Sharp Edge-Homotopy on Spatial Graphs

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ABSTRACT
A sharp-move is known as an unknotting operation for knots. A self sharp-move is a sharp-move on a spatial graph where all strings in the move belong to the same spatial edge. We say that two spatial embeddings of a graph are sharp edge-homotopic if they are transformed into each other by self sharp-moves and ambient isotopies. We investigate how is the sharp edge-homotopy strong and classify all spatial theta curves completely up to sharp edge-homotopy. Moreover we mention a relationship between sharp edge-homotopy and delta edge (resp. vertex)-homotopy on spatial graphs.

Key words: spatial graph, sharp move, delta move.

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1. Introduction
Let $G$ be a finite graph which is considered as a topological space in the usual way. An embedding $f : G \to S^3$ is called a spatial embedding of $G$ or simply a spatial graph. We call a subgraph of $G$ a cycle if it is homeomorphic to $S^1$ and denote the set of all cycles of $G$ by $\Gamma(G)$. A graph $G$ is said to be planar if there exists an embedding of $G$ into $S^2$, and a spatial embedding of a planar graph $G$ is said to be trivial if it is ambient isotopic to an embedding of $G$ into $S^2 \subset S^3$.

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A graph is said to be oriented if an orientation is given for each edge. A sharp-move is a local move on a spatial oriented graph as illustrated in Fig. 1. It is known that the sharp-move is an unknotting operation [10]. We say that a sharp-move is a self sharp-move if all four strings in the move belong to the same spatial edge. We say that two spatial embeddings of a graph \( G \) are sharp edge-homotopic if they are transformed into each other by self sharp-moves and ambient isotopies. It is easy to see that this definition does not depend on the edge orientations. If \( G \) is homeomorphic to the disjoint union of 1-spheres, then it coincides with self sharp-equivalence on oriented links [14, 22, 23, 25].

First we investigate how strong is sharp edge-homotopy. In [27], equivalence relations cobordism, isotopy, I-equivalence and edge-homotopy on spatial graphs were introduced. A delta move is a local move on a spatial graph as illustrated in Fig. 2. It is known that the delta move is also an unknotting operation [7, 12]. We say that a delta move is a self delta move if all three strings in the move belong to the same spatial edge and a quasi adjacent-delta move if it is on exactly two adjacent spatial edges. Two spatial embeddings of a graph are said to be delta edge (resp. vertex)-homotopic if they are transformed into each other by self delta moves (resp. quasi adjacent-delta moves) and ambient isotopies [15]. This is a generalization of delta link-homotopy (or self delta-equivalence) on oriented links [13, 24].
Theorem 1.1. The following implications hold:

Moreover the converse of each implication does not hold and if there does not exist any arrow between two equivalence relations then there does not exist any implication between them.

We note that the dotted parts have already proved in [27] and the gray parts have already proved by the author [15]. We prove Theorem 1.1 in the next section.

In section 3, we show that there exists a theta curve which is not trivial up to sharp edge-homotopy, where a theta curve is a spatial embedding of the graph $\theta$ as illustrated in Fig. 3.

Moreover, we classify all theta curves completely up to sharp edge-homotopy. For a theta curve $f$, let $\tilde{\alpha}(f)$ be the modulo two reduction of $\sum_{\gamma \in \Gamma(\theta)} a_2(f(\gamma))$, where $a_i(L)$ denotes the $i$-th coefficient of the Conway polynomial $\nabla_L(z)$ of a link $L$. We note that the modulo two reduction of $a_2(K)$ coincides with the Arf invariant [21] of a knot $K$. From the beginning this was introduced as a delta edge-homotopy invariant of theta curves [15]. We give the following classification in section 3.

Theorem 1.2. Two theta curves $f$ and $g$ are sharp edge-homotopic if and only if $\tilde{\alpha}(f) = \tilde{\alpha}(g)$.

A theta curve is said to be almost unknotted if $f(\gamma)$ is a trivial knot for any cycle $\gamma \in \Gamma(\theta)$. As a corollary of Theorem 1.2, we have the following:

Figure 3

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Corollary 1.3. Any almost unknotted theta curve is trivial up to sharp edge-homotopy.

It is known that there exist infinitely many oriented links up to delta edge-homotopy which are mutually sharp edge-homotopic [14]. On the other hand, the author showed that there exist infinitely many almost unknotted theta curves up to delta edge-homotopy [17]. Thus we have the following by Corollary 1.3.

Corollary 1.4. There exist infinitely many theta curves up to delta edge-homotopy which are mutually sharp edge-homotopic.

This implies that in general there exist infinitely many spatial embeddings of a graph $G$ up to delta edge-homotopy which are mutually sharp edge-homotopic even in the case that $G$ does not contain any disjoint cycles. In section 4, we give such an example in case $G$ is the complete graph on four vertices $K_4$ as follows.

Theorem 1.5. There exist spatial embeddings $f_m$ of $K_4$ for $m \in \mathbb{N} \cup \{0\}$ such that

(i) $f_i$ and $f_j$ are not delta edge-homotopic for $i \neq j$.

(ii) For any subgraph $H$ of $K_4$ which is homeomorphic to $\theta$, $f_i|_H$ and $f_j|_H$ are delta edge-homotopic for $i \neq j$.

(iii) $f_i$ and $f_j$ are sharp edge-homotopic for $i \neq j$.

Besides we show that there exist infinitely many spatial embeddings of the complete graph on five vertices $K_5$ up to delta vertex-homotopy which are mutually sharp edge-homotopic.

Theorem 1.6. There exist spatial embeddings $f_m$ of $K_5$ for $m \in \mathbb{N} \cup \{0\}$ such that

(i) $f_i$ and $f_j$ are not delta vertex-homotopic for $i \neq j$.

(ii) For any subgraph $H$ of $K_5$ which is homeomorphic to $K_4$, $f_i|_H$ and $f_j|_H$ are delta vertex-homotopic for $i \neq j$.

(iii) $f_i$ and $f_j$ are sharp edge-homotopic for $i \neq j$.

In particular, by Theorem 1.1 we have the following:

Corollary 1.7. There exist infinitely many spatial embeddings of $K_5$ up to isotopy which are mutually sharp edge-homotopic.

Remark 1.8. Classification of spatial graphs up to sharp-moves (resp. pass-moves) and ambient isotopies was studied by Y. Ohyama [19], but all strings in the move does not have to belong to the same spatial edge. The author and R. Shinjo have already classified all boundary spatial embeddings of a graph completely up to self sharp-moves (resp. self pass-moves) and ambient isotopies [18].
2. Proof of Theorem 1.1

In this section we prove Theorem 1.1. We give the definitions of cobordism, isotopy
and \( I \)-equivalence on spatial graphs roughly. We refer the reader to [27] for precise
definitions. Two spatial embeddings \( f \) and \( g \) of a graph \( G \) are said to be \( I \)-equivalent
if there exists an embedding \( \Phi : G \times [0,1] \to S^3 \times [0,1] \) between \( f \) and \( g \). On that
occasion, if \( \Phi \) is locally flat then \( f \) and \( g \) are said to be cobordant. On the other hand
if \( \Phi \) is level preserving then \( f \) and \( g \) are said to be isotopic. We remark here that \( f \)
and \( g \) are ambient isotopic if \( \Phi \) is locally flat and level preserving.

We give the definition of edge-homotopy on spatial graphs precisely. We say
that a crossing change is a self crossing change if all two strings in the move belong
to the same spatial edge. Two spatial embeddings of a graph are said to be edge-
homotopic [27] if they are transformed into each other by self crossing changes and
ambient isotopies. (In [27], edge-homotopy is called simply a homotopy.) This is a
generalization of link-homotopy on oriented links [8].

\textbf{Lemma 2.1.} Let \( f \) and \( g \) be spatial embeddings of a graph.

(i) If \( f \) and \( g \) are ambient isotopic then they are sharp edge-homotopic.

(ii) If \( f \) and \( g \) are delta edge-homotopic then they are sharp edge-homotopic.

(iii) If \( f \) and \( g \) are sharp edge-homotopic then they are edge-homotopic.

\textbf{Proof.} (i) and (iii) are clear. It is known that a delta move is realized by sharp-
moves on the strings in the delta move and ambient isotopies [14]. Thus we have (ii)
immediately.

\textbf{Lemma 2.2.} If two spatial embeddings of a graph are cobordant then they are sharp
edge-homotopic.

To prove Lemma 2.2, we prepare two kinds of specific local moves on a spatial
graph. A pass-move on a spatial oriented graph is a local move as illustrated in
Fig. 4 [5]. We say that a pass-move is a self pass-move if all four strings in the move
belong to the same spatial edge. It is known that a pass-move is realized by sharp moves on the strings in the pass-move and ambient isotopies [12]. Thus we have the following:

**Lemma 2.3.** A self pass-move is realized by sharp edge-homotopy.

A Γ-move [5] is a local move on a spatial oriented graph as illustrated in Fig. 5. We call a Γ-move a self Γ-move if all three strings in the move belong to the same spatial edge. It is known that a Γ-move is realized by a pass-move on the strings in the Γ-move and ambient isotopies [5]. Thus by Lemma 2.3 we have the following:

**Lemma 2.4.** A self Γ-move is realized by sharp edge-homotopy.

The following is a normalization of cobordism between two cobordant spatial embeddings of a graph.

**Lemma 2.5 ([27]).** Let $f$ and $g$ be cobordant spatial embeddings of a graph $G$. Then there exists an embedding $\Phi : G \times [0,1] \to S^3 \times [0,1]$ between $f$ and $g$ satisfying the following conditions:

(i) $\pi \circ \Phi|_{v \times [0,1]} : v \times [0,1] \to [0,1]$ is a homeomorphism for any vertex $v$ of $G$, where $\pi : S^3 \times [0,1] \to [0,1]$ is the natural projection.

(ii) The image of $\Phi$ has only finitely many critical points in $\text{int}(e \times [0,1])$ for any edge $e$ of $G$, consisting of minimal points, saddle points and maximal points.

(iii) All of the minimal points lie in $S^3 \times \{\frac{1}{6}\}$ and all of the maximal points lie in $S^3 \times \{\frac{5}{6}\}$.

(iv) All of the saddle points lie in $S^3 \times \{\frac{1}{3}\}$ and $S^3 \times \{\frac{2}{3}\}$ such that the cross-section $\Phi(G \times [0,1]) \cap S^3 \times \{\frac{1}{2}\}$ is homeomorphic to $G$.

**Proof of Lemma 2.2.** Let $f$ and $g$ be cobordant spatial embeddings of a graph $G$ and $\Phi$ is a normalization of cobordism between $f$ and $g$ as in Lemma 2.5. We can regard the cross-section $\Phi(G \times [0,1]) \cap S^3 \times \{\frac{1}{2}\}$ as the image of a spatial embedding of $G$ which is denoted by $h_{1/2}$. 

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**Figure 5**
Claim 1. \( f \) and \( h_{1/2} \) are sharp edge-homotopic.

We remark here that the number of minimal points in \( S^3 \times \{ \frac{1}{3} \} \) equals the number of saddle points in \( S^3 \times \{ \frac{1}{3} \} \) as it was pointed out in [27]. We can deform \( \Phi(G \times [0, 1]) \) by an ambient isotopy of \( S^3 \times [0, 1] \) so that the minimal points in \( S^3 \times \{ \frac{1}{3} \} \) and the saddle points in \( S^3 \times \{ \frac{1}{3} \} \) change into \textit{minimal bands} \( D_1, D_2, \ldots, D_l \) and \textit{saddle bands} \( b_1, b_2, \ldots, b_l \), respectively and each saddle band intersects the spatial graph by an arc. Then we have that \( h_{1/2} \) is a band fusion of an \( l \)-component trivial link \( L = J_1 \cup J_2 \cup \cdots \cup J_l \) and \( f \). Then by using self \( \Gamma \)-moves and ambient isotopies if necessary, we can shrink each band with the component \( J_i \) one by one, see Fig. 6. By shrinking all bands in such a way, we obtain the spatial embedding \( f \). Therefore by Lemma 2.4 we have that \( f \) and \( h_{1/2} \) are sharp edge-homotopic.

Claim 2. \( g \) and \( h_{1/2} \) are sharp edge-homotopic.

We can see Claim 2 in the same way as Claim 1. This completes the proof.

Remark 2.6. The proof of Lemma 2.2 shows that if two spatial embeddings of a graph are cobordant then they are transformed into each other by self pass-moves and ambient isotopies.

Proof of Theorem 1.1. By Lemmas 2.1 and 2.2, we have the desired implications. Let \( f \) and \( g \) be theta curves as illustrated in Fig. 7. It is known that any theta curve is trivial up to isotopy [27]. But \( f \) is not trivial up to sharp edge-homotopy, see Example 3.6. We note that \( g \) is an almost unknotted theta curve which is called \textit{Kinoshita’s theta curve}. It is known that \( g \) is not trivial up to delta edge-homotopy and cobordism [17]. But we have that \( g \) is trivial up to sharp-edge homotopy by Corollary 1.3. Let \( L \) be an oriented link as illustrated in Fig. 7. By a calculation we have that \( \nabla_L(z) = -4z^3 - 5z^5 - z^7 \). Since the first non-vanishing coefficient of the Conway polynomial of a link is an \( I \)-equivalence invariant [1], we have that \( L \) is not trivial up to \( I \)-equivalence and isotopy. Since the third coefficient of the Conway polynomial of a 2-component algebraically split link is a delta link-homotopy invariant [13], we have that \( L \) is not trivial up to delta vertex-homotopy. But \( L \) is trivial up to sharp edge-homotopy, see Fig. 8. Therefore we have that the converse of each implication does not hold and if there does not exist any arrow between two equivalence relations then there does not exist any implication between them. This completes the proof.

3. Sharp edge-homotopy classification of theta curves

In this section we prove Theorem 1.2. We first show the sharp edge-homotopy invariance of the \( \tilde{\alpha} \)-invariant of theta curves.

Proposition 3.1. If two theta curves \( f \) and \( g \) are sharp edge-homotopic then \( \tilde{\alpha}(f) = \tilde{\alpha}(g) \).
Figure 6
To prove Proposition 3.1, we recall the \( a_3 \)-invariant of theta curves [17]. For a theta curve \( f \), the associated 3-component link \( L_f \) is defined, which is the boundary of a compact, connected and orientable surface \( S_f \) with zero Seifert linking form having \( f \) as a spine [6], see Fig. 9. We order and orient \( L_f = K_1^f \cup K_2^f \cup K_3^f \) so that \( K_i^f \) is freely homotopic to \( f(e_i+1) - f(e_{i+2}) \), where suffixes are taken modulo 3. We denote the sublink \( K_{i+1}^f \cup K_{i+2}^f \) by \( l_i(f) \) \((i = 1, 2, 3)\). Since it is known that \( a_3(l_1(f)) = a_3(l_2(f)) = a_3(l_3(f)) \) for any theta curve \( f \) [4, 26], we can define that \( a_3(f) = a_3(S) \), where \( S \) is arbitrary 2-component sublink of \( L_f \). We remark here that the \( a_3 \)-invariant is a complete delta edge-homotopy invariant of theta curves.

**Theorem 3.2 ([17]).** Two theta curves \( f \) and \( g \) are delta edge-homotopic if and only if \( a_3(f) = a_3(g) \).

We also recall the calculation of the Arf invariant of totally proper links. The following is a direct consequence of the results in [2, 9, 29].

**Proposition 3.3.** Let \( L = J_1 \cup J_2 \cup \cdots \cup J_n \) be an \( n \)-component totally proper link, namely \( \text{lk}(J_i \cup J_j) \equiv 0 \pmod{2} \) for any \( i \neq j \), where \( \text{lk} \) denotes the linking number. Then we have the following:

(i) \( a_{n+1}(L) \equiv \sum_{m=1}^{n} \sum_{1 \leq i_1 < i_2 < \cdots < i_m \leq n} \text{Arf}(J_{i_1} \cup J_{i_2} \cup \cdots \cup J_{i_m}) \pmod{2} \),

(ii) \( \text{Arf}(L) \equiv \sum_{m=1}^{n} \sum_{1 \leq i_1 < i_2 < \cdots < i_m \leq n} a_{m+1}(J_{i_1} \cup J_{i_2} \cup \cdots \cup J_{i_m}) \pmod{2} \).
Lemma 3.4. $\tilde{\alpha}(f) \equiv a_3(f) \pmod{2}$ for any theta curve $f$.

Proof. By Proposition 3.3 we have that

$$0 = a_4(L_f) \equiv \text{Arf}(L_f) + \sum_{i=1}^{3} \text{Arf}(l_i(f)) + \sum_{i=1}^{3} \text{Arf}(K_i^j)$$

$$\equiv \text{Arf}(L_f) + \sum_{i=1}^{3} a_3(l_i(f)) + \sum_{i=1}^{3} a_2(K_i^j)$$

$$\equiv \text{Arf}(L_f) + a_3(f) + \tilde{\alpha}(f) \pmod{2}.$$ 

Since we can see that $\text{Arf}(L_f) = 0$, we have the result.

Lemma 3.5. (i) ([23]) $\overline{\text{Arf}}(L)$ is a sharp edge-homotopy invariant of a 2-component proper link $L$.

(ii) $a_3(L) \equiv \overline{\text{Arf}}(L) \pmod{2}$.

Proof. We show (ii). By Proposition 3.3 (ii) we have that

$$\overline{\text{Arf}}(L) \equiv \text{Arf}(L) - \text{Arf}(J_1) - \text{Arf}(J_2)$$

$$\equiv a_3(L) + a_2(J_1) + a_2(J_2) - a_2(J_1) - a_2(J_2)$$

$$= a_3(L) \pmod{2}.$$ 

Proof of Proposition 3.1. Let $f$ and $g$ be theta curves such that $g$ is obtained from $f$ by a single self sharp move on $f(e_1)$. Then we have that $l_3(g)$ is obtained from $l_3(f)$.
Then by Lemmas 3.4 and 3.5 (i) and (ii), we have that $f$ quasi adjacent-delta move on two of the three spatial edges $e$ by a single self sharp-move on $K$ realized by quasi adjacent-delta moves on $f$. Let Lemma 3.7. For quasi adjacent-delta moves, we have the following:

\[ \tilde{\alpha}(f) \equiv a_3(f) = a_3(l_3(f)) \equiv \overline{\alpha_3}(l_3(f)) \\
= \overline{\alpha_3}(l_3(g)) \equiv a_3(l_3(g)) = a_3(g) \equiv \tilde{\alpha}(g) \pmod{2}. \]

This completes the proof. \qed

Example 3.6. Let $f$ be the theta curve as illustrated in Fig. 7. Then by a calculation we have that $\tilde{\alpha}(f) = 1$. Thus $f$ is not trivial up to sharp edge-homotopy.

In the rest of this section we show that the converse of Proposition 3.1 is also true. For quasi adjacent-delta moves, we have the following:

**Lemma 3.7.** Let $G$ be a graph and $v$ a vertex of $G$ such that exactly three edges $e_1, e_2$ and $e_3$ of $G$ are incident to $v$. Let $f$ be a spatial embedding of $G$. Then, any quasi adjacent-delta move on two of the three spatial edges $f(e_1), f(e_2)$ and $f(e_3)$ are realized by quasi adjacent-delta moves on $f(e_1)$ and $f(e_2)$ and sharp edge-homotopies.

**Proof.** It is sufficient to show that a quasi adjacent-delta move on $f(e_2)$ and $f(e_3)$ can be realized by quasi adjacent-delta moves on $f(e_1)$ and $f(e_2)$ and sharp edge-homotopies.

**Step 1.** If two of the three strings in a quasi adjacent-delta move belong to $f(e_2)$, by an application of the move in Fig. 10 with respect to $f(e_3)$, we have that it is realized by a self delta move on $f(e_2)$, namely self sharp-moves on $f(e_2)$, and a quasi adjacent-delta move on $f(e_1)$ and $f(e_2)$ and ambient isotopies. So we have the result.

**Step 2.** If two of the three strings in a quasi adjacent-delta move belong to $f(e_3)$, by an application of the move in Fig. 10 with respect to $f(e_3)$, we have that it is realized by a quasi adjacent-delta move on $f(e_2)$ and $f(e_3)$, a delta move on exactly three spatial edges $f(e_1), f(e_2)$ and $f(e_3)$ and ambient isotopies. The former move can be realized by a quasi adjacent-delta move on $f(e_1)$ and $f(e_2)$ and sharp edge-homotopies in the same way as Step 1. For the latter move, by an application of the move in Fig. 10 with respect to $f(e_3)$, we have that it is realized by two quasi adjacent-delta moves on $f(e_1)$ and $f(e_2)$ and ambient isotopies. This completes the proof. \qed

**Proof of Theorem 1.2.** We have the ‘only if’ part by Proposition 3.1. We show the ‘if’ part. Let $f$ and $g$ be two theta curves such that $\tilde{\alpha}(f) = \tilde{\alpha}(g)$. It is known that any two theta curves are delta vertex-homotopic [15]. Thus we have that $f$ and $g$ are delta vertex-homotopic. Then by Lemma 3.7 we have that $f$ can be obtained from $g$ by quasi adjacent-delta moves on $g(e_1)$ and $g(e_2)$ and sharp edge-homotopies. We note that a delta move can be regarded as a band fusion of Borromean rings, see Fig. 11. So we have that $f$ is sharp edge-homotopic to $f_0$ which is a band fusion of Borromean rings and $g$, where the roots of three fusion bands with each Borromean rings belong to $g(e_1)$ and $g(e_2)$. We note that $\tilde{\alpha}(f) = \tilde{\alpha}(f_0)$ by Lemma 3.1.
Figure 10

Figure 11
All pairs of adjacent two Borromean rings attached to \( g(e_1) \) and \( g(e_2) \) with respect to \( f_0(e_1) \) are divided into the six patterns as illustrated in Fig. 12, where the gray spatial edge is \( f_0(e_1) \) and the black spatial edge is \( f_0(e_2) \). By sliding the roots of a fusion band with the Borromean rings if necessary, we can reduce (4), (5), and (6) to (1). Then we can see that each pair of adjacent two Borromean rings attached to \( g(e_1) \) and \( g(e_2) \) as illustrated in Fig. 12 (1), (2), and (3) can be omitted by self \( \Gamma \)-moves and ambient isotopies, namely up to sharp edge-homotopy by Lemma 2.4 as illustrated in Figs. 13, 14, and 15, respectively. Therefore we have that \( f_0 \) is sharp edge-homotopic to \( f_1 \) which is a band fusion of Borromean rings and \( g \), where the number of the Borromean rings is at most one. Assume that there exists exactly one Borromean rings. We remark here that if a knot \( K \) can be obtained from \( J \) by a single delta move then \( |a_2(K) - a_2(J)| = 1 \) [20]. Then we have that \( |a_2(f_1(e_1 \cup e_2)) - a_2(g(e_1 \cup e_2))| = 1 \), \( a_2(f_1(e_2 \cup e_3)) = a_2(g(e_2 \cup e_3)) \) and \( a_2(f_1(e_2 \cup e_3)) = a_2(g(e_2 \cup e_3)) \). This implies that \( \tilde{\alpha}(f) = \tilde{\alpha}(f_0) = \tilde{\alpha}(f_1) \neq \tilde{\alpha}(g) \). This is a contradiction. Thus we have that \( f_1 \) is sharp edge-homotopic to \( g \). Therefore \( f \) and \( g \) are sharp edge-homotopic.

\[\square\]

4. Sharp edge-homotopy vs. delta edge (resp. vertex)-homotopy

In this section we give some examples of infinitely many spatial embeddings of \( K_4 \) (resp. \( K_5 \)) up to delta edge (resp. vertex)-homotopy which are mutually sharp edge-homotopic. To detect the non-trivial delta edge (resp. vertex)-homotopy classes, we recall the \textit{n-invariant} [15] of spatial embeddings of \( K_4 \) and \( K_5 \). A cycle of a graph \( G \) is called a \( k \)-cycle if it contains exactly \( k \) edges. Let \( \omega_4 : \Gamma(K_4) \rightarrow \mathbb{Z} \) be a map defined by \( \omega_4(\gamma) = 1 \) if \( \gamma \) is a 4-cycle and \( -1 \) if \( \gamma \) is a 3-cycle. Let \( \omega_5 : \Gamma(K_5) \rightarrow \mathbb{Z} \) be a map defined by \( \omega_5(\gamma) = 1 \) if \( \gamma \) is a 5-cycle, \( -1 \) if \( \gamma \) is a 4-cycle and 0 if \( \gamma \) is a 3-cycle. For
Figure 14
Figure 15
a spatial embedding \( f \) of \( K_i \) for \( i = 4, 5 \), we put

\[
n_{\omega_i}(f) = \frac{1}{18} \sum_{\gamma \in \Gamma(K_i)} \omega_i(\gamma) V^{(3)}_{f(\gamma)}(1),
\]

where \( V^{(i)}_L(1) \) denotes the \( i \)-th derivative at 1 of the Jones polynomial \( V_L(t) \) of a link \( L \). (In this paper we calculate the Jones polynomial of a link by the skein relation

\[
t V_{J_+}(t) - t^{-1} V_{J_-}(t) = (t^{-\frac{1}{2}} - t^{\frac{1}{2}}) V_{J_0}(t).
\]

Then the following was shown in [15] by the author:

**Theorem 4.1.**

(i) \( n_{\omega_4}(f) \) is an integer-valued delta edge-homotopy invariant of a spatial embedding \( f \) of \( K_4 \).

(ii) \( n_{\omega_5}(f) \) is an integer-valued delta vertex-homotopy invariant of a spatial embedding \( f \) of \( K_5 \).

We prepare the formula of the variation of \( V^{(3)}(1) \) of knots which differed by a single crossing change. The following formula is convenient to calculate the \( n \)-invariant.

**Proposition 4.2.**

\[
V^{(3)}_{J_+}(1) - V^{(3)}_{J_-}(1) = 36a_2(J_+) + 18\{\text{lk}(J_0)\}^2 - 36\{a_2(K_1) + a_2(K_2)\},
\]

where \( J_+, J_- \), and \( J_0 = K_1 \cup K_2 \) are two knots and a 2-component link as illustrated in Fig. 16.

To prove Proposition 4.2, we use the following results.

**Lemma 4.3.** Let \( L = J_1 \cup J_2 \cup \cdots \cup J_n \) be an \( n \)-component link.

(i) ([3]) \( V_L(1) = (-2)^{n-1} \).

(ii) ([3, 11])

\[
V^{(1)}_L(1) = \begin{cases} 
0 & \text{if } n = 1, \\
3(-2)^{n-2} \sum_{1 \leq i < j \leq n} \text{lk}(J_i \cup J_j) & \text{if } n \geq 2.
\end{cases}
\]
(iii) ([11])

\[
V_L^{(2)}(1) = \begin{cases} 
-6a_2(L) & \text{if } n = 1, \\
12\{a_2(J_1) + a_2(J_2)\} - 6\{\text{lk}(L)\}^2 - 3\text{lk}(L) - \frac{1}{2} & \text{if } n = 2.
\end{cases}
\]

Proof of Proposition 4.2. By differentiating both sides of the skein relation at 1 about three times and Lemma 4.3, it is easy to check that

\[
V_{J^+}^{(3)}(1) - V_{J^-}^{(3)}(1) = -3V_{J^+}^{(2)}(1) - 3V_{J^-}^{(2)}(1) + 3V_J^{(1)}(1) - 3V_J^{(2)}(1) - \frac{3}{2}. \tag{1}
\]

By Lemma 4.3 we have that

\[
\begin{align*}
V_{J^+}^{(2)}(1) &= -6a_2(J_+), \\
V_{J^-}^{(2)}(1) &= -6a_2(J_-), \\
V_J^{(1)}(1) &= 3\text{lk}(J_0), \\
V_J^{(2)}(1) &= 12\{a_2(K_1) + a_2(K_2)\} - 6\{\text{lk}(J_0)\}^2 - 3\text{lk}(J_0) - \frac{1}{2}.
\end{align*}
\]

Besides it is well known that ([5])

\[
a_2(J_+) - a_2(J_-) = \text{lk}(J_0). \tag{2}
\]

Thus by substituting them to (1) we have the result.

Proof of Theorem 1.5. Let \( f_m (m \in \mathbb{N} \cup \{0\}) \) be a spatial embedding of the complete graph on four vertices \( K_4 \) as illustrated in Fig. 17.

We first show (iii). We can see that \( f_m \) is sharp edge-homotopic to a trivial spatial embedding for any \( m \in \mathbb{N} \cup \{0\} \) as illustrated in Fig. 18.
Figure 18

Figure 19
Next we show (i). It is not hard to see that $f_m$ contains exactly two non-trivial knots $J_1 = K_m$ and $J_2$ as illustrated in Fig. 19. We note that $J_2$ is ambient isotopic to $K_0$ and

$$\nabla_{J_2}(z) = 1 + 2z^2,$$

$$V_{J_2}(t) = -t^{-6} + t^{-5} - t^{-4} + 2t^{-3} - t^{-2} + t^{-1}. \tag{3}$$

For a skein triple $(K_i, K_{i-1}, L_0)$ $(i = 1, 2, \ldots, m)$ as illustrated in Fig. 20, we have that $a_2(K_m) = a_2(K_{m-1}) = \cdots = a_2(K_0) = 2$ because $\text{lk}(L_0) = 0$. We note that the components $M_1^0$ and $M_2^0$ of $L_0$ are trivial knots. Then by Proposition 4.2 we have that

$$\frac{1}{18}V^{(3)}_{K_i}(1) - \frac{1}{18}V^{(3)}_{K_{i-1}}(1) = 2a_2(K_i) = 4$$

for $i = 1, 2, \ldots, m$. Thus we have that

$$\frac{1}{18}V^{(3)}_{K_m}(1) = \frac{1}{18}V^{(3)}_{K_{m-1}}(1) + 4 = \cdots = \frac{1}{18}V^{(3)}_{K_0}(1) + 4m.$$

Therefore we have that

$$n_{\omega_4}(f_m) = \frac{1}{18}V^{(3)}_{J_1}(1) - \frac{1}{18}V^{(3)}_{J_2}(1) = \frac{1}{18}V^{(3)}_{K_m}(1) - \frac{1}{18}V^{(3)}_{K_0}(1) = 4m.$$

This implies that $f_i$ and $f_j$ are not delta edge-homotopic for $i \neq j$.

Finally we show (ii). Fig. 21 illustrates all constituent theta curves of $f_m$. We can see that $f_m|_{K_{4-e_1}}$ and $f_m|_{K_{4-e_2}}$ are trivial theta curves for any $m \in \mathbb{N} \cup \{0\}$ and $f_i|_{K_{4-e_2}}$ and $f_j|_{K_{4-e_2}}$ are ambient isotopic for $i \neq j$. 

Revista Matemática Complutense
2005, 18; Núm. 1, 181–207
We take a 2-component sublink $L_3^m$, $L_4^m$, and $L_5^m$ of the associated 3-component link of $f_m|_{K_4-e_3}$, $f_m|_{K_4-e_4}$, and $f_m|_{K_4-e_5}$, respectively as illustrated in Fig. 22. Since $L_3^i$ and $L_3^j$ are ambient isotopic for $i \neq j$, we have that $a_3(f_i|_{K_4-e_3}) = a_3(f_j|_{K_4-e_3})$ for $i \neq j$. It is easy to see that $L_4^m$ is delta edge-homotopic to a trivial theta curve. Thus we have that $a_3(f_m|_{K_4-e_4}) = 0$ for any $m \in \mathbb{N} \cup \{0\}$. For $L_5^m$, we can see that the knot which can be obtained from $L_5^m$ by a smoothing on any crossing point of full twists is trivial. Then it is not hard to see that $a_3(L_5^m) = -4$. Thus we have that $a_3(f_m|_{K_4-e_5}) = -4$ for any $m \in \mathbb{N} \cup \{0\}$. Hence by Theorem 3.2 we have that $f_i|_{K_4-e_k}$ and $f_j|_{K_4-e_k}$ are delta edge-homotopic for $i \neq j$ and $k = 3, 4, 5$. This completes the proof.

Proof of Theorem 1.6. Let $f_m (m \in \mathbb{N} \cup \{0\})$ be a spatial embedding of the complete graph on five vertices $K_5$ as illustrated in Fig. 23.

We first show (iii). We can see that $f_m$ is sharp edge-homotopic to $f_0$ for any $m \in \mathbb{N}$ in a similar way as in Fig. 18.

Next we show (i). It is not hard to see that $f_m (m \neq 0)$ contains exactly six non-trivial knots $J_1, J_2, \ldots, J_6$ as illustrated in Fig. 24. We note that each of $J_1, J_3,$ and $J_5$ is ambient isotopic to the $m$ times connected sum of the knot $J_2$ in Fig. 19. Since $V^{(3)}(1)$ is additive under the connected sum, by (3) we have that

$$\frac{1}{18} V^{(3)}_{J_1}(1) = \frac{1}{18} V^{(3)}_{J_2}(1) = \frac{1}{18} V^{(3)}_{J_3}(1) = 8m.$$  

(4)
Figure 22
Besides we can see that $J_2$, $J_4$, and $J_6$ are mutually ambient isotopic. Let us consider a skein tree of $J_2 = K_m$ as illustrated in Fig. 25. Then by Proposition 4.2 and (2), we have that

\[
\frac{1}{18} V_{K_m}^{(3)}(1) - \frac{1}{18} V_{M_2}^{(3)}(1) = 2a_2(K_m),
\]
\[
\frac{1}{18} V_{M_2}^{(3)}(1) - \frac{1}{18} V_{M_3}^{(3)}(1) = 2a_2(M_2) + 2,
\]
\[
\frac{1}{18} V_{M_3}^{(3)}(1) - \frac{1}{18} V_{M_4}^{(3)}(1) = -2a_2(M_4),
\]
\[
\frac{1}{18} V_{M_4}^{(3)}(1) - \frac{1}{18} V_{M_5}^{(3)}(1) = -2a_2(M_5) - 4,
\]

Thus we have that

\[
\frac{1}{18} V_{K_m}^{(3)}(1) = \frac{1}{18} V_{K_{m-1}}^{(3)}(1) - 6 \cdots = \frac{1}{18} V_{K_0}^{(3)}(1) - 6m = -6m. \quad (5)
\]
Therefore by (4) and (5) we have that

\[ n_{\omega_5}(f_m) = - \left\{ \frac{1}{18} V_{J_1}^{(3)}(1) + \frac{1}{18} V_{J_2}^{(3)}(1) \right\} \]
\[ + \left\{ \frac{1}{18} V_{J_3}^{(3)}(1) + \frac{1}{18} V_{J_4}^{(3)}(1) + \frac{1}{18} V_{J_5}^{(3)}(1) + \frac{1}{18} V_{J_6}^{(3)}(1) \right\} \]
\[ = - (8m - 6m) + (8m - 6m + 8m - 6m) \]
\[ = 2m. \]

This implies that \( f_i \) and \( f_j \) are not vertex-homotopic for \( i \neq j \).

Finally we show (ii). Since \( f_i \) and \( f_j \) are sharp edge-homotopic for \( i \neq j \), by Theorem 1.1 we have that \( f_i \) and \( f_j \) are edge-homotopic for \( i \neq j \). We note that \( f_i|_H \) and \( f_j|_H \) are also edge-homotopic for any subgraph \( H \) of \( K_5 \). It is known that if two spatial embeddings of \( K_4 \) are edge-homotopic then they are delta vertex-homotopic [16]. This completes the proof. \( \square \)

**Remark 4.4.** This spatial embedding \( f_m \) of \( K_5 \) contains a theta curve \( T_m \) as illustrated in Fig. 26. We note that \( T_m \) is the \( m \) times vertex connected sum of the theta curve \( T_1 \). Then by a direct calculation we have that \( a_3(T_1) = -4 \). Since the \( a_3 \)-invariant of theta curves is a cobordism invariant and additive under the vertex connected sum of theta curves [17], we have that \( a_3(T_m) = -4m \) and \( f_i \) and \( f_j \) are not cobordant for \( i \neq j \).
Ryo Nikkuni

Sharp edge-homotopy on spatial graphs

Figure 25
References


