# $.121221222\ldots$ Is Not Quadratic

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**ABSTRACT**

In this note, we show that if $b > 1$ is an integer, $f(X) \in \mathbb{Q}[X]$ is an integer valued quadratic polynomial and $K > 0$ is any constant, then the $b$-adic number

$$\sum_{n\geq 0} \frac{a_n}{b^{f(n)}},$$

where $a_n \in \mathbb{Z}$ and $1 \leq |a_n| \leq K$ for all $n \geq 0$, is neither rational nor quadratic.

*Key words:* irrationality, applications of sieve methods.

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## Introduction

There are many studies dealing with criteria to decide, from their base $b$ expansions, the irrationality or transcendence of real numbers. For example, it is an easy application of Ridout's Theorem in Diophantine approximations that a number of the form

$$\sum_{n\geq 0} \frac{a_n}{b^{u_n}}$$

is transcendental whenever $b > 1$ is an integer, $a_n$ are integers which are bounded, $(u_n)_{n\geq 0}$ is a sequence of positive integers with $\liminf_n u_{n+1}/u_n > 1$ and $a_n \neq 0$ for

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infinitely many $n$. A stronger result appears in [2], where it is shown that if $(u_n)_{n\geq 0}$ is a sequence of positive integers such that the estimate

$$\#\{\, n \leq x \,:\, u_n < x \,\} < cx^{\delta} \tag{1}$$

holds for large enough values of $x$, where $c$ is some constant, then

$$\sum_{n\geq 0} \frac{1}{2^{u_n}} \tag{2}$$

cannot be algebraic of degree smaller than $1/\delta$. In particular, if estimate (1) holds for a sequence of $\delta$ tending to zero, then the number shown at (2) is transcendental. While the above result is too weak to allow one to decide if

$$z = \sum_{n\geq 0} \frac{1}{2^{n^2}}$$

is quadratic or not, in [2] it is shown that most binary digits of $z^2$ are 0 and therefore $z$ is not quadratic (the stronger assertion that $z$ is fact transcendental follows from known results about about the transcendence of values of theta functions at algebraic arguments, as is shown in [3, 5]). In this note, we generalize the above result in two ways: by replacing $n^2$ with any quadratic polynomial which is integer valued, and by allowing arbitrary coefficients subject to the restriction that they are bounded and nonzero.

In what follows, for a positive integer $n$ we write $\tau(n)$ for the number of divisors of $n$. For a real number $x > 1$ we write $\log x$ for the natural logarithm of $x$. We use $p$ to denote a prime number. We use the Vinogradov symbols $\gg$ and $\ll$, as well as the Landau symbols $O$, $o$, and $\asymp$, with their regular meanings. Recall that $A \ll B$, $B \gg A$, and $A = O(B)$ are all equivalent and mean that $|A| \leq c|B|$ holds with some positive constant $c$, and that $A \asymp B$ means that both $A \ll B$ and $B \ll A$ hold. Finally, $A = o(B)$ means that the ratio $A/B$ tends to zero. All implied constants may depend on the given data.

## 1. The main result

Our main result is the following:

**Theorem 1.1.** *Let $b > 1$ be an integer, $f(X)$ be an integer valued quadratic polynomial with positive leading term and $K > 1$ be any real number. Then the number*

$$z = \sum_{n\geq 0} \frac{a_n}{b^{f(n)}} \tag{3}$$

*is not algebraic of degree $\leq 2$ provided that $a_n \in \mathbb{Z}$ is such that $1 \leq |a_n| \leq K$ holds for all positive integers $n$.*

Since the number which appears in the title of this article is

$$.2222\ldots - .101001000\ldots = \frac{2}{9} - \sum_{k\geq 0} \frac{1}{10^{k(k+3)/2+1}},$$

we get that this number is not rational or quadratic. We point out that it is not known whether this number is transcendental. (See also [1] where a base 3 variant of this number appears.) We also mention that Duverney [4] proved that if $q \geq 2$ is an integer, then

$$\sum_{n\geq 0} \frac{1}{q^{n(n+1)/2}}$$

is neither rational nor quadratic, which is a particular instance of our Theorem 1.1.

## 2. The proof

The proof of our Theorem 1.1 uses some elements from [7], although it is somewhat easier.

We begin by simplifying the problem. Replacing $n$ by $n+n_0$ where $n_0$ is any fixed positive integer, we may assume that $a_n \neq 0$ for all $n \geq n_0$. Multiplying $z$ by $b^{f(0)}$, we may assume that $f(0) = 0$. Let

$$Uz^2 + Vz + W = 0 \tag{4}$$

be an equation with integer coefficients $U$, $V$, $W$, not all zero. Let us prove that $U \neq 0$. Indeed, if $U = 0$, then $V \neq 0$ because otherwise $U = V = W = 0$. Replacing $z$ by $Vz+W$ (which can be done by replacing $a_0$ by $Va_0+W$, $a_n$ by $Va_n$ for all $n > 0$ and $K$ by $K(|V| + |W|)$), it follows that it suffices to show that $z \neq 0$. However, if $z = 0$, we then get the equation

$$\frac{A_n}{b^{f(n)}} = -\frac{a_{n+1}}{b^{f(n+1)}} - \sum_{m\geq n+2} \frac{a_m}{b^{f(m)}}, \tag{5}$$

where $A_n$ is an integer. Since $f(n+1) - f(n)$ is a linear polynomial with positive leading term which is integer valued, it follows that $f(n+1) - f(n) \geq n + c$ where $c$ is a constant. Equation (5) now shows that

$$|A_n| \ll \frac{1}{b^{f(n+1)-f(n)}} \ll \frac{1}{b^n},$$

which for large $n$ implies that $A_n = 0$. Using again equation (5) we deduce that

$$\frac{a_{n+1}}{b^{f(n+1)}} = -\sum_{m\geq n+2} \frac{a_m}{b^{f(m)}},$$

which, by the same argument as above, leads to

$$|a_{n+1}| \ll \frac{1}{b^{f(n+2)-f(n+1)}} \ll \frac{1}{b^n},$$

which is impossible for large $n$ because $|a_{n+1}| \geq 1$.

We may therefore assume that $U \neq 0$. The above equation (4) is equivalent to

$$(Uz + V)^2 + (4UW - V^2) = 0.$$

By replacing $z$ by $Uz+V$ (which can be done by replacing $a_0$ by $Ua_0+V$, $a_n$ by $Ua_n$ for all $n > 0$ and $K$ by $K(|U| + |V|)$), it follows that it suffices to show that $z^2 \notin \mathbb{Z}$. Let $d = \gcd\{ f(n) \mid n \geq 0 \}$. By replacing $f(x)$ by $f(x)/d$ and $b$ by $b^d$, it follows that we may assume that $d = 1$. Since $d = 1$ and $f(0) = 0$, it follows that we may write $f(X) = aX(X + 1)/2 + bX$, where $a = f(2) - f(1) \in \mathbb{Z} \setminus \{0\}$, $b = 2f(1) - f(2) \in \mathbb{Z}$, $a$ and $b$ are coprime if $a$ is odd and $a/2$ and $b$ are coprime if $a$ is even. The relation $z^2 \in \mathbb{Z}$ is equivalent to

$$\sum_{n\geq 0} \frac{c_n}{b^n} \in \mathbb{Z}, \tag{6}$$

where

$$c_n := \sum_{\substack{(u,v)\\ f(u)+f(v)=n}} a_u a_v.$$

We observe that $n = f(u) + f(v)$ if and only if

$$4an + (a + 2b)^2 = (au + av + a + 2b)^2 + (au - av)^2.$$

Thus, if we denote by $r_2(n)$ the number of ways of writing $n$ as a sum of two squares of integers, we have $c_n \ll r_2(4an + (a + 2b)^2)$. Furthermore, it is known that if

$$n = \prod_{i=1}^{k} p_i^{\alpha_i}$$

is the prime factorization of $n$, then $r_2(n) = 0$ if there exists $j \in \{1, \dots, k\}$ such that $p_j \equiv 3 \pmod 4$ and $\alpha_j$ is odd, and

$$r_2(n) \leq \{ (u, v) \in \mathbb{Z}^2 \, : \, u^2 + v^2 = n \} = 4 \prod_{\substack{1\leq i\leq k\\ p_i\equiv 1 \pmod 4}} (\alpha_i + 1) \leq \tau(n),$$

otherwise. In particular, $c_n \ll \tau(4an + (a + 2b)^2)$. We shall prove the theorem only when $a$ is odd, and we will indicate how to adapt the proof when $a$ is even. Our strategy will be to prove that Theorem 1.1 is a consequence of the following lemma.

**Lemma 2.1.** *For a positive real number $x$ we write $t(x) := \log x$, $m(x) := \lfloor t(x)^{1/3} \rfloor$ and $s(x) := \lfloor t^2(x) \rfloor$. There exists an infinite set $\mathcal{A}$ of positive integers $n$ such that the following properties hold.*

(i) $1 \le |c_n| \ll 1$,

(ii) $c_{n \pm i} = 0$ *for all* $i = 1, \dots, m(n)$,

(iii) $\tau(4a(n+i) + (a+2b)^2) < \exp(t(n)^{1/4})$ *for all* $i = 1, \dots, s(n)$.

Let us start by showing that, as claimed, in case $a$ is odd Theorem 1.1 is a consequence of this lemma. Suppose, hence, Lemma 2.1 proved. For $n \in \mathcal{A}$ write equation (6) as

$$\sum_{m<n} \frac{c_m}{b^m} \in -\sum_{m \ge n} \frac{c_m}{b^m} + \mathbb{Z}.$$

By condition (ii) of Lemma 2.1, the above equation leads to an equation of the form

$$\frac{B_n}{b^{n-m(n)}} = \frac{c_n}{b^n} + \sum_{n+m(n) \le m \le n+s(n)} \frac{c_m}{b^m} + \sum_{m > n+s(n)} \frac{c_m}{b},$$

where $B_n$ is an integer. Clearly, by condition (iii) of Lemma 2.1, we have

$$\sum_{n+m(n) \le m \le n+s(n)} \frac{c_m}{b^m} \ll \sum_{n+m(n) \le m \le n+s(n)} \frac{\tau(4am + (a+2b)^2)}{b^m}$$
$$\ll \frac{s(n) \exp(t(n)^{1/4})}{b^{n+m(n)}} < \frac{1}{b^{n+\frac{m(n)}{2}}},$$

where the last inequality above holds for large $n$. Once $m$ is large, we have that $\tau(4am + (a+2b)^2) < m$; hence,

$$\sum_{m > n+s(n)} \frac{c_m}{b^m} \ll \sum_{m > n+s(n)} \frac{\tau(4am + (a+2b)^2)}{b^m} < \sum_{m > n+s(n)} \frac{m}{b^m}$$
$$\ll \frac{n}{b^{n+s(n)}} < \frac{1}{b^{n+\frac{m(n)}{2}}},$$

where the last inequality holds for large values of $n$. Thus, for large $n \in \mathcal{A}$, we have

$$\frac{B_n}{b^{n-m(n)}} = \frac{c_n}{b^n} + O\left(\frac{1}{b^{n+\frac{m(n)}{2}}}\right). \tag{7}$$

Using the fact that $|c_n| \ll 1$ (see (i) of Lemma 2.1), the above relation shows that

$$|B_n| \ll \frac{1}{b^{m(n)}},$$

and since $B_n$ is an integer, it follows that $B_n = 0$ for large $n$. This together with equation (7) leads to

$$c_n = O\Big(\frac{1}{b^{m(n)/2}}\Big),$$

which is impossible for large $n$ because by (i) of Lemma 2.1 we have that $c_n$ is a nonzero integer.

It remains to prove Lemma 2.1.

*Proof of Lemma 2.1.* Let $g(X,Y) \in \mathbb{Q}[X,Y]$ be the quadratic polynomial given by $g(X,Y) = (2aX+2b)^2 + a^2(2Y+1)^2$. We recall the following result from [6].

**Theorem 2.2.** *Let $P(X,Y) \in \mathbb{Z}[X,Y]$ be a polynomial of degree two of the form*

$$P(X,Y) = AX^2 + BXY + CY^2 + DX + EY + F$$

*with* $\gcd(A,B,C,D,E,F) = 1$*, irreducible in* $\mathbb{Q}[X,Y]$*, which represents arbitrarily large odd numbers and depends essentially on two variables. Then*

(i)
$$\frac{x}{\log x} \ll \sum_{\substack{p\le x\\ p=P(r,s)}} 1,$$
*if* $\Delta = AF^2 - BEF + CE^2 + (B^2-4AC)G = 0$ *or* $\Delta_1 = B^2 - 4AC$ *is a perfect square,*

(ii)
$$\frac{x}{(\log x)^{3/2}} \asymp \sum_{\substack{p\le x\\ p=P(r,s)}} 1,$$
*otherwise.*

One checks now immediately that

$$g(X,Y) = (4a^2)X^2 + (4a^2)Y^2 + (8ab)X + (4a^2)Y + (4b^2+a^2)$$

satisfies all the conditions (i) of the above Theorem 2.2. Let

$$\mathcal{C}(x) := \{\, p > x \,:\, p \text{ prime}, \quad p = g(r,s) \text{ for some } r,s \in \mathbb{Z}_{>0}\}.$$

It then follows that for large enough $x$, we have $\#\mathcal{C}(x) \gg x/\log x$. Of the primes in $\mathcal{C}(x)$, only a subset $\mathcal{C}_1(x)$ of cardinality $O(x^{1/2})$ satisfies that $|r-s| \le n_0 + 1 + 2|b/a|$ and $2r \le 1 + 2|b/a|$. Thus, we may look only at the primes $p \in \mathcal{C}(x) \setminus \mathcal{C}_1(x)$. Such primes satisfy the conditions $|r-s| > n_0 + 1 + 2|b/a|$ and $2r > 1 + 2|b/a|$.

If we restrict our attention to such primes, we see that the integer $r-s$ takes the same sign in a subset $\mathcal{C}_2(x)$ of them with $\#\mathcal{C}_2(x) \gg x/\log x$. We will assume that $r > s$, for the case $r < s$ can be dealt with in a similar way. Setting $u = r + s$ and $v = r - s - 1$, we note that both $u$ and $v$ are positive integers greater than $n_0$, $au + av + a + 2b = 2ar + 2b$ and $au - av = a(2s + 1)$. Thus, $p = g(r, s) = (au + av + a + 2b)^2 + (au - av)^2$. Therefore, if we set $n(p) = (p - (a + 2b)^2)/4a$ for the primes $p \in \mathcal{C}_2(x)$, we have that $n(p) = f(u) + f(v)$. We now show that for most primes in $\mathcal{C}_2(x)$, the two pairs $(u, v)$ and $(v, u)$ with $u$ and $v$ constructed as above are the only ones such that $n(p) = f(u) + f(v)$.

Since $p$ is prime, it follows that the only integer solutions $(\alpha, \beta)$ of the equation $p = \alpha^2 + \beta^2$ are $(\alpha, \beta) = (\pm\lambda, \pm\nu)$, where $\lambda = 2ar + 2b$ and $\mu = a(2s + 1)$. We may hence assume that $au_1 + av_1 + a + 2b = \varepsilon_1\lambda$ and $au_1 - av_1 = \varepsilon_2\mu$, where $\varepsilon_1, \varepsilon_2 \in \{\pm 1\}$. When $(\varepsilon_1, \varepsilon_2) \in \{(1, 1), (1, -1)\}$, we get $(u_1, v_1) = (u, v)$ and $(v, u)$, respectively, which are already accounted for. When $(\varepsilon_1, \varepsilon_2) = (-1, -1)$, we get $au_1 + av_1 + a + 2b = -2ar - 2b$, therefore $u_1 + v_1 = -2r - 1 - 2b/a$, which is impossible because the right hand side of this equation is negative and the left hand side of it is positive, while when $(\varepsilon_1, \varepsilon_2) = (-1, 1)$, we get $u_1 = s - r - 2b/a$, which is again negative because $r - s$ is positive and $> 2b/a$. Thus, for the primes $p \in \mathcal{C}_2(x)$, the corresponding numbers $n(p)$ satisfy that $c_{n(p)} = 2a_u a_v$, and since both $u$ and $v$ are larger than $n_0$, it follows that $c_{n(p)}$ fulfills (i) of Lemma 2.1. We now show that most of the numbers $n(p)$ constructed from the primes $p \in \mathcal{C}_2(x)$ fulfill both (ii) and (iii) of Lemma 2.1 when $x$ is large.

For (ii), it suffices to show that $p \pm 4ai$ is not a sum of two squares for $i = 1, \dots, \lfloor t(x)^{1/3} \rfloor$. Fix a number $i$. If $p \pm 4ai$ is a sum of two squares, then it either is coprime to all primes $q > t^2(x)$ which are congruent to 3 modulo 4, or it is divisible by the square of one such prime.

For every prime number $q$ let

$$\rho(q) = \begin{cases} 2, & \text{if} \quad t^2(x) < q < x \quad \text{and} \quad q \equiv 3 \pmod 4, \\ 1, & \text{otherwise.} \end{cases}$$

By Brun's Sieve, the number $N_i$ of primes $p \in \mathcal{C}_2(x)$ such that $p + 4ai$ is free of primes $q > t^2(x)$ which are congruent to 3 modulo 4 is

$$N_i \ll x \prod_{q<x} \left(1 - \frac{\rho(q)}{q}\right) \ll \frac{x \log\log x}{(\log x)^{3/2}},$$

and the same is true for $p - 4ai$. On the other hand, the number $N_i'$ of primes $p \in \mathcal{C}_2(x)$ such that $p + 4ai < x + 4a \log x < 2x$ is a multiple of $q^2$ for some $q > t^2(x)$, certainly does not exceed

$$N_i' \le \sum_{q>t^2(x)} \frac{2x}{q^2} \ll \frac{x}{t^2(x)} < \frac{x \log\log x}{(\log x)^{3/2}},$$

and the same is true for $p - 4ai$. If we let $i$ vary from 1 to $\lfloor t(x)^{1/3} \rfloor$, we get that the number $M$ of primes $p \in \mathcal{C}_2(x)$ such that $n(p)$ does not satisfy condition (ii) of

Lemma 2.1, verifies

$$M \leq 2 \sum_{i \leq t(x)^{1/3}} (N_i + N_i') \ll \frac{xt(x)^{1/3} \log\log x}{(\log x)^{3/2}} = \frac{x \log\log x}{(\log x)^{7/6}}.$$

Since $\#\mathcal{C}_2(x) \gg x/\log x$, we get that for most of the primes $p \in \mathcal{C}_2(x)$, the number $n(p)$ satisfies (ii) of Lemma 2.1. Finally, we will take care of condition (iii) of Lemma 2.1. Since $\#\mathcal{C}_2(x) \gg x/\log x$ and there are $O(x/\log^2 x)$ primes $p \leq x/\log x$, we may assume that every prime $p$ in $\mathcal{C}_2(x)$ satisfies $p > x/\log x$. When $p \in \mathcal{C}_2(x)$, we have that the inequality

$$\sqrt{x} < \frac{x}{\log x} < p + 4ai < x + 4at^2(x) < 2x$$

holds for all $i \leq t^2(x)$. Fix a value for $i$. Since

$$\sum_{n<2x} \tau(n) = O(x \log x),$$

it follows that only $O(x \log x \exp(-(0.5 \log x)^{1/4}))$ primes $p < x$ can exist such that

$$\tau(p + 4ai) > \exp\bigl((\log(p + 4ai))^{1/4}\bigr) > \exp((0.5 \log x)^{1/4}) \tag{8}$$

holds. Summing over $i$, we get that only $O(x(\log x)^3 \exp(-(0.5 \log x)^{1/4}))$ primes $p < x$ can exist such that inequality (8) holds for some positive integer $i \leq t^2(x)$. Since this last function is $o(x/\log x)$, and our set $\mathcal{C}_2(x)$ of primes satisfies $\#\mathcal{C}_2(x) \gg x/\log x$, it follows that for most of the primes $p \in \mathcal{C}_2(x)$, the number $n(p)$ satisfies both conditions (ii) and (iii) of Lemma 2.1. Putting $n(p)$ in $\mathcal{A}$ for such primes $p \leq x$ and letting $x$ tend to infinity, we complete the proof of the lemma. □

We end with some indications about how to proceed in the case in which $a$ is even. The proof in such case is similar to the one we have just described for $a$ odd. Only the polynomial $g(X, Y)$ is different. For example, when $a/2$ and $b$ are of different parities, then $n = f(u)+f(v)$ if and only if $an+(a/2+b)^2 = (a(u+v+1)/2+b)^2+(a(u-v)/2)^2$. We may then take $g(X, Y) = (aX + b)^2 + (a(2Y + 1)/2)^2$, and setting $u = r + s$ and $v = r - s - 1$, one checks easily that $ar + b = a(u + v + 1)/2 + b$ and $a(u - v)/2 = a(2s + 1)/2$. Hence, $an + (a/2 + b)^2 = (a(u + v + 1)/2 + b)^2 + (a(u - v)/2)^2$ whenever $an + (a/2 + b)^2 = g(r, s)$. Finally, when $a/2$ and $b$ are both odd, we then have $an/2 + ((a + 2b)/4)^2 = (au/2 + (a + 2b)/4)^2 + (av/2 + (a + 2b)/4)^2$ and we may take $g(X, Y) = (aX/2 + (a + 2b)/4)^2 + (aY/2 + (a + 2b)/4)^2$. In both cases above, one checks that condition (i) from the statement of Theorem 2.2 is fulfilled and so the previous argument extends in these cases as well.

## 3. Remarks

It can be seen that the number shown at (3) is irrational under the weaker condition that $a_n \neq 0$ for infinitely many $n$. It is probably true that the number shown at (3) is not quadratic under this weaker condition either, but we could not find a proof of this fact. It can also be seen that the present proof of Theorem 1.1 shows that our result remains also valid if instead of $a_n$ remaining bounded we impose the condition that $a_n$ does not grow too fast with respect to $n$. (For example, the conclusion of Theorem 1.1 remains true when $|a_n|$ stays smaller than a fixed power of $\log n$.) Our proof also shows that

$$\sum_{n \text{ perfect power}} \frac{a_n}{b^n},$$

where $a_n$ satisfy the hypothesis from the statement of Theorem 1.1 is not algebraic of degree at most 2. (For this, note that if $x$ is large, then there are at most $O(x^{5/6})$ positive integers $n < x$ which are a sum of two perfect powers but not a sum of two squares.) A similar method can be used to show that

$$\sum_{n \text{ powerful}} \frac{a_n}{b^n},$$

where $a_n$ satisfy again the hypothesis from the statement of Theorem 1.1 is not algebraic of degree at most 2, but we shall provide the details of such an argument with a different occasion.

More generally, one can ask if it is true that given a polynomial $f(X) \in \mathbb{Q}[X]$ which is integer valued and of degree $d \geq 3$ then

$$z = \sum_{n \geq 0} \frac{a_n}{b^{f(n)}}$$

is not algebraic of degree smaller than $d$ whenever $|a_n| \leq K$ assuming either $a_n \neq 0$ for all $n$ or just for infinitely many of them. We do not know how to deal with such problems.

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