Very Ampleness of Multiples of Principal Polarization on Degenerate Abelian Surfaces

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ABSTRACT

Quite recently, Alexeev and Nakamura proved that if Y is a stable semi-Abelic variety (SSAV) of dimension g equipped with the ample line bundle $\mathcal{O}_Y(1)$, which deforms to a principally polarized Abelian variety, then $\mathcal{O}_Y(n)$ is very ample as soon as $n \geq 2g+1$, that is $n \geq 5$ in the case of surfaces. Here it is proved, via elementary methods of projective geometry, that in the case of surfaces this bound can be improved to $n \geq 3$.

Key words: very ampleness, degenerate Abelian surface, principal polarization, special projective embeddings.

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1. Introduction

In the past, there have been many attempts to construct suitable compactifications of the (coarse) moduli space of Abelian varieties, both in the principally polarized and in the non-principally polarized cases (see [7] for a detailed review). Let us restrict our attention to the principally polarized case. In this case, the first solution was given by Satake (see [9]) who constructed a projective normal variety $\bar{\mathcal{A}}_g$, which is highly singular along the boundary (the boundary $\partial\bar{\mathcal{A}}_g$ of $\mathcal{A}_g$ is not a divisor in this case and it is set-theoretically the union of the moduli $\mathcal{A}_i$ for $i \leq g-1$). Subsequently, by blowing up along the boundary, Igusa constructed a partial desingularization of Satake's

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compactification: in his compactification the boundary has codimension 1 (see [8]). These ideas were the starting point for Mumford's theory of toroidal compactifications of quotients of bounded symmetric domains (see [3] for a detailed description of this theory). Namikawa proved that Igusa's compactification is a toroidal compactification in Mumford' s sense. Unfortunately, toroidal compactifications are not unique, since they depend on the choice of cone decompositions. Ideally, one would like to construct a compactification which is meaningful for moduli, so as to obtain a space whose boundary points can be described in terms of Abelian varieties and well-understood degenerations.

Of course, the model is the Deligne-Mumford compactification of $\mathcal{M}_g$. The fact that toroidal compactifications are not unique has made very difficult to select the right compactification (if there is one). In spite of this, quite recently, Alexeev and Nakamura (see [1, 2]), building up on previous works of Nakamura and Namikawa, have shown that the toroidal compactification $\mathcal{A}_g^{\mathrm{Vor}}$, associated to the second Voronoi decomposition has a nice interpretation in terms of degenerations. This means that $\mathcal{A}_g^{\mathrm{Vor}}$ can be considered as the canonical compactification of the moduli space $\mathcal{A}_g$ of principally polarized Abelian varieties, as the Deligne-Mumford compactification represents the canonical compactification for the moduli space of curves (this point is investigated in [1]).

More specifically, $\mathcal{A}_g^{\mathrm{Vor}}$ parameterizes stable semi-Abelic varieties (SSAV) together with a divisor. Let us recall that a semi-Abelian variety G is just an extension $1 \to T \to G \to A \to 1$, where A is an Abelian variety and T is an algebraic torus $T = (\mathbb{C}^*)^r$, for some r. Then, in Alexeev's construction a SSAV Y is a good degeneration of an Abelian variety (corresponding to a boundary point in $\mathcal{A}_g^{\mathrm{Vor}}$) and on it there is an action of a semi-Abelian variety G in such a way that there are finitely many orbits. We will not recall the whole construction (see [2]), we want just to remark that any SSAV Y is a projective, semi-normal variety (i.e. Y is isomorphic to its semi-normalization Y' in Y^{nor}: $Y^{\mathrm{nor}} \to Y' \xrightarrow{\pi} Y$ and Y' is maximal such that for each $x \in Y$ there is a unique $x' \in Y'$ with $\pi(x') = x$ and $\mathbb{C}(x) \cong \mathbb{C}(x')$), equipped with an ample line bundle $\mathcal{O}_Y(1)$.

In this paper, we study very ampleness of line bundles coming from multiples of principal polarization on degenerate Abelian surface (over $\mathbb{C}$), corresponding to boundary components of $\mathcal{A}_2^{\mathrm{Vor}}$, that is on SSAV's. A well-known theorem of Lefschetz states that if A is a *smooth* Abelian variety of dimension g and $\mathcal{O}_A(1)$ is a principal polarization, then $\mathcal{O}_A(3)$ is very ample, (in fact, the theorem of Lefschetz is true for all polarizations, not just for principal polarizations). We want to understand how far is this statement if we replace A, with a SSAV Y (restricting to the case of surfaces). Indeed, in [2] it is proved that for a SSAV Y of genus g, $\mathcal{O}_Y(n)$ is very ample as soon as $n \geq 2g + 1$, that is, in the case of surfaces, as soon as $n \geq 5$. We improve this bound, showing that already $\mathcal{O}_Y(3)$ is very ample (that is Lefschetz's theorem still holds for a SSAV of dimension 2, which deforms to a principally polarized Abelian variety). The proof of this result is elementary in spirit and it is based on proving the analogous statement for each degeneration type, providing a careful description of $\mathcal{O}_Y(3)$ and of its sections. Indeed, in the principally polarized case, there are only three types of degenerations for surfaces. Two degenerations where there is no remaining Abelian part (which in the following sections are called degenerations of second and third type), corresponding to the two standard Delaunay decompositions of $\mathbb{Z}^2$ (lattice of rank 2) and one degeneration (of the first type) where there is still an Abelian part surviving (an elliptic curve) and which corresponds to the unique Delaunay decomposition of a lattice of rank one.

The case of the $\mathbb{P}^1$-bundle over an elliptic curve (degeneration of the first type) is the most general, since it depends on two moduli, i.e. the moduli of the elliptic curve and the glueing parameter. The second degeneration type depends on one moduli, namely the glueing parameter, while the third degeneration type depends on no moduli at all.

2. Very ampleness on the first type of degeneration

The first type of degeneration for smooth principally polarized Abelian surfaces can be constructed from a $\mathbb{P}^1$-bundle X, with two sections, over an elliptic curve E, $\pi : X \to E$. In this case, in the degenerate surface, there is still an Abelian part surviving and the smooth model is $X := \mathbb{P}(\mathcal{O}_E \oplus \mathcal{O}_E(e - o))$ (e is a point on E and o is the zero of the group law on E); it has two sections C_0 and C_1, corresponding to the fact that the ruled surface X comes from a split rank two vector bundle on E. We can identify the two sections by saying what is their normal bundle, so we define C_0 so that $\mathcal{N}_{C_0/X} = \mathcal{O}_{C_0}(C_0) = \mathcal{O}_E(e - o)$, and correspondingly C_1 so that $\mathcal{N}_{C_1/X} = \mathcal{O}_{C_1}(C_1) = \mathcal{O}_E(o - e)$.

It is well known that $\operatorname{Pic}(X) = \pi^* \operatorname{Pic}(E) \oplus \mathbb{Z}C_0$, while the Néron-Severi group of X is $NS(X) = \mathbb{Z}F \oplus \mathbb{Z}C_0$, where F is any fiber of X over E and the intersection pairing is $C_i^2 = 0$, $F^2 = 0$ and $C_i \cdot F = 1$ (see for instance [6]).

The degenerate Abelian surface Y is obtained by identifying each point $x \in C_0$ with the point $x + p_0 \in C_1$ for some parameter $p_0 \in E$.

Let ν be the desingularization map of Y: $\nu : X \to Y$. Let $\mathcal{L} \in \operatorname{Pic}(Y)$ and $\mathcal{L}' := \nu^*\mathcal{L}$. Then $\mathcal{L}'$ is numerically equivalent to $aC_0 + bF$, for some a and b. We want $\mathcal{L}$ to represent a principal polarization: since on an Abelian surface a principal polarization has self-intersection 2, and the self-intersection does not change in a flat family, we require $\mathcal{L}^2 = 2$, which pulling back to the normalization, implies $\mathcal{L}'^2 = 2$. Thus we get $(aC_0 + bF)^2 = 2$, which implies $a = b = 1$. This means that $\mathcal{L}' = \mathcal{O}_X(C_0 + F)$, for some $F = \pi^{-1}(p)$, $p \in E$, or equivalently $\mathcal{L}' = \mathcal{O}_X(C_0) \otimes \pi^*\mathcal{M}$, $\mathcal{M} \in \operatorname{Pic}^1(E)$, $\mathcal{M} = \mathcal{O}_E(p)$. Now we ask under which conditions on the glueing process, such a line bundle $\mathcal{L}'$ descends to Y, or equivalently, when there exists $\mathcal{L} \in \operatorname{Pic}(Y)$ such that $\nu^*\mathcal{L} = \mathcal{L}'$. The answer is given by the following:

Lemma 2.1. *Let $\mathcal{L}'$, $X = \mathbb{P}(\mathcal{O}_E \oplus \mathcal{O}_E(e - o))$ and Y as above; then $\mathcal{L}'$ descends to Y if and only if $e + p_0 \sim o$ (where $\sim$ stands for linear equivalence and p_0 is the glueing* *parameter described above).*

Proof. Let $\varphi : C_0 \xrightarrow{\approx} C_1$ be the isomorphism between the two sections, given by $x \mapsto x + p_0$. Then it is clear that $\mathcal{L}'$ descends to Y iff

$$\varphi^* \mathcal{L}'_{|C_1} = \mathcal{L}'_{|C_0}. \tag{1}$$

$\mathcal{L}'_{|C_0} = \mathcal{O}_X(C_0 + F) \otimes \mathcal{O}_{C_0}$ and this is equal to $\mathcal{N}_{C_0/X}(p)$, where $p = C_0 \cap F$. Since $C_0 \cong E \cong C_1$, and $\mathcal{N}_{C_0/X} = \mathcal{O}_{C_0}(C_0) = \mathcal{O}_E(e-o)$, we get that $\mathcal{L}'_{|C_0} = \mathcal{O}_E(e-o+p)$, where we identify C_0 with E and the point $p = C_0 \cap F$ with its projection to E. Analogously, $\mathcal{L}'_{|C_1} = \mathcal{O}_X(C_0+F) \otimes \mathcal{O}_{C_1}$ which is equal to $\mathcal{O}_{C_1}(p')$ for $p' = C_1 \cap F$ since $\mathcal{O}_{C_1}(C_0) = \mathcal{O}_{C_1}$ (due to the fact that $C_1 \cap C_0 = \emptyset$). Then $\mathcal{O}_{C_1}(p') \cong \mathcal{O}_E(p)$, (since if π is the projection to E, then $\pi(p) = \pi(p')$) so that we can translate condition (1) as a relation on line bundles on E: $\mathcal{O}_E(p - p_0) = \mathcal{O}_E(e - o + p)$, which is equivalent to $e + p_0 \sim 0$. □

Fixed e, that is fixed X, from now on we assume that the parameter p_0 has been chosen in order to satisfy the condition $e+p_0 \sim o$. Under this assumption, not only $\mathcal{L}'$, but also all line bundles of the form $\mathcal{O}_X(nC_0+nF)$ descend to Y. Let us call $\mathcal{O}_Y(1)$ the line bundle $\mathcal{L}$ on Y such that $\nu^*\mathcal{L} = \mathcal{O}_X(C_0 + F)$ so that $\nu^*\mathcal{O}_Y(n) = \mathcal{O}_X(nC_0 + nF)$. Then, by the results of Alexeev and Nakamura ([2, Theorem 4.7]), it turns out that the map on Y associated to the line bundle $\mathcal{O}_Y(5)$ gives an embedding. We have:

Proposition 2.2. *Let Y and $\mathcal{O}_Y(n)$ as above. If $n \geq 0$ and $k \geq 0$, then we have that $h^0(X, \mathcal{O}_X(nF + kC_0)) = (k+1)n$ and $h^0(Y, \mathcal{O}_Y(n)) = n^2$.*

Proof. This is just a particular case of Theorem 4.3 and Theorem 4.4, in [2]. □

Collecting the results of Alexeev and Nakamura ([2, Theorem 4.7]) and the previous proposition, we have that $|\mathcal{O}_Y(5)|$ gives an embedding of Y as a linearly normal surface in $\mathbb{P}^{24}$.

Theorem 2.3. *Let Y and $\mathcal{O}_Y(n)$ as above. Then the complete linear system $|\mathcal{O}_Y(3)|$ is base-point free and the associate morphism $\phi_{|\mathcal{O}_Y(3)|} : Y \hookrightarrow \mathbb{P}^8$ is an embedding.*

Proof. Assume $e \neq o$. Indeed, if $e = 0$, then $Y = E \times C$ and the proof is immediate (see Remark 2.4, at the end of the proof).

First of all we prove that $|\mathcal{O}_Y(3)|$ has no fixed component. Assume the contrary, and let K be an irreducible component of the 1-dimensional fixed locus. Now observe that any curve on the smooth model which is not equal to C_1, intersects C_0. Then K corresponds to a locus on the smooth model, which will intersect C_0 or C_1. Thus, there exists always an $x \in K \cap C_0$ (or C_1 which is identified to C_0 on Y).

Now the restriction morphism $H^0(Y, \mathcal{O}_Y(3)) \to H^0(C_0, \mathcal{O}_{C_0}(3o))$ is surjective. Indeed, using the exact sequence

$$0 \to \nu_* \mathcal{O}_X(-C_0 - C_1) \to \mathcal{O}_Y \to \mathcal{O}_C \to 0,$$

twisting by $\mathcal{O}_Y(3)$ and taking cohomology, it is sufficient to prove that the group $H^1(Y, \nu_*\mathcal{O}_X(-C_0 - C_1) \otimes \mathcal{O}_Y(3))$ vanishes. By Leray spectral sequence, this cohomology group is equal to $H^1(X, \mathcal{O}_X(3C_0 + 3F_o - C_0 - C_1))$. Using the fact that $C_1 \sim C_0 + (F_o - F_e)$ and $K_X \sim -2C_0 + F_e - F_o$, where K_X is the canonical class, this cohomology group is equal to $H^1(X, \mathcal{O}_X(3C_0 + 3F_o) \otimes \mathcal{O}(K_X))$, which is zero, by Kawamata-Viehweg vanishing.

Thus that there exists a $t \in H^0(C_0, \mathcal{O}_{C_0}(3o))$ such that $t(x) \neq 0$ (since the complete linear system $|\mathcal{O}_{C_0}(3o)|$ embeds C_0 in $\mathbb{P}^2$ and we can always find a hyperplane section of this embedded curve which does not hit the point x); this implies that there exists $s \in H^0(Y, \mathcal{O}_Y(3))$, such that $s(x) \neq 0$ so that x is not a point in the fixed component: this is a contradiction. Thus $|\mathcal{O}_Y(3)|$ has no fixed component and $Bs|\mathcal{O}_Y(3)|$ is at most a *finite set* of points.

Let $s \in H^0(Y, \mathcal{O}_Y(1))$ and consider $D := (s)_0$, the zero scheme of s (clearly $Bs|\mathcal{O}_Y(3)| \subset D$). Since we assume $e \neq o$, D is irreducible and its pull-back to X is numerically equivalent to $C_0 + F$. Now $s^{\otimes 3} \in H^0(Y, \mathcal{O}_Y(3))$ and $red[(s^{\otimes 3})_0] = D$ (where red is the reduced scheme structure on the zero scheme $(s^{\otimes 3})_0$). Since $|\mathcal{O}_Y(3)|$ has no fixed-component, we can always choose $s' \in H^0(Y, \mathcal{O}_Y(3))$ such that $s'_{|D} \neq 0$.

Let $D' = (s')_0$, then the pull-back of D' to X is numerically equivalent to $3C_0+3F$, so that $D \cdot D' = (C_0 + F) \cdot (3C_0 + 3F) = 6$. By this computation we have that $Bs|\mathcal{O}_Y(3)| \subset (s')_0 \cdot (s^{\otimes 3})_0$, which consists of (at most) 6 *distinct* points, possibly with multiplicities. On the other hand, let us consider the subgroup $G^{(3)}$ of 3-torsion points in the semi-Abelian group variety G, where $G \simeq X - (C_0 \cup C_1)$ and G is a $\mathbb{C}^*$ extension of E:

$$1 \to \mathbb{C}^* \to G \to E \to 0.$$

There is a natural action of $G^{(3)}$ on the base locus and a natural action of its $\mathbb{C}^*$-extension on $\mathcal{O}_Y(3)$. In view of this action, it turns out that if $|\mathcal{O}_Y(3)|$ has one fixed point (possibly with multiplicities), then it has to have at least 27 *distinct* base-points (possibly with multiplicities) since $|G^{(3)}| = 27$. On the other hand, by the previous computation $Bs|\mathcal{O}_Y(3)|$ has at most 6 distinct points. Contradiction. Thus $|\mathcal{O}_Y(3)|$ is base-point free and defines a morphism of Y to $\mathbb{P}^8$.

Now we have to prove that $|\mathcal{O}_Y(3)|$ separates points. We have to deal with different cases.

First case: the inverse images P_1 and P_2 of the two distinct points $x, y \in Y$ on the smooth model X belong to distinct fibers and $x, y \notin \mathrm{Sing}(Y)$; then consider the sections $s \in H^0(X, \mathcal{O}_X(3C_0 + 3F_o))$ such that $s_{|C_0} = s_{|C_1} = 0$, which certainly descend onto Y. These sections are in one to one correspondence with the sections of $\mathcal{O}_X(3C_0 + 3F_o - C_0 - C_1)$. Since we have $C_1 \sim C_0 + (F_o - F_e)$, we have that $\mathcal{O}_X(3C_0 + 3F_o - C_0 - C_1) = \mathcal{O}_X(C_0) \otimes \pi^*\mathcal{O}_E(B)$, where B is a divisor of degree 3 on E. Since we can always find on E a divisor of degree 3, containing $\pi(P_1)$, but not $\pi(P_2)$ (recall that P_1 and P_2 belong to different fibers), it is always possible to find out a section of $\mathcal{O}_X(3C_0 + 3F_o - C_0 - C_1)$, which vanishes on P_1, but not on P_2 and consequently a section of $\mathcal{O}_X(3C_0 + 3F_o)$, which descends to Y and which separates x, y.

Second case: the two distinct points $x, y \in Y$ are on the glued sections C_0, C_1. In this case, since the restriction morphism $H^0(Y, \mathcal{O}_Y(3)) \to H^0(C_0, \mathcal{O}_{C_0}(3))$ is surjective by construction, and the complete linear system $|\mathcal{O}_{C_0}(3)|$ gives an embedding of C_0 into $\mathbb{P}^2$, we are done immediately.

Third case: the point $x \in C_0$, $y \notin C_0$. In this case we can do as in the first case or just observe that there are certainly sections which vanish on x (those sections which on the smooth model vanish on C_0 and C_1), but not on y.

Fourth case: the inverse images P_1 and P_2 of the two distinct points $x, y \in Y$ on the smooth model X belong to the same fiber F_p and are outside the two sections C_0 and C_1. In this case, consider the sections of $\mathcal{O}_X(C_0 + 2F_o + F_e)$, which are in one to one correspondence with the sections of $\mathcal{O}_X(3C_0 + 3F_o)$ which vanish on C_0 and C_1, that is those sections which descend automatically onto Y. We have the exact sequence

$$0 \to \mathcal{O}_X(C_0 + 2F_o + F_e - F_p) \to \mathcal{O}_X(C_0 + 2F_o + F_e) \to \mathcal{O}_{F_p}(1) \to 0 \qquad (2)$$

twisting the defining sequence of F_p, since $(C_0 + 2F_o + F_e) \cdot F_p = 1$ and $F_p \cong \mathbb{P}^1$. Now, $H^1(X, \mathcal{O}_X(C_0 + 2F_o + F_e - F_p)) = H^1(E, \pi_*(\mathcal{O}_X(C_0 + 2F_o + F_e - F_p))) = 0$, so that taking the long exact cohomology sequence induced from (2), we get that the restriction morphism $H^0(X, \mathcal{O}_X(C_0 + 2F_o + F_e)) \to H^0(F_p, \mathcal{O}_{F_p}(1))$ is surjective. Then we can always find a section s of $\mathcal{O}_{F_p}(1)$ which vanishes on P_1, but not on P_2; we lift s to a section of $\mathcal{O}_X(C_0 + 2F_o + F_e)$, which corresponds to a section t of $\mathcal{O}_X(3C_0 + 3F_o)$ vanishing on C_0 and C_1; this section descends to a section of $\mathcal{O}_Y(3)$ and vanishes on x, but not on y. Thus, the linear system $|\mathcal{O}_Y(3)|$ separates points also in this case. So we have proved that the map $\phi_{|\mathcal{O}_Y(3)|} : Y \to \mathbb{P}^8$ is injective, since there are clearly no other cases for the relative position of the points x and y.

$|\mathcal{O}_Y(3)|$ separates tangent directions: to prove this we distinguish two different cases: $p \in Y$ is a smooth point (first case), or $p \in Y$ belongs to the the image of C_0 and so it is singular (second case).

First case: let $v \in T_pY \cong \mathbb{A}^2$. To prove that $|\mathcal{O}_Y(3)|$ separates tangent directions it is sufficient to find out a curve $C' \in |\mathcal{O}_Y(3)|$, passing through p and smooth at p such that $T_pC' \neq v$. If $v \neq T_pF$, then we consider any smooth curve C'' on X, inside the linear equivalence class of $2C_0 - C_1 + F_1 + F_2 + F_3$, where $p \in F_1$, $p \notin F_2$, $p \notin F_3$ and such that the fibers F_1, F_2 and F_3 are arranged so that $2C_0 - C_1 + F_1 + F_2 + F_3 \sim 2C_0 - C_1 + 3F_o$. In this case, $|2C_0 - C_1 + F_1 + F_2 + F_3|$ can be viewed as a subsystem of $|\mathcal{O}_X(3)|$, corresponding to sections vanishing on C_0 and C_1. All these sections clearly descend to Y and correspondingly any curve $C'' \in |2C_0 - C_1 + F_1 + F_2 + F_3|$. Thus it is sufficient to set $C' := \nu(C'')$, where $\nu : X \to Y$ is the desingularization map.

If, instead, $v = T_pF$, it is sufficient to prove that the morphism $H^0(Y, \mathcal{O}_Y(3)) \to H^0(F, \mathcal{O}_F(3))$ is surjective, since the complete linear system $|\mathcal{O}_F(3)|$ defines an embedding of $\mathbb{P}^1$ as a twisted cubic. Assume that this is not the case; then, since we already know that the map $\phi_{|\mathcal{O}_Y(3)|}$ is injective, it means that the image in $\mathbb{P}^8$ of F

is a plane rational curve, having at most a cusp as a singularity. Since the image is a plane curve, then there exists a unique plane $V \cong \mathbb{P}^2$ containing it. Now consider all hyperplanes of $\mathbb{P}^8$ containing this V: by a standard argument, they are parameterized by a $\mathbb{P}^5$. Observe that the rational curve intersects the image Z of the singular locus of Y (an elliptic curve) in two distinct points (because $p_0 \neq o$), which are obviously contained in V.

Let x and y be these two points and fix another point z on the image of the elliptic curve Z in $\mathbb{P}^8$ so that $x + y + z$ is not linearly equivalent to a hyperplane section of Z. Then, all hyperplanes containing V and z have also to contain Z; these hyperplanes are parameterized by a $\mathbb{P}^4$, but their pull-back to X cut out a divisor D such that $3C_0+3F-D$ is numerically equivalent to C_0+2F. Now, by Proposition 2.2, $h^0(\mathcal{O}_X(C_0 + 2F)) = 4$, which implies that these hyperplanes should span a $\mathbb{P}^3$, not a $\mathbb{P}^4$. Contradiction. This happens because we have assumed that the image of F is a *plane* rational curve. Thus $\phi_{|\mathcal{O}_Y(3)|}$ separates tangent directions also in this case.

Second case: the point p belongs to the singular locus $\mathrm{Sing}(Y)$ of Y and clearly $T_pY \cong \mathbb{A}^3$. In this case, to prove that $|\mathcal{O}_Y(3)|$ separates tangent directions it is sufficient to prove that the image of T_pY in $\mathbb{P}^8$ is 3-dimensional. Assume that the image is not 3-dimensional; then its image in $\mathbb{P}^8$ is at most a $V \cong \mathbb{P}^2$. Then look at the hyperplanes of $\mathbb{P}^8$ containing this V; by a standard argument, they are parameterized by a $\mathbb{P}^5$. The pull-back of any of these hyperplanes to the smooth model X determines a divisor on X having multiplicity 2 at the point x_0 and x_1, where $x_0 \in C_0$ and $x_1 \in C_1$ (the two points x and y are just the preimages in X of the point $p \in \mathrm{Sing}(Y)$). Then choose an other point q on the image of the singular locus Z of Y (the image of $\mathrm{Sing}(Y)$ in $\mathbb{P}^8$ is just a plane elliptic curve, since $H^0(Y, \mathcal{O}_Y(3)) \to H^0(C_0, \mathcal{O}_{C_0}(3))$ is surjective). Choose q such that $2p + q$ is not linearly equivalent to a hyperplane section of Z, the image of $\mathrm{Sing}(Y)$ in $\mathbb{P}^8$.

Then we obtain a $\mathbb{P}^4$ of hyperplanes, containing V and q. Since we have chosen q in this way, it turns out these hyperplanes have to contain Z, hence on the smooth model they cut out a divisor containing C_0 and C_1 and the points x_0 and x_1 with multiplicity 2. Finally, choose on the smooth model X two other points: y_0 on the fiber passing through x_0 and y_1 on that passing through x_1. On the hyperplanes of $\mathbb{P}^8$ satisfying the previous conditions, impose also to pass through the images of y_0 and y_1: in this way we get a $\mathbb{P}^2$ of these hyperplanes. The pull-back of any of these hyperplanes to the smooth model X cut out C_0, C_1 and two fibers F_1 and F_2 and the remaining divisor in $3C_0+3F$ is numerically equivalent to C_0+F. On the other hand, by Proposition 2.2, $h^0(X, \mathcal{O}_X(C_0+F)) = 2$ and this is true if replace C_0+F with any other divisor numerically equivalent to it. This is a contradiction, because we have a $\mathbb{P}^2$ of these hyperplanes, while $|\mathcal{O}_X(C_0 + F)| = \mathbb{P}^1$. The contradiction arises from the fact that we have assumed that the image of T_pY in $\mathbb{P}^8$ is at most 2-dimensional.

Thus, we have proved that $\phi_{|\mathcal{O}_Y(3)|} : Y \hookrightarrow \mathbb{P}^8$ is an embedding. □

Remark 2.4. In the proof of Theorem 2.3, we have assumed that $e \neq o$ (and consequently $p_0 \neq o$). Indeed, if $e = o$, by Lemma 2.1 we take $p_0 = o$ and in this case

$Y = E \times C$, where E is an elliptic curve and C is a nodal cubic curve. Then we get $E \times C \hookrightarrow \mathbb{P}^2 \times \mathbb{P}^2 \hookrightarrow \mathbb{P}^8$, where the last embedding is a Segre map. Since $h^0(Y, \mathcal{O}_Y(3))$ is independent of the parameter e (hence p_0), it turns out that for *generic* e the map $\phi_{|\mathcal{O}_Y(3)|}$ gives an embedding. However, to prove this for any e we have to do as above.

Remark 2.5. The image of Y in $\mathbb{P}^8$ is a linearly normal surface. Here, by linearly normal surface we just mean that the linear system used to embed Y in $\mathbb{P}^8$ is complete: so Y can not be obtained as a projection from a surface Z isomorphic to Y, sitting in a higher dimensional space. In principle, however, Y can be obtained as a projection from a less singular surface $\tilde{Z}$, which is no more isomorphic to Y.

Assume $e \neq o$, then its singular locus is a smooth plane elliptic curve Z and through each point of this elliptic curve there are two twisted cubic curve, intersecting transversally each other and also transversally with Z. The degree is 18, since $(3C_0 + 3F)^2 = 18$.

3. Very ampleness on the second type of degeneration

The second type of degeneration of smooth principally polarized Abelian surfaces we are going to consider is obtained by a smooth quadric $X = \mathbb{P}^1 \times \mathbb{P}^1$ (with homogeneous coordinates $[x_0, x_1] \times [y_0, y_1]$), identifying the points of coordinates $[x_0, x_1] \times [1, 0]$ with those of coordinates $[x_0, x_1 T] \times [0, T]$ and the points of coordinates $[1, 0] \times [y_0, y_1]$ with those of coordinates $[0, T] \times [y_0, y_1 T]$ for some parameter $T \in \mathbb{C}^*$; in particular the points corresponding to coordinates $[1, 0] \times [1, 0]$, $[1, 0] \times [0, T]$, $[0, T] \times [0, T]$ and $[0, T] \times [1, 0]$ are all identified.

Observe that this type of degeneration depends on 1 moduli, namely the glueing parameter T.

Let us call Y (strictly speaking Y_T, since it depends on the parameter T) the image of X under these identifications, $\pi : X \to Y$. Y is one of the degeneration type of smooth principally polarized Abelian surfaces represented by a point in the boundary of $\mathcal{A}_2^{\mathrm{Vor}}$.

Recall that $\mathrm{Pic}(X) = \mathbb{Z}L_1 \oplus \mathbb{Z}L_2$, where L_1 and L_2 generate the two rulings on X, while the intersection pairing is $L_i^2 = 0$, $L_1 \cdot L_2 = 1$. Recall also that the self-intersection of a principal polarization on a smooth Abelian surface is 2 and that self-intersection does not change in a flat family. Having recalled this, it is natural to consider as a degenerate principal polarization on Y a line bundle $\mathcal{L}$ such that $\pi^* \mathcal{L} = \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1, 1)$ (simply because the corresponding divisor class is of the form $L_1 + L_2$ and $(L_1 + L_2)^2 = 2$). In this light, proving a sort of Lefschetz theorem for this type of degeneration is equivalent to prove the following:

Theorem 3.1. *Let $\mathcal{L}$ be a line bundle on Y such that $\pi^* \mathcal{L} = \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(3, 3)$. Then the complete linear system $|\mathcal{L}|$ is base-point free and the corresponding map $\phi_{|\mathcal{L}|} : Y \hookrightarrow \mathbb{P}^8$ defines an embedding of the singular model Y.*

Proof. First of all, we have to exhibit a basis of $H^0(Y,\mathcal{L})$, that is we have to understand which sections of $\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(3,3)$ descend to sections of $\mathcal{L}$.

Since $H^0(X,\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(3,3)) = H^0(\mathcal{O}_{\mathbb{P}^1}(3))\otimes H^0(\mathcal{O}_{\mathbb{P}^1}(3))$ any section σ of the line bundle $\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(3,3)$ can be written as

$$\begin{aligned}\sigma = a_1x_0^3y_0^3 + a_2x_0^3y_0^2y_1 + a_3x_0^3y_0y_1^2 + a_4x_0^3y_1^3 +&\\ + a_5x_0^2x_1y_0^3 + a_6x_0^2x_1y_0^2y_1 + a_7x_0^2x_1y_0y_1^2 + a_8x_0^2x_1y_1^3 +&\\ + a_9x_0x_1^2y_0^3 + a_{10}x_0x_1^2y_0^2y_1 + a_{11}x_0x_1^2y_0y_1^2 + a_{12}x_0x_1^2y_1^3 +&\\ + a_{13}x_1^3y_0^3 + a_{14}x_1^3y_0^2y_1 + a_{15}x_1^3y_0y_1^2 + a_{16}x_1^3y_1^3.&\end{aligned}$$

The necessary and sufficient condition for a section to descend is that it satisfies some compatibility conditions under the glueing process described above. In particular, it obvious that the sections represented by $x_0^2x_1y_0^2y_1$, $x_0^2x_1y_0y_1^2$, $x_0x_1^2y_0^2y_1$ and $x_0x_1^2y_0y_1^2$ always descend since they are identically zero on the points which are going to be identified (hence $h^0(Y,\mathcal{L})\geq 4$).

The compatibility conditions for σ can be expressed as

$$\sigma|_{[1,0]\times[y_0,y_1]} = \lambda\sigma|_{[0,1]\times[y_0,Ty_1]}, \tag{3}$$

and

$$\sigma|_{[x_0,x_1]\times[1,0]} = \tilde{\lambda}\sigma|_{[x_0,Tx_1]\times[0,1]}, \tag{4}$$

for *some* $\lambda,\tilde{\lambda}\in\mathbb{C}^*$. Since we have not fixed any λ, but we just say that there is some λ such that (3) holds, the equation (3) is equivalent to the vanishing of three 2×2 minors of the matrix

$$\begin{pmatrix} a_1 & a_2 & a_3 & a_4\\ a_{13} & Ta_{14} & T^2a_{15} & T^3a_{16}\end{pmatrix}.$$

This imposes 3 conditions on the coefficients a_i, while the equation (4) is equivalent (always because we have not fixed any $\tilde{\lambda}$) to the vanishing of three 2×2 minors of the matrix:

$$\begin{pmatrix} a_1 & a_5 & a_9 & a_{13}\\ a_4 & Ta_8 & T^2a_{12} & T^3a_{16}\end{pmatrix},$$

and this imposes three other conditions on the coefficients a_i, which are not all independent of the previous ones; it is immediate to check that only 2 of these conditions are independent of the previous ones, so that we get a total of 5 conditions. To these 5 conditions we have to add 2 other independent conditions determined by the glueing parameters $\lambda,\tilde{\lambda}$, so that we get $h^0(Y,\mathcal{L}) = h^0(X,\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(3,3)) - (5+2) = 16-7 = 9$. Now we determine a basis for the 9-dimensional vector space $H^0(Y,\mathcal{L})$. From the condition (3), exchanging λ^{-1} with λ, we have

$$\lambda a_1 = a_{13} \quad \lambda a_2 = Ta_{14} \quad \lambda a_3 = T^2a_{15} \quad \lambda a_4 = T^3a_{16},$$

while from condition (4), exchanging $\tilde{\lambda}^{-1}$ with $\tilde{\lambda}$, we get

$$\tilde{\lambda}a_1 = a_4 \quad \tilde{\lambda}a_5 = Ta_8 \quad \tilde{\lambda}a_9 = T^2 a_{12} \quad \tilde{\lambda}a_{13} = T^3 a_{16},$$

so that we can choose $(a_1, a_2, a_3, a_5, a_6, a_7, a_9, a_{10}, a_{11})$ as coordinates on $H^0(Y, \mathcal{L})$, since, by the relations just described above, we have

$$a_{16} = \frac{\lambda\tilde{\lambda}}{T^3}a_1 \quad a_{15} = \frac{\lambda}{T^2}a_3 \quad a_{14} = \frac{\lambda}{T}a_2 \quad a_{13} = \lambda a_1,$$
$$a_{12} = \frac{\tilde{\lambda}}{T^2}a_9 \quad a_8 = \frac{\tilde{\lambda}}{T}a_5 \quad a_4 = \tilde{\lambda}a_1.$$

To write down an explicit basis of $H^0(Y, \mathcal{L})$ it is sufficient to substitute iteratively $(a_1, a_2, a_3, a_5, a_6, a_7, a_9, a_{10}, a_{11})$ equal to $(1, 0, \dots, 0)$, $(0, 1, \dots, 0)$,..., $(0, \dots, 1)$ in the expression of the general section σ, taking into account the relations which define a_{16}, a_{15}, a_{14}, a_{13}, a_{12}, a_8 and a_4 in terms of the other a_i' s. Completing the computation, we obtain as a basis of $H^0(Y, \mathcal{L})$:

$$x_0^3y_0^3 + \tilde{\lambda}x_0^3y_1^3 + \lambda x_1^3y_0^3 + \frac{\lambda\tilde{\lambda}}{T^3}x_1^3y_1^3, \quad (x_0^3 + \frac{\lambda}{T}x_1^3)y_0^2y_1, \quad (x_0^3 + \frac{\lambda}{T^2}x_1^3)y_0y_1^2,$$
$$x_0^2x_1(y_0^3 + \frac{\tilde{\lambda}}{T}y_1^3), \quad x_0^2x_1y_0^2y_1, \quad x_0^2x_1y_0y_1^2, \quad x_0x_1^2(y_0^3 + \frac{\tilde{\lambda}}{T^2}y_1^3),$$
$$x_0x_1^2y_0^2y_1, \quad x_0x_1^2y_0y_1^2.$$

Different choices of λ and $\tilde{\lambda}$ do not lead to the same line bundle. Indeed, these are the two parameters in $\mathrm{Pic}^0(Y) \cong (\mathbb{C}^*)^2$ and different choices leads to different line bundles. On the other hand, if we map x to λx and y to $\tilde{\lambda}y$, this defines an automorphism of the singular variety (recall that the torus $T = (\mathbb{C}^*)^2$ acts on Y and this is exactly that action). Now pulling back via this automorphism, identifies the line bundle given by λ, $\tilde{\lambda}$, with that given by $(1, 1)$. Thus, by acting with this automorphism, we may indeed assume $\lambda = \tilde{\lambda} = 1$. So from now on, we fix $\lambda = \tilde{\lambda} = 1$.

Hence we get a rational map $\phi_{|\mathcal{L}|} : Y \dashrightarrow \mathbb{P}^8$; now we prove that $|\mathcal{L}|$ is base-point free. To this aim, suppose that we have a point P of Y such that its image under the complete linear system $|\mathcal{L}|$ corresponds to the origin $(0, \dots, 0)$ of $H^0(Y, \mathcal{L})$ (which is not a point of the corresponding projective space). In particular this implies that $x_0^2x_1y_0^2y_1 = 0$, that is at least one of the coordinates is zero. For instance, let us assume that $y_0 = 0$; then substituting in the fixed basis for $H^0(Y, \mathcal{L})$, we obtain: $x_0^2x_1y_1^3 = 0$, $x_0x_1^2y_1^3 = 0$ and $x_0^3y_1^3 + \frac{1}{T^3}x_1^3y_1^3 = 0$. From the first two equalities, we get that either $y_1 = 0$ (but this is not possible, since $[y_0, y_1] = [0, 0]$ is not a point of the projective line), or $x_0 = 0$ or $x_1 = 0$. If $x_0 = 0$, from the last equality we have $x_1 = 0$ (but this again impossible since $[x_0, x_1] = [0, 0]$ is not a point of the projective line), while if $x_1 = 0$, then $x_0 = 0$ or $y_1 = 0$, and we conclude as before. Hence we have a *morphism* $\phi_{|\mathcal{L}|} : Y \to \mathbb{P}^8$.

To conclude we have to prove that $\phi_{|\mathcal{L}|}$ separates points and tangent lines. On the singular model Y we can distinguish three types of points: smooth points of Y and which have coordinates of the form $[1,\alpha]\times[1,\beta]$ for $\alpha,\beta\neq 0$ (first type) (these are the points which are not glued); those for which only one of the coordinates $[x_0,x_1]\times[y_0,y_1]$ is zero (second type); and finally those points for which two of the coordinates $[x_0,x_1]\times[y_0,y_1]$ are zero (third type), (actually, on Y there is just one point of the third type, which comes from the identification of 4 points on the smooth model). Let us consider the image under $\phi_{|\mathcal{L}|}$ of a point P of the first type of the form $[1,\alpha]\times[1,\beta]$. The sections $x_0^2x_1y_0^2y_1$ and $x_0x_1^2y_0^2y_1$ are never vanishing on the points of this form; moreover, from their ratio one gets immediately the homogeneous coordinates $[x_0,x_1]$; a completely analogous reasoning, using now the sections $x_0^2x_1y_0y_1^2$ and $x_0^2x_1y_0^2y_1$ gives the homogeneous coordinates $[y_0,y_1]$. This means that $\phi_{|\mathcal{L}|}^{-1}(\phi_{|\mathcal{L}|}(P))=P$, for a point of the first type.

Consider now a point P of the second type, for instance of the form $[1,\alpha]\times[1,0]$. Its image under $\phi_{|\mathcal{L}|}$ is given by $[1+\alpha^3,0,0,\alpha,0,0,\alpha^2,0,0]=Q\in\mathbb{P}^8$ and we have to prove that $\phi_{|\mathcal{L}|}^{-1}(Q)$ consists of two points on the smooth model X which are on the edges and which are going to be identified to a unique point on the singular model Y. Since $\alpha\neq 0$, from the expression of the coordinates of Q we get that either $y_0=0$ or $y_1=0$. If $y_0=0$, from the expression $x_0^2x_1\frac{1}{T}y_1^3=\alpha$, $x_0x_1^2\frac{1}{T^2}y_1^3=\alpha^2$, taking their ratios we have $\frac{x_1}{x_0}=\alpha T$, so that we obtain the point $P_2=[1,\alpha T]\times[0,1]$. If instead $y_1=0$, from the relations $x_0^2x_1y_0^3=\alpha$, $x_0x_1^2y_0^3=\alpha^2$ we are led to the point $P_1=P=[1,\alpha]\times[1,0]$. Now the points P_1 and P_2 are distinct on the model X, but they are identified under the glueing process, so that $\phi_{|\mathcal{L}|}^{-1}(\phi_{|\mathcal{L}|}(P))=P$, also for a point P of the second type.

Finally, as for the points of third type, we can consider $P=P_1=[1,0]\times[1,0]$ and its image $Q=[1,0,\dots,0]$ under $\phi_{|\mathcal{L}|}$. Now, $\phi_{|\mathcal{L}|}^{-1}(Q)$ can be computed immediately, since from the expression of the coordinates of Q we obtain the following four possibilities ($y_0=0$, $x_0=0$), ($y_0=0$, $x_1=0$), ($y_1=0$, $x_0=0$), and ($y_1=0$, $x_1=0$): again these four points are distinct on X, but are identified to a unique point on Y. This proves that $\phi_{|\mathcal{L}|}$ separates points on Y.

Now we prove that $\phi_{|\mathcal{L}|}$ separates tangent directions. First of all we prove that $d\phi_{|\mathcal{L}|_P}: T_PY\to T_P\mathbb{P}^8$ is injective for any point P of the first type. So $P=[1,\alpha]\times[1,\beta]$; since the question about the injectivity of $d\phi_{|\mathcal{L}|}$ is local, we can substitute $x_0=1$, $y_0=1$ in the expression of the sections we have fixed as a basis of $H^0(Y,\mathcal{L})$ (this is equivalent to consider x_1, y_1 al local affine coordinates such that $(x_1,y_1)(P)=(\alpha,\beta)$). The fifth basis vector has then the form x_1y_1 and is never vanishing for a point of the first type since $\alpha,\beta\neq 0$. Thus we can divide all the other sections by x_1y_1 obtaining a map to $\mathbb{A}^8$: $\rho:=\phi_{|\mathcal{L}|}|_U: U\to\mathbb{A}^8$ (where U is an open neighborhood of P), given by

$$\begin{aligned}(x_1,y_1)\mapsto\Big(&\frac{1+y_1^3+x_1^3+\frac{1}{T^3}x_1^3y_1^3}{x_1y_1},\quad \frac{1}{x_1}\Big(1+\frac{1}{T}x_1^3\Big),\quad \frac{y_1}{x_1}\Big(1+\frac{1}{T^2}x_1^3\Big),\\ &\frac{1}{y_1}\Big(1+\frac{1}{T}y_1^3\Big),\quad y_1,\quad \frac{x_1}{y_1}\Big(1+\frac{1}{T^2}y_1^3\Big),\quad x_1,\quad x_1y_1\Big).\end{aligned}$$

To check the injectivity of $d\phi_{|\mathcal{L}|}$ at the points of the first type, it is sufficient to compute the Jacobian matrix of the map ρ (with respect to x_1 and y_1) and evaluate it at (α, β) proving that it has rank 2. To prove that it has rank 2, just look at the form of the Jacobian matrix

$$\begin{pmatrix} - & - \\ - & - \\ - & - \\ - & - \\ 0 & 1 \\ - & - \\ 1 & 0 \\ - & - \end{pmatrix}.$$

The first column compute the derivative with respect to x_1 of the map ρ, the second column compute the derivative with respect to y_1, and the sign $-$ means that we have skipped the computation; however, it is clear that the rank of $d\phi_{|\mathcal{L}|}$ is 2, so that $\phi_{|\mathcal{L}|}$ separates tangent directions for the points of the first type.

To check the injectivity of the differential for points of the second and third type, we have to understand the singularities of Y on these kinds of points. From the toric description of Y and the corresponding Delaunay decomposition, we have that $T_P Y \cong \mathbb{A}^3$ if P is a point of second type, while $T_P Y \cong \mathbb{A}^4$ if P is of the third type. Indeed, the Delaunay decomposition from which Y arises can be represented as four squares $S_1, \dots, S_4$ in $\mathbb{A}^2$ with vertices $[(0,0),(1,0),(1,1),(0,1)]$ for S_1, $[(0,0),(0,1),(-1,1),(-1,0)]$ for S_2, $[(0,0),(-1,0),(-1,-1),(0,-1)]$ for S_3, and $[(0,0),(0,-1),(1,-1),(1,0)]$ for S_4. They all meet in $(0,0)$ (the singular point of the third type) and their edges are pairwise identified according to the rule used to construct Y, starting from $\mathbb{P}^1 \times \mathbb{P}^1$.

Let us study for example the singularity of P, the point of third type. Consider the lattice points u_1, u_2, u_3 and u_4 each of which belongs to a different square. Since the lattice points u_1, u_2, u_3 and u_4 are not cell-mates, from the associated toric construction it turns out that we have in $\mathbb{C}[u_1, u_2, u_3, u_4]$ the relations $u_1 u_2 = 0$, $u_3 u_4 = 0$; these correspond to four 2-planes meeting in one point (the point P). These four 2-planes are $\pi_1 = \{(u_1 = 0, u_3 = 0)\}$, $\pi_2 = \{(u_1 = 0, u_4 = 0)\}$, $\pi_3 = \{(u_2 = 0, u_3 = 0\}$ and $\pi_4 = \{(u_2 = 0, u_4 = 0)\}$. Clearly the intersection of all these four planes is just the origin (the point P) and there are pairs of these planes which intersect along a line, as π_1 and π_2. Moreover, $T_P Y$ can be spanned just by a pair of planes, which intersect each other just in P, such as (π_1, π_4). These two planes can be represented in the Delaunay decomposition as two opposite squares, meeting just in the origin. This indeed proves that $T_P Y \cong \mathbb{A}^4$. A completely analogous reasoning proves that $T_P Y \cong \mathbb{A}^3$ for a point P of the second type.

To prove injectivity of $d\phi_{|\mathcal{L}|}$ for a point P of the second type, we prove that the vector space spanned by the image of the differential at the two points P_1 and P_2 (these two points are on the edges of the square and are identified to the unique point P on Y) is at least 3-dimensional.

Let us consider as $P_1 = [1, \alpha] \times [1, 0]$ and $P_2 = [1, T\alpha] \times [0, T]$ which correspond to the unique (singular) point P on Y. As before, on the sections forming a basis of $H^0(Y, \mathcal{L})$, we substitute $x_0 = 1$, $y_0 = 1$, and we divide by the fourth basis section, which is not vanishing on P. In this way we get a map $\tilde{\rho}$ to $\mathbb{A}^8$ (centered at $(\alpha, 0)$) given by

$$(x_1, y_1) \mapsto \Bigg(\frac{1 + y_1^3 + x_1^3 + \frac{1}{T^3} x_1^3 y_1^3}{x_1(1 + \frac{1}{T} y_1^3)}, \quad \frac{(1 + \frac{1}{T} x_1^3) y_1}{x_1(1 + \frac{1}{T} y_1^3)}, \quad \frac{(1 + \frac{1}{T^2} x_1^3) y_1^2}{x_1(1 + \frac{1}{T} y_1^3)}, \\ \frac{y_1}{1 + \frac{1}{T} y_1^3}, \quad \frac{y_1^2}{1 + \frac{1}{T} y_1^3}, \quad \frac{x_1(1 + \frac{1}{T^2} y_1^3)}{1 + \frac{1}{T} y_1^3}, \quad \frac{x_1 y_1}{1 + \frac{1}{T} y_1^3}, \quad \frac{x_1 y_1^2}{1 + \frac{1}{T} y_1^3} \Bigg).$$

Computing the Jacobian matrix of the map $\tilde{\rho}$ and evaluating at $(\alpha, 0)$ we have

$$\begin{pmatrix} - & 0 \\ 0 & - \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 1 & 0 \\ 0 & - \\ 0 & 0 \end{pmatrix}.$$

Now consider the point $P_2 = [1, \alpha T] \times [0, T]$ and substitute $x_1 = \alpha T$ and $y_1 = T$ in the expression of the sections forming a basis of $H^0(Y, \mathcal{L})$. We then divide all sections again by the fourth section (which is not vanishing on P_2) and we obtain a map to $\mathbb{A}^8$ (this time centered at $(1, 0)$) given by

$$(x_0, y_0) \mapsto \Bigg(\frac{x_0^3 y_0^3 + T^3 x_0^3 + \alpha^3 T^3 y_0^3 + \alpha^3 T^3}{x_0^2 \alpha T (y_0^3 + T^2)}, \frac{y_0^2 T (x_0^3 + \alpha^3 T^2)}{x_0^2 \alpha T (y_0^3 + T^2)}, \frac{y_0 T^2 (x_0^3 + \alpha^3 T)}{x_0^2 \alpha T (y_0^3 + T^2)}, \\ \frac{y_0^2 T}{y_0^3 + T^2}, \quad \frac{y_0 \alpha T^3}{\alpha T (y_0^3 + T^2)}, \quad \frac{\alpha^2 T^2 (y_0^3 + T)}{x_0 \alpha T (y_0^3 + T^2)}, \quad \frac{y_0^2 \alpha T^2}{x_0 (y_0^3 + T^2)}, \quad \frac{y_0 \alpha T^3}{x_0 (y_0^3 + T^2)} \Bigg).$$

Now we just take the partials of this map with respect to y_0 and evaluate at $(1, 0)$ obtaining

$$\begin{pmatrix} - \\ - \\ - \\ - \\ 1 \\ - \\ - \\ - \end{pmatrix}.$$

Comparing with the image of the differential at P_1, we see that the rank of $d\phi_{|\mathcal{L}|}$ at the point of the second type is at least 3, and this is sufficient to conclude that $\phi_{|\mathcal{L}|}$ separates tangent directions for the points of second type.

As for the point of the third type, we have to check that the rank of $d\phi_{|\mathcal{L}|}$ is 4. From the toric description of the singularity around P it turns out that this can be seen as the intersection of four 2-planes, meeting in one point (the point P) and along some other lines (which corresponds to the points of the second type). In particular there are two 2-planes just meeting in P which span $T_P Y \cong \mathbb{A}^4$. So it is sufficient to check that the image of these two planes under $d\phi_{|\mathcal{L}|}$ spans again an $\mathbb{A}^4$. To this aim, we compute $d\phi_{|\mathcal{L}|}$ at $[1,0] \times [1,0]$ and at $[0,T] \times [0,T]$.

As before, for the point $P_1 = [1,0] \times [1,0]$, we substitute $x_0 = 1$, $y_0 = 1$ in the sections forming a basis of $H^0(Y, \mathcal{L})$, and we divide by the first basis section, which is not vanishing on P_1. In this way, we get a map (centered in $(0,0)$) to $\mathbb{A}^8$, given by

$$\begin{aligned}(x_1, y_1) \mapsto \Bigg(&\frac{y_1(1 + \frac{1}{T}x_1^3)}{1 + y_1^3 + x_1^3 + \frac{1}{T^3}x_1^3y_1^3}, \frac{y_1^2(1 + \frac{1}{T^2}x_1^3)}{1 + y_1^3 + x_1^3 + \frac{1}{T^3}x_1^3y_1^3}, \frac{x_1(1 + \frac{1}{T}y_1^3)}{1 + y_1^3 + x_1^3 + \frac{1}{T^3}x_1^3y_1^3},\\ &\frac{x_1y_1}{1 + y_1^3 + x_1^3 + \frac{1}{T^3}x_1^3y_1^3}, \frac{x_1y_1^2}{1 + y_1^3 + x_1^3 + \frac{1}{T^3}x_1^3y_1^3}, \frac{x_1^2(1 + \frac{1}{T^2}y_1^3)}{1 + y_1^3 + x_1^3 + \frac{1}{T^3}x_1^3y_1^3},\\ &\frac{x_1^2y_1}{1 + y_1^3 + x_1^3 + \frac{1}{T^3}x_1^3y_1^3}, \frac{x_1^2y_1^2}{1 + y_1^3 + x_1^3 + \frac{1}{T^3}x_1^3y_1^3}\Bigg).\end{aligned}$$

Taking the Jacobian matrix of this map and evaluating at $(0,0)$ we have

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Finally, to construct the corresponding map to $\mathbb{A}^8$ centered at $P_2 = [0,T] \times [0,T]$, we substitute $x_1 = T$ and $y_1 = T$ in the sections forming a basis of $H^0(Y, \mathcal{L})$ and we divide again by the first basis section (as we have done above), which is not vanishing on P_2. By so doing we get the map

$$(x_0,y_0)\mapsto\Bigg(\frac{y_0^2T(x_0^3+T^2)}{x_0^3y_0^3+x_0^3T^3+y_0^3T^3+T^3},\frac{y_0T^2(x_0^3+T)}{x_0^3y_0^3+x_0^3T^3+y_0^3T^3+T^3},\\ \frac{x_0^2T(y_0^3+T^2)}{x_0^3y_0^3+x_0^3T^3+y_0^3T^3+T^3},\frac{x_0^2y_0^2T^2}{x_0^3y_0^3+x_0^3T^3+y_0^3T^3+T^3},\\ \frac{x_0^2y_0T^3}{x_0^3y_0^3+x_0^3T^3+y_0^3T^3+T^3},\frac{x_0T^2(y_0^3+T)}{x_0^3y_0^3+x_0^3T^3+y_0^3T^3+T^3},\\ \frac{x_0y_0^2T^3}{x_0^3y_0^3+x_0^3T^3+y_0^3T^3+T^3},\frac{x_0y_0T^4}{x_0^3y_0^3+x_0^3T^3+y_0^3T^3+T^3}\Bigg).$$

Again, taking the Jacobian matrix and evaluating at $(0,0)$ we get

$$\begin{pmatrix}0&0\\0&1\\0&0\\0&0\\0&0\\1&0\\0&0\\0&0\end{pmatrix}$$

Thus, we see that the two 2-planes spanned by the image of $d\phi_{|\mathcal{L}|}$ at P_1 and P_2 are independent (in $\mathbb{A}^8$), so that the rank of $d\phi_{|\mathcal{L}|}$ at the point P of the third type on Y is 4. Then $\phi_{|\mathcal{L}|}$ separates tangent directions.

We thus have an embedding $\phi_{|\mathcal{L}|}:Y\hookrightarrow\mathbb{P}^8$. □

Remark 3.2. For the value $T=1$, the singular model Y is just the product of two nodal curves C and C'; each of these curves is embedded into $\mathbb{P}^2$ and then via a Segre map into $\mathbb{P}^8$: $C\times C'\hookrightarrow\mathbb{P}^2\times\mathbb{P}^2\hookrightarrow\mathbb{P}^8$. Since the dimension of $H^0(Y_T,\mathcal{L})$ is independent of T, this automatically implies that for *generic* T, $\phi_{|\mathcal{L}|}$ is an embedding; however to show this for any $T\in\mathbb{C}^*$ we have to give a proof as above.

4. Very ampleness on the third degeneration type and conclusion

The third degeneration type Y for smooth principally polarized Abelian surfaces is constructed via a glueing of two disjoint copies of $\mathbb{P}^2$. It does not depend on any moduli, i.e. it is rigid.

Indeed, from the toric construction associated to the corresponding Delaunay decomposition, it turns out that Y is obtained by glueing two disjoint $\mathbb{P}^2$' s along the following pairs of lines ($[x_0,x_1,x_2]$ denote homogeneous coordinates on the first $\mathbb{P}^2$,

$[y_0, y_1, y_2]$ on the second): $\{x_0 = 0\}$ and $\{y_0 = 0\}$, $\{x_1 = 0\}$ and $\{y_1 = 0\}$, $\{x_2 = 0\}$ and $\{y_2 = 0\}$ and identifying moreover the fundamental points to a unique point.

The desingularization of Y clearly consists of $X := \mathbb{P}^2 \amalg \mathbb{P}^2$, $\pi : X \to Y$. A line bundle on X is just the union of two line bundles, one on each copy of $\mathbb{P}^2$. Obviously, the divisor class group of X is generated by L_1 and L_2 (each of which is a line in $\mathbb{P}^2$, such that $L_1 \cdot L_2 = 0$). Since the self-intersection of a principal polarization on a smooth Abelian surface is 2, and this does not change in a flat family, it turns out that we can consider as a degenerate principal polarization on the smooth model X the line bundle given by $\mathcal{O}_{\mathbb{P}^2}(1)$ on each $\mathbb{P}^2$. For simplicity, let us call this bundle $\mathcal{O}_X(1;1)$, and observe that $h^0(X, \mathcal{O}_X(1;1)) = 6$. If $\mathcal{L}$ is a line bundle on Y such that $\pi^*\mathcal{L} = \mathcal{O}_X(n;n)$, then we denote $\mathcal{L}$ as $\mathcal{O}_Y(n)$.

By the results of Alexeev and Nakamura, it turns out that the complete linear system $|\mathcal{O}_Y(5)|$, gives an embedding of Y into some $\mathbb{P}^N$. This is concretely realized embedding each disjoint copy of $\mathbb{P}^2$, via $|\mathcal{O}_{\mathbb{P}^2}(5)|$, in such a way that they are glued along the prescribed lines and points. Indeed, the embedding of Y can be described by determining which sections of $\mathcal{O}_X(5;5)$ descend to Y:

$$\begin{tikzcd}
\mathbb{P}^2 \amalg \mathbb{P}^2 \arrow[rr, shift left=0.5ex] \arrow[rr, shift right=0.5ex] \arrow[dr, "\pi"'] & & \mathbb{P}^N \\
& Y \arrow[ur, "|\mathcal{O}_Y(5)|"'] &
\end{tikzcd}$$

Also in this case, to prove an analogue of Lefschetz theorem is equivalent to prove the following:

Theorem 4.1. *With the notations as above, the complete linear system $|\mathcal{O}_Y(3)|$ is base-point free and the map $\phi_{|\mathcal{O}_Y(3)|} : Y \hookrightarrow \mathbb{P}^8$ is an embedding.*

Proof. As previously noticed, the map $\phi_{|\mathcal{O}_Y(3)|}$ is induced by the sections of the line bundles $\mathcal{O}_{\mathbb{P}^2}(3)$ on each $\mathbb{P}^2$, imposing the glueing conditions. Let us call x_0, x_1, x_2 the homogeneous coordinates on the first $\mathbb{P}^2$ and y_0, y_1, y_2 those on the second. Then the general section $\sigma_x \in H^0(\mathcal{O}_{\mathbb{P}^2}(3))$ can be written as

$$\begin{aligned}\sigma_x = a_0x_0^3 + a_1x_1^3 + a_2x_2^3 + a_3x_0^2x_1 + a_4x_0^2x_2 \\ + a_5x_1^2x_0 + a_6x_1^2x_2 + a_7x_2^2x_0 + a_8x_2^2x_1 + a_9x_0x_1x_2,\end{aligned} \tag{5}$$

and analogously for

$$\sigma_y = b_0y_0^3 + b_1y_1^3 + b_2y_2^3 + b_3y_0^2y_1b_4y_0^2y_2 + b_5y_1^2y_0 + b_6y_1^2y_2 + b_7y_2^2y_0 + b_8y_2^2y_1 + b_9y_0y_1y_2.$$

In order to determine which sections descend onto Y, we express the glueing conditions as

$$\sigma_x|_{\{x_0=0\}} = \lambda\sigma_y|_{\{y_0=0\}}, \tag{6}$$
$$\sigma_x|_{\{x_1=0\}} = \tilde{\lambda}\sigma_y|_{\{y_1=0\}}, \tag{7}$$
$$\sigma_x|_{\{x_2=0\}} = \hat{\lambda}\sigma_y|_{\{y_2=0\}}, \tag{8}$$
$$\sigma_x|_{[1,0,0]} = \mu\sigma_x|_{[0,1,0]}, \tag{9}$$
$$\sigma_x|_{[1,0,0]} = \nu\sigma_x|_{[0,0,1]}, \tag{10}$$

for *some* (not fixed!) parameters $\lambda, \tilde{\lambda}, \hat{\lambda}, \mu, \nu \in \mathbb{C}^*$. The equations (6), (7), and (8), express the glueing conditions for the coordinate lines, while the remaining equations express the fact that the coordinate points have to be identified to a unique point.

From the first three equations we get $\sigma_x|_{[1,0,0]} = \tilde{\lambda}\sigma_y|_{[1,0,0]} = \hat{\lambda}\sigma_y|_{[1,0,0]}$ and $\sigma_x|_{[0,1,0]} = \lambda\sigma_y|_{[0,1,0]} = \hat{\lambda}\sigma_y|_{[0,1,0]}$; from these relations we get $\lambda = \tilde{\lambda} = \hat{\lambda}$ and $b_0 = \lambda a_0$, $b_1 = \lambda a_1$ and $b_2 = \lambda a_2$. On the other hand, from (9) and (10), we obtain $a_2 = \nu^{-1}a_0$ and $a_1 = \mu^{-1}a_0$. Combining these relations, we see that all coordinates a_2, a_3, b_0, b_1, b_2 are multiples of a_0. Now notice that the equations (6), (7), and (8) are determinantal and can be written as (just assuming that there are *some*, not fixed parameters $\lambda, \tilde{\lambda}, \hat{\lambda}$):

$$rk\begin{pmatrix} a_1 & a_2 & a_6 & a_8 \\ b_1 & b_2 & b_6 & b_8 \end{pmatrix} \leq 1, \tag{11}$$
$$rk\begin{pmatrix} a_0 & a_2 & a_4 & a_7 \\ b_0 & b_2 & b_4 & b_7 \end{pmatrix} \leq 1, \tag{12}$$
$$rk\begin{pmatrix} a_0 & a_1 & a_3 & a_5 \\ b_0 & b_1 & b_3 & b_5 \end{pmatrix} \leq 1. \tag{13}$$

From these and from the previous relations, we find immediately all the compatibility conditions:

$$b_0 = \lambda a_0, \quad b_1 = \lambda\mu^{-1}a_0, \quad b_2 = \lambda\nu^{-1}a_0, \quad a_2 = \nu^{-1}a_0, \quad a_1 = \mu^{-1}a_0,$$
$$b_3 = \lambda a_3, \quad b_4 = \lambda a_4, \quad b_5 = \lambda a_5, \quad b_6 = \lambda a_6, \quad b_7 = \lambda a_7, \quad b_8 = \lambda a_8.$$

Then we can choose as coordinates for determining a basis for the sections of $\mathcal{O}_Y(3)$, the coefficients $(a_0, a_3, a_4, a_5, a_6, a_7, a_8, b_9)$. Now observe that changing the parameter λ, does not change the line bundle, since the two planes are different and one can make a corresponding choice of coordinates on one of the planes, so as to cancel out the effect of changing λ. Thus we can set $\lambda = 1$. On the other hand, different choices of μ and ν lead to different line bundles; indeed, $Pic^0(Y) \cong (\mathbb{C}^*)^2$ and the parameters μ and ν are coordinates on $Pic^0(Y)$. But again modulo the action of the torus $T = (\mathbb{C}^*)^2$, all line bundles are the same, so one can choose one of them, e.g. by setting $\mu = \nu = 1$. Thus, from now on, we set $\lambda = \mu = \nu = 1$.

In the light of this analysis, the map $\phi_{|\mathcal{O}_Y(3)|}$ can be precisely described via a pair of maps $(\phi_x, \phi_y) : \mathbb{P}^2 \amalg \mathbb{P}^2 \rightrightarrows \mathbb{P}^8$, which is explicitly given by

$$\big[(x_0^3 + x_1^3 + x_2^3; y_0^3 + y_1^3 + y_2^3), \quad (x_0^2 x_1; y_0^2 y_1), \quad (x_1^2 x_0; y_1^2 y_0), \quad (x_0^2 x_2; y_0^2 y_2), \\ (x_1^2 x_2; y_1^2 y_2), \quad (x_2^2 x_0; y_2^2 y_0), \quad (x_2^2 x_1; y_2^2 y_1), \quad (x_0 x_1 x_2; 0), \quad (0; y_0 y_1 y_2)\big].$$

If $[z_0, z_1, \ldots, z_8]$ denotes homogeneous coordinates in $\mathbb{P}^8$, then the image of the first $\mathbb{P}^2$ is contained in the hyperplane $\{z_8 = 0\}$, while that of the second is contained in $\{z_7 = 0\}$. Now we prove that $\phi_{|\mathcal{O}_Y(3)|}$ is an embedding, checking the suitable properties on the pair of maps (ϕ_x, ϕ_y).

First of all, we check that $|\mathcal{O}_Y(3)|$ has no base points: just take into account one of the maps of the pair, for instance ϕ_x. If there is a point on Y such that all sections of $\mathcal{O}_Y(3)$ vanish, then for the corresponding point(s) in X, we have $x_0 x_1 x_2 = 0$, $x_1^2 x_0 = 0$, $x_0^2 x_2 = 0$, and $x_1^2 x_2 = 0$. Then at least two of the x_i's are zero, but since also $x_0^3 + x_1^3 + x_2^3 = 0$, then all x_i' s are zero. This is clearly impossible since this is not a point of $\mathbb{P}^2$. Thus $\phi_{|\mathcal{O}_Y(3)|} : Y \to \mathbb{P}^8$ is a morphism.

Let us distinguish, also for this type of degeneration, three kinds of points: smooth points (first type), singular points obtained by glueing pairs of points on the edges of the triangles, which are not vertices (second type), and the unique point which comes from the glueing of the vertices (third type).

Now we prove that $\phi_{|\mathcal{O}_Y(3)|}$ separates points. For the points of the first type, let us consider a point in the first copy of $\mathbb{P}^2$, of homogeneous coordinates $[1, \alpha, \beta]$, where $\alpha, \beta \in \mathbb{C}^*$. Then the image of this point under ϕ_x is given by: $[1 + \alpha^3 + \beta^3, \alpha, \alpha^2, \beta, \alpha^2\beta, \beta^2, \beta^2\alpha, \alpha\beta, 0]$. From this expression, since $\alpha\beta \neq 0$, then this point in $\mathbb{P}^8$ is never the image of a point of the second $\mathbb{P}^2$, and it is clear that one can recover the homogeneous coordinates $[1, \alpha, \beta]$, just taking ratios, so that the map $\phi_{|\mathcal{O}_Y(3)|}$ separates points of the first type.

Now consider a point of the second type, which can be represented on the smooth model X, by a pair of points of the form (for example) $[\alpha, \beta, 0], [\tilde{\alpha}, \tilde{\beta}, 0]$ such that $\alpha, \tilde{\alpha}, \beta, \tilde{\beta} \in \mathbb{C}^*$ and $\beta/\alpha = \tilde{\beta}/\tilde{\alpha}$. Rescaling the homogeneous coordinates, one can represent these as $[1, \gamma, 0], [1, \gamma, 0]$. These two points have the same image under the two maps (ϕ_x, ϕ_y): $[1 + \gamma^3, \gamma, \gamma^2, 0, 0, 0, 0, 0, 0]$. Again taking ratios one sees that this point corresponds exactly to the pair of points, which are going to be identified on Y to a point of the second type (indeed, looking at the zero entries, the point $[1 + \gamma^3, \gamma, \gamma^2, 0, 0, 0, 0, 0, 0]$ can not be the image of a smooth point or of a point on a different singular line).

Finally, the unique point of the third type is represented by three pairs of points on X. Any of these pairs has image in $\mathbb{P}^8$: $[1, 0, 0, 0, 0, 0, 0, 0, 0] = P$. Then it is clear that $(\phi_x, \phi_y)^{-1}(P)$ consists exactly of these three pairs, which correspond to the unique point of the third type on Y. Thus $\phi_{|\mathcal{O}_Y(3)|}$ is injective on Y.

To conclude the proof it remains to show that this map separates tangent directions. If Q is a point of the first type on Y, then it corresponds to a unique point on one of the two copies of $\mathbb{P}^2$. It is not restrictive to assume that the homogeneous coordinates of this point belong to the first $\mathbb{P}^2$ and are of the form $[1,\alpha,\beta]$ $(\alpha,\beta\in\mathbb{C}^*)$. Since $T_QY\cong\mathbb{A}^2$ for this type of points, it is sufficient to prove that the rank of $d\phi_x$ is 2 in a neighborhood of $\pi^{-1}(Q)=[1,\alpha,\beta]$. Substituting $x_0=1$ and dividing by $x_0x_1x_2$ the entries of the map ϕ_x we obtain a map from a neighborhood U of $\pi^{-1}(Q)$ to $\mathbb{A}^8$: $\tilde{\phi}_x: U\to\mathbb{A}^8$, centered at (α,β), which is explicitly given by

$$\left(\frac{1+x_1^3+x_2^3}{x_1x_2},\quad \frac{1}{x_2},\quad \frac{x_1}{x_2},\quad \frac{1}{x_1},\quad x_1,\quad \frac{x_2}{x_1},\quad x_2,\quad 0\right).$$

It is immediate to check that the differential of this map computed at (α,β) has rank 2.

If Q is a point of the second type on Y, then since it is obtained by glueing two $\mathbb{P}^2$'s along lines, it is clear that $T_QY\cong\mathbb{A}^3$. Moreover, $\pi^{-1}(Q)=(P_1,P_2)$ where each P_i belongs to a $\mathbb{P}^2$. Without loosing generality, we can assume that $P_1=[1,\alpha,0]$ (x-coordinates) and $P_2=[1,\alpha,0]$ (y-coordinates). Then it is sufficient to prove that $d\phi_x|T_{P_1\mathbb{P}^2}$ and $d\phi_y|T_{P_2\mathbb{P}^2}$ span a vector space of dimension at least 3. To compute $d\phi_x$ we substitute $x_0=1$ and divide all entries by $x_0^2x_1$, obtaining an explicit map $\tilde{\phi}_x$ to $\mathbb{A}^8$ of the form

$$\left(\frac{1+x_1^3+x_2^3}{x_1},\quad x_1,\quad \frac{x_2}{x_1},\quad x_2x_1,\quad \frac{x_2^2}{x_1},\quad x_2^2,\quad x_2,\quad 0\right).$$

An analogous reasoning for the y-coordinates, gives an explicit map $\tilde{\phi}_y$ to $\mathbb{A}^8$ given by

$$\left(\frac{1+y_1^3+y_2^3}{y_1},\quad y_1,\quad \frac{y_2}{y_1},\quad y_2y_1,\quad \frac{y_2^2}{y_1},\quad y_2^2,\quad 0,\quad y_2\right).$$

Computing the Jacobian of the two maps at the point $(\alpha,0)$ (i.e. $(x_1,x_2)=(\alpha,0)$ for the x-coordinates and $(y_1,y_2)=(\alpha,0)$ for the y-coordinates) we get the two matrices:

$$\begin{pmatrix} - & - \\ 1 & 0 \\ - & - \\ - & - \\ - & - \\ - & - \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad\qquad \begin{pmatrix} - & - \\ 1 & 0 \\ - & - \\ - & - \\ - & - \\ - & - \\ 0 & 0 \\ 0 & 1 \end{pmatrix},$$

which span together at least a 3-dimensional vector space in $\mathbb{A}^8$.

Finally, we have to prove that $\phi_{|\mathcal{O}_Y(3)|}$ separates tangent directions for the unique point Q of the third type. This point corresponds to the vertices of the triangles, which are all identified.

A neighborhood of Q can be represented from the toric description as six triangles, having all a common vertex in Q and either a common edge or nothing else in common, besides Q.

They can be constructed like this: consider four squares $S_1, \dots, S_4$ in $\mathbb{A}^2$ with vertices $[(0,0),(1,0),(1,1),(0,1)]$ for the first square, $[(0,0),(0,1),(-1,1),(-1,0)]$ for the second square, $[(0,0),(-1,0),(-1,-1),(0,-1)]$ for the third square, and finally $[(0,0),(0,-1),(1,-1),(1,0)]$ for the fourth.

They all meet in $(0,0)$, which represents the point Q. If we subdivide these four squares drawing the antidiagonal lines, we get the six triangles meeting in $(0,0)$, describing the local geometry around Q. Then a neighborhood of Q can be described as six copies of $\mathbb{A}^2$ (the six triangles), meeting along lines according to the above pattern. We call this copies of $\mathbb{A}^2$ as V_1, V_2, V_3, V_4, V_5, V_6. Now $V_1 \cap V_2$ is a line so that they span together a 3-dimensional vector space W_1. Then $W_1 \cap V_3$ is again a line, so that the span of W_1 and V_3 is 4-dimensional vector space W_2. Again $W_2 \cap V_4$ is a line and they together span W_3 which is 5-dimensional and finally $W_3 \cap V_5$ is a line and they span W_4 which is 6-dimensional. Then observe that $V_6 \subset W_4$ since they have in common 2 lines. This implies that $T_Q Y = \mathbb{A}^6$, showing that Q is an extremely nasty singularity.

We can give a cleaner proof of the fact that the dimension of the tangent space is actually equal to six, via toric geometry. Indeed, consider lattice points u_1, u_2, u_3, u_4, u_5, u_6, each of which belongs to one of the six triangles; since they are not cell-mates, we get in $\mathbb{C}[u_1, \dots, u_6]$, the following exhaustive set of relations: $u_1u_3 = 0$, $u_1u_4 = 0$, $u_1u_5 = 0$, $u_2u_4 = 0$, $u_2u_5 = 0$, $u_2u_6 = 0$, $u_3u_6 = 0$, $u_3u_5 = 0$, $u_4u_6 = 0$. The ideal generated by these relations in $\mathbb{C}[u_1, \dots, u_6]$, corresponds to six 2-planes: $\pi_1 = \{u_1 = u_2 = u_3 = u_4 = 0\}$, $\pi_2 = \{u_1 = u_2 = u_3 = u_6 = 0\}$, $\pi_3 = \{u_3 = u_4 = u_5 = u_6 = 0\}$, $\pi_4 = \{u_2 = u_3 = u_4 = u_5 = 0\}$, $\pi_5 = \{u_1 = u_2 = u_5 = u_6 = 0\}$ and $\pi_6 = \{u_1 = u_4 = u_5 = u_6 = 0\}$. All these six planes intersect just in the origin and the span of three of them, such as π_1, π_3 and π_5 is $\mathbb{A}^6$. This just proves that $T_Q Y \cong \mathbb{A}^6$.

Then, to conclude it is enough to show that the images of $d\phi_x$ at the three coordinate points span altogether a 6-dimensional vector space. Indeed, to span $T_Q Y$ it is sufficient to take three copies of $\mathbb{A}^2$ around O, which meet only in O, and having no edge in common (they correspond to the 2-planes π_1, π_3 and π_5). These copies correspond to the three vertices of just one copy of $\mathbb{P}^2$, let us say the x-copy. Then we have just to compute $d\phi_x$ at the points $P_1 = [1,0,0]$, $P_2 = [0,1,0]$ and $P_3 = [0,0,1]$. To compute $d\phi_x$ at P_1 we set $x_0 = 1$ and divide by the entry $x_0^3 + x_1^3 + x_2^3$ all other entries, getting the map ϕ_{x,P_1} as follows:

$$\left(\frac{x_1}{1+x_1^3+x_2^3}, \quad \frac{x_1^2}{1+x_1^3+x_2^3}, \quad \frac{x_2}{1+x_1^3+x_2^3}, \quad \frac{x_1^2 x_2}{1+x_1^3+x_2^3}, \quad \frac{x_2^2}{1+x_1^3+x_2^3}, \quad \frac{x_2^2 x_1}{1+x_1^3+x_2^3}, \quad \frac{x_1 x_2}{1+x_1^3+x_2^3}, \quad 0\right).$$

Computing the differential of ϕ_{x,P_1} at $(0,0)$ we get

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Repeating the procedure with the point P_2 (this time setting $x_1 = 1$ but *always* dividing by $x_0^3 + x_1^3 + x_2^3$) we obtain the map ϕ_{x,P_2}:

$$\Big(\frac{x_0^2}{1+x_0^3+x_2^3}, \quad \frac{x_0}{1+x_0^3+x_2^3}, \quad \frac{x_0^2 x_2}{1+x_0^3+x_2^3}, \quad \frac{x_2}{1+x_0^3+x_2^3}, \\ \frac{x_2^2 x_0}{1+x_0^3+x_2^3}, \quad \frac{x_2^2}{1+x_0^3+x_2^3}, \quad \frac{x_0 x_2}{1+x_0^3+x_2^3}, \quad 0\Big),$$

and also we get the differential at $(0,0)$:

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Finally, considering the point P_3, we have the map ϕ_{x,P_3}:

$$\Big(\frac{x_0^2 x_1}{1+x_0^3+x_1^3}, \quad \frac{x_1^2 x_0}{1+x_0^3+x_1^3}, \quad \frac{x_0^2}{1+x_0^3+x_1^3}, \quad \frac{x_1^2}{1+x_0^3+x_1^3}, \\ \frac{x_0}{1+x_0^3+x_1^3}, \quad \frac{x_1}{1+x_0^3+x_1^3}, \quad \frac{x_0 x_1}{1+x_0^3+x_1^3}, \quad 0\Big),$$

the differential of which at $(0,0)$ is given by

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

This implies that $d\phi_{|\mathcal{O}_Y(3)|}$ is always injective on Y, also for the point of third type, since the rank of $d\phi_x$ is 6. Thus $\phi_{|\mathcal{O}_Y(3)|} : Y \hookrightarrow \mathbb{P}^8$ is an embedding. $\square$

Recalling all the results of the previous sections, we get immediately the following main result:

Theorem 4.2. *Let Y be a SSAV of dimension 2, which is a degeneration of a principally polarized Abelian surface and let $\mathcal{O}_Y(1)$ the associated ample line bundle. Then $\mathcal{O}_Y(3)$ is already very ample.*

Proof. Immediate in the light of the previous results, since any SSAV Y of dimension 2, coming from a principally polarized Abelian surface belongs to one of the three degeneration types studied in the previous sections. $\square$

Remark 4.3. In [5] it is proved that if C is an irreducible curve, having only nodes as singularities, then on the compactified Jacobian (considered as the moduli scheme parameterizing torsion-free, rank 1 sheaves of Euler characteristic 0 on C), there is a line bundle $\mathcal{L}$ representing a principal polarization, such that $\mathcal{L}^{\otimes 3}$ is already very ample. This theorem covers partially our result, but the proof given there is less elementary. Our proof is based on a concrete study of the projective map induced by a principal polarization and is completely based on projective geometry.

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