}_1| + |\tilde{v}_2 - \tilde{v}_1|)(t,x)\, dx\, d\tau.$$

for every $t \in (0,T)$. In the same way, we obtain for every $t \in (0,T)$

$$\int_\Omega |\tilde{u}_2 - \tilde{u}_1|(t,x) dx \leq C \int_0^t \int_\Omega (|\tilde{u}_2 - \tilde{u}_1| + |\tilde{v}_2 - \tilde{v}_1|)(t,x)\, dx\, d\tau.$$

Finally, it suffices to add the last two inequalities and to apply Gronwall's lemma, which completes the proof of this lemma. □

The uniqueness of the entropy solutions is then a consequence of the above lemma. Indeed, Let $(\tilde{u}_1, \tilde{v}_1)$ and $(\tilde{u}_2, \tilde{v}_2)$ be two entropy solutions of (27)–(28). Consider $(\tilde{u}_3, \tilde{v}_3)$ be the limit of solution obtained by approximation of $(\tilde{u}_{3,\varepsilon})_\varepsilon$, $(\tilde{v}_{3,\varepsilon})_\varepsilon$ solution of equations (11) and (12). According to Lemma 5.1, we have $(\tilde{u}_1, \tilde{v}_1) = (\tilde{u}_3, \tilde{v}_3)$ and $(\tilde{u}_2, \tilde{v}_2) = (\tilde{u}_3, \tilde{v}_3)$, which establishes Theorem 2.3.

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