# Variations on Yano's Extrapolation Theorem

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**ABSTRACT**

We give very short and transparent proofs of extrapolation theorems of Yano type in the framework of Lorentz spaces. The decomposition technique developed in [4] enables us to obtain known and new results in a unified manner.

*Key words:* extrapolation, Lebesgue space, Orlicz space, Zygmund space, Zygmund-Lorentz space

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## 1. Introduction and preliminaries

The well-known extrapolation theorem of Yano (see [11] and [12, Theorem XII.4.41]) states that if $(\Omega, \mu)$ is a finite measure space and for all $p$ near 1, $p > 1$, $T$ is a bounded linear map from $L_p(\Omega)$ to $L_p(\Omega)$ with norm not exceeding $C(p-1)^{-\alpha}$ for some positive $C$ and $\alpha$, then $T$ maps the Zygmund space $L(\log L)^{\alpha}(\Omega)$ boundedly into $L_1(\Omega)$. In fact, the result holds if $T$ is quasilinear rather than linear; and the theorem can also be put into the framework of abstract extrapolation theory (see [7] and [9]). The aim of the present paper is, using the decompositions developed in [4] for the extrapolation characterisation of exponential Orlicz spaces, to give extremely simple proofs of theorems of Yano type in the setting of Lorentz spaces. These theorems include not only the classical Yano theorem and some recent variants of it, but also new results.

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Throughout the paper $\Omega$ will be a subset of the $n$-dimensional Euclidean space $\mathbf{R}^n$ with finite Lebesgue measure $|\Omega|$; to simplify the formulae we shall suppose that $|\Omega| = 1$. Given a locally integrable (real-valued) function $f$ on $\Omega$, its distribution function $m_f$ is defined by

$$m_f(\lambda) = |\{ x \in \Omega : |f(x)| > \lambda \}|, \quad \lambda \geq 0,$$

and the non-increasing rearrangement $f^*$ of $f$ is given by

$$f^*(t) = \inf\{ \lambda : m_f(\lambda) \leq t \}, \quad t \geq 0.$$

For $1 \leq p \leq \infty$ the Lebesgue space $L_p = L_p(\Omega)$ is defined in the usual way and the norm of a function $f$ in this space will be denoted by $\|f\|_p$. If we need to specify the underlying set, say $A \subset \Omega$, then we write $\|f\|_{p,A}$ for the $L_p$ norm of $f$ over this set. Recall that $f$ and $f^*$ are equimeasurable and that $\|f\|_p = \|f^*\|_{p,(0,1)}$; for shortness we shall write $\|f^*\|_p$ for $\|f^*\|_{p,(0,1)}$. For $p, q \in [1, \infty]$ the Lorentz space $L_{p,q} = L_{p,q}(\Omega)$ is defined to be the space of all functions $f$ such that the (quasi-)norm

$$\|f\|_{p,q} := \Big( \int_0^1 \big\{ t^{1/p} f^*(t) \big\}^q \frac{dt}{t} \Big)^{1/q}$$

(appropriately modified if $q = \infty$ and/or $p = \infty$) is finite. For any $\alpha > 0$, $L(\log L)^\alpha = L(\log L)^\alpha(\Omega)$ is the Orlicz space generated by any Young function equivalent to $t \mapsto t(\log t)^\alpha$ near infinity. It is well known that since the measure of $\Omega$ is finite, all such Young functions give the same space (up to equivalence of norms) and that one can introduce a quasinorm on $L(\log L)^\alpha$ by the formula

$$\|f\|_{L(\log L)^\alpha} = \int_0^1 f^*(t) \Big( \log \frac{1}{t} \Big)^\alpha dt.$$

For this we refer to [2].

For $k \in \mathbf{N}$ let $I_k = (e^{-k}, e^{-k+1})$. In [4] we used the behaviour of $f^*$ on the intervals $I_k$ to characterise exponential Orlicz spaces; see also [3] and [5] for use of this localisation technique. It turns out that this is also useful in the present extrapolation context. We need to find functions $f_k$ such that $f = \sum_1^\infty f_k$ and $f_k^*$ has appropriate behaviour. To this end, observe that if $0 \leq a < b < \infty$, then

$$|\{ x \in \Omega : a \leq |f(x)| \leq b \}| = |\{ t \geq 0 : a \leq f^*(t) \leq b \}|.$$

Hence for each $k \in \mathbf{N}$, the measure $|I_k|$ of $I_k$ is not greater than the measure of $A_k := \{ x \in \Omega : f^*(e^{-k+1}) \leq |f(x)| \leq f^*(e^{-k}) \}$. Since the function $r \mapsto |B(0,r) \cap A_1|$ is continuous (here $B(0,r)$ is the open ball in $\mathbf{R}^n$ with centre 0 and radius $r$), there exists $r_1 > 0$ such that the measure of $\Omega_1 := B(0, r_1) \cap A_1$ equals $|I_1|$. Put $\tilde{\Omega}_2 = A_2 \backslash \Omega_1$. Repeating the above continuity argument for the set $B(0,r) \cap \tilde{\Omega}_2$, we find $r_2 > 0$

such that $\Omega_2 := B(0, r_2) \cap \tilde{\Omega}_2$ has measure equal to $|I_2|$. In this way we generate a sequence $\{\Omega_k\}$ of disjoint subsets of $\Omega$, with union differing from $\Omega$ by only a set of zero measure, such that for each $k \in \mathbf{N}$, $|\Omega_k| = |I_k|$. Now let $f_k := f\chi_{\Omega_k}$ $(k \in \mathbf{N})$, where $\chi_{\Omega_k}$ denotes the characteristic function of $\Omega_k$. On $\Omega_k$, $|f(x)| \le f^*(e^{-k})$ and so if $\lambda > f^*(e^{-k})$, then $m_{f_k}(\lambda) = 0$; thus $f_k^*(t) \le f^*(e^{-k})$ for all $t > 0$. On the other hand, if $\lambda < f^*(e^{-k+1})$ and $0 < t < |I_k|$, then

$$|\{\, x \in \Omega_k : |f(x)| > \lambda \,\}| = |\Omega_k| = |I_k| > t;$$

hence $f_k^*(t) \ge f^*(e^{-k+1})$ if $0 < t < |I_k|$. Moreover, if $t \ge |I_k|$, then plainly $f_k^*(t) = 0$. The corresponding decomposition $f = \sum_{k=1}^{\infty} f_k$ of $f$ thus has the property that for all $k \in \mathbf{N}$ and all $t \in (0, |I_k|)$, $f_k^*(t) \in [f^*(e^{-k+1}), f^*(e^{-k})]$. This will be crucial in what follows.

We write $A \lesssim B$ if $A \le cB$ for some positive constant $c$ independent of appropriate quantities involved in the expressions $A$ and $B$, and $A \sim B$ if $A \lesssim B$ and $B \lesssim A$.

## 2. The classical setting

Throughout we shall assume that $T$ is a sublinear operator, which means that its domain is the set of all scalar-valued measurable functions on $\Omega$ occurring in the assumptions of the respective theorems, and that for all such functions $f, g$, $|T(\mathrm{f+g})| \le |Tf| + |Tg|$. To explain the basic idea we start with the classical Yano theorem. Unlike the procedure in e.g. [12, Chapter 12] we directly discretise the rearrangement-invariant quasinorm in the operator domain.

**Theorem 2.1.** *Suppose that for all $p$ near 1 with $p > 1$, $T : L_p \to L_p$ is bounded, with $\|T \mid L_p \to L_p\| \le C(p-1)^{-\alpha}$ for some $\alpha > 0$ and $C$ independent of $p$. Then $T : L(\log L)^{\alpha} \to L_1$ is bounded.*

*Proof.* Use of our decomposition makes the proof is very simple. Let $f \in L \log L$, write $f = \sum f_k$ as above, and observe that if $t \in I_k$ then $\log(1/t) \sim k$, where the constants implicit in the equivalence estimates can be chosen independent of $k \in \mathbf{N}$. Hence the norm of $f$ in $L \log L$ is

$$\int_0^1 f^*(t)\Big(\log \frac{1}{t}\Big)^{\alpha}\, dt \sim \sum_{k=1}^{\infty} k^{\alpha} e^{-k} f^*(e^{-k}).$$

Put $p_k = 1 + \frac{1}{k}$ $(k \in \mathbf{N})$. Then using the subadditivity (see [2, Theorem 2.3.4]) of the operation $g \mapsto g^{**}$, where $g^{**}(t) = t^{-1} \int_0^t g^*(s) ds$, together with the properties of $T$

and Hölder's inequality, we have

$$\begin{aligned}\int_0^1 (Tf)^*(t)\,dt &\le \sum_k \int_0^1 (Tf_k)^*(t)\,dt \le \sum_k \Big(\int_0^1 \big((Tf_k)^*(t)\big)^{p_k}\Big)^{1/p_k} \\ &\le C\sum_k (p_k-1)^{-\alpha}\|f_k\|_{p_k} \lesssim \sum_k k^\alpha \|f_k^*\|_{p_k} \\ &\le \sum_k k^\alpha e^{-k/p_k} f^*(e^{-k}).\end{aligned}$$

Since $k/p_k = k - k/(k+1)$, we conclude that

$$\int_0^1 (Tf)^*(t)\,dt \lesssim \sum_k k^\alpha e^{-k} f^*(e^{-k}) \lesssim \int_0^1 f^*(t)\Big(\log\frac{1}{t}\Big)^\alpha\,dt,$$

and the proof is complete. □

*Remark* 2.2. The proof makes it plain that the target space $L_p$ in the assumption on $T$ can be replaced by $L_1$. This result is known: see [7] for a proof using the machinery of abstract extrapolation theory. For convenience we formulate this separately.

**Theorem 2.3.** *Suppose that for all $p$ near 1 with $p > 1$, $T : L_p \to L_1$ is bounded, with $\|T \mid L_p \to L_1\| \le C(p-1)^{-\alpha}$ for some $\alpha > 0$ and $C$ independent of $p$. Then $T : L(\log L)^\alpha \to L_1$ is bounded.*

*Remark* 2.4. These theorems (as well as those in the next sections) have obvious dual counterparts. We shall not state these explicitly.

## 3. Variations in the assumed target spaces

We shall need the following simple lemma on the embedding of $L_{p,q}$ in $L_1$.

**Lemma 3.1.** *If $p, q \in (1,\infty)$, then for all $f \in L_{p,q}$,*

$$\|f\|_1 \le \Big(\frac{p}{q'(p-1)}\Big)^{1/q'} \|f\|_{p,q},$$

*where $1/q' = 1 - 1/q$.*

*Proof.* If $f \in L_1$, then

$$\begin{aligned}\|f\|_1 &= \int_0^1 t^{1/p} f^*(t)\cdot t^{1-1/p}\,dt \\ &\le \Big(\int_0^1 \big(t^{1/p}f^*(t)\big)^q \frac{dt}{t}\Big)^{1/q}\Big(\int_0^1 t^{(1-1/p)q'}\frac{dt}{t}\Big)^{1/q'},\end{aligned}$$

which leads to the desired result. □

This immediately gives

**Lemma 3.2.** *Suppose that for some* $q \in (1,\infty)$ *and all* $p$ *near* 1 *with* $p > 1$, $T : L_p \to L_{p,q}$ *is bounded, with* $\|T \mid L_p \to L_{p,q}\| \le C(p-1)^{-\alpha}$ *for some* $\alpha > 0$ *and* $C$ *independent of* $p$*. Then* $T : L(\log L)^{\alpha+1/q'} \to L_1$ *is bounded.*

*Proof.* Let $f \in L(\log L)^{\alpha+1/q'}$. Then

$$\|Tf\|_1 \lesssim (p-1)^{-1/q'}\|Tf\|_{p,q} \lesssim (p-1)^{-\alpha-1/q'}\|f\|_p$$

and the conclusion follows from Theorem 2.3. □

To deal with the end point cases $q = 1$ and $q = \infty$ we proceed as follows.

**Lemma 3.3.** *Suppose that for all* $p$ *near* 1 *with* $p > 1$, $T : L_p \to L_{p,\infty}$ *is bounded, with* $\|T \mid L_p \to L_{p,\infty}\| \le C(p-1)^{-\alpha}$ *for some* $\alpha > 0$ *and* $C$ *independent of* $p$*. Then* $T : L(\log L)^{\alpha+1} \to L_1$ *is bounded.*

*Proof.* As before, put $p_k = 1 + \frac{1}{k}$ ($k \in \mathbf{N}$). Then for all $f \in L(\log L)^{\alpha+1}$,

$$\begin{aligned}
\|Tf\|_1 &= \int_0^1 (Tf)^*(t)\,dt \le \sum_k \int_0^1 (Tf_k)^*(t)\,dt \\
&= \sum_k \int_0^1 t^{1/p_k}(Tf_k)^*(t)\cdot t^{-1/p_k}\,dt \lesssim \sum_k (p_k-1)^{-1}\|Tf_k\|_{p_k,\infty} \\
&\lesssim \sum_k (p_k-1)^{-\alpha-1}\|f_k\|_{p_k} \lesssim \sum_k k^{\alpha+1}e^{-k/p_k}f^*(e^{-k}) \\
&\lesssim \int_0^1 f^*(t)\Big(\log\frac{1}{t}\Big)^{\alpha+1}\,dt.
\end{aligned}$$

□

**Lemma 3.4.** *Suppose that for all* $p$ *near* 1, $p > 1$, $T : L_p \to L_{p,1}$ *is bounded, with* $\|T \mid L_p \to L_{p,1}\| \le C(p-1)^{-\alpha}$ *for some* $\alpha > 0$ *and* $C$ *independent of* $p$*. Then* $T : L(\log L)^{\alpha} \to L_1$ *is bounded.*

*Proof.* Since $\|Tf\|_p \le \|Tf\|_{p,1}$ (see [2, Proposition 4.4.2]), the claim follows immediately from Theorem 2.1. □

**Corollary 3.5.** *Let* $1 \le q \le \infty$ *and suppose that for all* $p$ *near* 1 *with* $p > 1$, $T : L_p \to L_{p,q}$ *is bounded, with* $\|T \mid L_p \to L_{p,q}\| \le C(p-1)^{-\alpha}$ *for some* $\alpha > 0$ *and* $C$ *independent of* $p$*. Then* $T : L(\log L)^{\alpha+1/q'} \to L_1$ *is bounded, where* $1/q' = 1 - 1/q$ *(*$1' = \infty$ *and* $\infty' = 1$*).*

## 4. Variations in the assumed domain space

Here we shall look at what happens if $T$ operates on $L_{p,\infty}$ instead of $L_p$.

**Lemma 4.1.** *Suppose that for all $p$ near 1 with $p > 1$, $T : L_{p,\infty} \to L_1$ is bounded, with $\|T \mid L_{p,\infty} \to L_1\| \le C(p-1)^{-\alpha}$ for some $\alpha > 0$ and $C$ independent of $p$. Then $T : L(\log L)^\alpha \to L_1$ is bounded.*

*Proof.* Proceeding as before, if $f \in L(\log L)^\alpha$ we have

$$\begin{aligned}\int_\Omega |Tf(x)|\,dx &\le \sum_k \|Tf_k\|_1 \lesssim \sum_k (p_k-1)^{-\alpha}\|f_k\|_{p_k,\infty} \\ &= \sum_k k^\alpha \sup_{0<t<|I_k|} \big\{ t^{1/p_k} f_k^*(t) \big\} \le \sum_k k^\alpha e^{k-k/p_k} e^{-k} f^*(e^{-k}),\end{aligned}$$

and this last series is equivalent to the quasinorm on $L(\log L)^\alpha$. □

Lemma 4.1 gives a corresponding result for all target spaces embedded in $L_1$ and in particular for $L_{p,1}$. Nevertheless, just as in the last section we can do better and summarise the position in the following Theorem. Since the proofs are similar to those already given we omit them.

**Theorem 4.2.** *Suppose that for some $q \in [1,\infty]$ and all $p$ near 1 with $p > 1$, $T : L_{p,\infty} \to L_{p,q}$ is bounded, with $\|T \mid L_{p,\infty} \to L_{p,q}\| \le C(p-1)^{-\alpha}$ for some $\alpha > 0$ and $C$ independent of $p$. Then $T : L(\log L)^{\alpha+1/q'} \to L_1$ is bounded.*

*Remark* 4.3. Our techniques do not seem to give a short proof of the result of Soria [10] (see also [7, 5.7]) that if $T : L_{p,1} \to L_{p,\infty}$ is bounded with norm blowing up like $(p-1)^{-\alpha}$ for some $\alpha > 0$, then $T : L(\log L)^\alpha(\log\log L) \to L_{1,\infty}$ is bounded. We shall return to this point in a forthcoming paper.

## 5. More logarithms

It is well known (see, for example, [9, 2.4]) that some of the theorems we have been discussing continue to hold if the initial and target spaces are further logarithmically tuned. For example, if for all $p$ near 1, $p > 1$, $T : L_p \to L_p$ is bounded, with norm $\|T \mid L_p \to L_p\| \le C(p-1)^{-\alpha}$ for some $\alpha > 0$, then for all $\beta > 0$, $T : L(\log L)^{\alpha+\beta} \to L_1(\log L)^\beta$ is bounded. Our theorems in Sections 2, 3 and 4 all have conclusions to the effect that $T$ maps a space of the form $L(\log L)^\gamma$ boundedly into $L_1$, and it turns out that they can be refined in this way by the addition of logarithms.

**Theorem 5.1.** *Suppose that the hypotheses of any one of the preceding theorems hold, and that consequently there exists $\gamma \ge 0$ such that $T : L(\log L)^\gamma \to L_1$ is bounded. Then for all $\beta > 0$, $T : L(\log L)^{\gamma+\beta} \to L(\log L)^\beta$ is bounded.*

*Proof.* In all cases we use the same decomposition idea. For shortness we simply give the proof for the case in which $T$ maps $L_p$ boundedly into $L_{p,\infty}$ for all $p$ near 1, $p > 1$, with blow-up of the norms of $T$ of order $(p-1)^{-\alpha}$, for some $\alpha > 0$. We therefore know that $L(\log L)^{\alpha+1} \to L_1$ is bounded. As before we write $p_k = 1 + 1/k$ $(k \in \mathbf{N})$ and use the decomposition $f = \sum f_k$ of $f \in L_1(\log L)^\beta$. Then

$$\begin{aligned}\int_0^1 (Tf)^*(t)(\log(1/t))^\beta\,dt &\le \sum_k \int_0^1 t^{1/p_k}(Tf_k)^*(t)t^{-1/p_k}(\log(1/t))^\beta\,dt \\ &\le \sum_k \|Tf_k\|_{p_k,\infty} \int_0^1 t^{-1/p_k}(\log(1/t))^\beta\,dt \\ &\lesssim \sum_k (p_k-1)^{-\alpha}\|f_k\|_{p_k} \int_0^1 t^{-1/p_k}(\log(1/t))^\beta\,dt \\ &\le \sum_k k^{\alpha+\beta}e^{-k/p_k}f^*(e^{-k})\Big(\int_0^1 t^{-1/p_k}(\log(1/t))^\beta\,dt\Big)k^{-\beta}e^{k(1-1/p_k)}.\end{aligned}$$

Change of variables gives

$$\int_0^1 t^{-1/p_k}(\log(1/t))^\beta\,dt \le ck^{\beta+1}\Gamma(\beta+1),$$

where $\Gamma$ is the Gamma function. Hence

$$\int_0^1 (Tf)^*(t)(\log(1/t))^\beta\,dt \lesssim \sum_k k^{\beta+\alpha+1}e^{-k/p_k}f^*(e^{-k})\Gamma(\beta+1),$$

and this last expression is equivalent to the quasinorm on $L(\log L)^{\beta+\alpha+1}$ since $e^{-k/p_k} \sim e^{-k}$ and $\Gamma(\beta)$ is a finite positive constant. □

*Remark* 5.2. It is clear from the above proof that in some cases the range for $\beta$ can be bigger than stated. In particular, the assumptions for which the proof is given permit $\beta > -1$ provided that $L(\log L)^\beta$ is defined as the space of $f$ such that $\int_0^1 f^*(t)(\log(e/t))^\beta\,dt$ for $\beta < 0$.

*Remark* 5.3. The decomposition technique can be also used to prove extrapolation results for couples of spaces $(X, Y)$, where $X$ and $Y$ result from logarithmic or Lorentz (perhaps both) tuning. For instance, an amalgam of proofs of theorems in sections 3 and 4 with the proof of Theorem 5.1 (with some technical changes) easily yields $T : L(\log L)^{\alpha+\beta} \to L_r(\log L)^\beta$ provided $T : L_{p,1} \to L_{r,p}$ for some fixed $r \in [1,\infty)$, $p > 1$ and close to 1, and with the blow-up $(p-1)^\alpha$. This is a generalization of the archetypal extrapolation theorem from [6], which was generalized in the recent paper [8]. Problems of this type will be dealt with in detail elsewhere.

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