Homogeneity of Dynamically Defined Wild Knots

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Received: May 12, 2005
Accepted: June 9, 2005

ABSTRACT

In this paper we prove that a wild knot K which is the limit set of a Kleinian group acting conformally on the unit 3-sphere, with its standard metric, is homogeneous: given two points $p, q \in K$ there exists a homeomorphism f of the sphere such that $f(K) = K$ and $f(p) = q$. We also show that if the wild knot is a fibered knot then we can choose an f which preserves the fibers.

Key words: wild knots and Kleinian groups.

2000 Mathematics Subject Classification: Primary: 57M30. Secondary: 57M45, 57Q45, 30F14.

Introduction

The birth of wild topology was in the 1920's with works of Alexander, Antoine, Artin, and Fox, among others. At that time one of the main problems was to generalize the Schoenflies Theorem. Let S be a simple closed surface in $\mathbb{R}^3$ which is homeomorphic to the unit sphere $\mathbb{S}^2$. Let h be a homeomorphism of S onto the unit sphere $\mathbb{S}^2$ in $\mathbb{R}^3$. Is there an extension $\tilde{h}$ of h such that $\tilde{h}$ is a homeomorphism of $\mathbb{R}^3$ onto

The first author was partially supported by PROMEP (SEP) and CONACyT (Mexico), grant G36357-E. The second author was partially supported by CONACyT (Mexico), grant G36357-E, a joint project CNRS-CONACyT and PAPIIT (UNAM) grant ES11140.

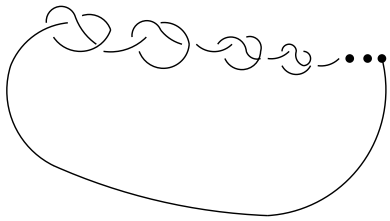


Figure 1: An example of a wild knot.

itself? Alexander proved this result in the special case that S is a finite polytope. At the same time, however, he gave his famous example, the *Alexander horned sphere*, where its unbounded complement in $\mathbb{R}^3$ is not simply connected and, in fact, its fundamental group is infinitely generated. Since the complement of $\mathbb{S}^2$ in $\mathbb{R}^3$ is not simply connected, it follows that no homeomorphism of $\mathbb{R}^3$ onto itself will send the horned sphere onto $\mathbb{S}^2$ (see [5, 9]). The work of Alexander was published in 1924.

In 1948, Artin and Fox gave the definition of *tame embeddings* and *wild embeddings* and constructed a number of surprising examples. For instance, see figure 1.

Many works by Antoine, Bing, Harold, Moise, Mazur, Brown, Montesinos, among others have contributed much to the understanding of wild sets in $\mathbb{R}^3$.

Let $K \subset \mathbb{S}^3$ be a knot. We say that a point $x \in K$ is *locally flat* if there exists an open neighborhood U of x such that there is a homeomorphism of pairs: $(U, U \cap K) \sim (\mathrm{Int}(\mathbb{B}^3), \mathrm{Int}(\mathbb{B}^1))$. Otherwise, x is said to be a *wild* point. A knot K is a wild knot if it contains at least one wild point.

We say that a knot $K \subset \mathbb{S}^3$ is *homogeneous* if given two points p, $q \in K$, there exists a homeomorphism $\psi : \mathbb{S}^3 \to \mathbb{S}^3$ such that $\psi(K) = K$ and $\psi(p) = q$.

The wild knot K given by Artin and Fox (figure 1) is not homogeneous. In fact, it contains just one wild point p, hence it is not possible to give a homeomorphism $\psi : \mathbb{S}^3 \to \mathbb{S}^3$ such that $\psi(K) = K$ and $\psi(p) = q$, $q \neq p$, since any homeomorphism sends wild points into wild points. In general, wild knots are not homogeneous.

The purpose of this paper is to show that dynamically defined wild knots (see section 1) are homogeneous. In section 2, we will give a proof of this fact.

1. Preliminaries

In this section, we will describe the construction of dynamically defined wild knots. We will begin with some basic definitions.

Let $\mathrm{Möb}(\mathbb{S}^n)$ denote the group of Möbius transformations of the n-sphere $\mathbb{S}^n = \mathbb{R}^n \cup \{\infty\}$, i.e., the group of diffeomorphisms of the n-sphere that preserves angles with respect to the standard metric. Let $\Gamma \subset \mathrm{Möb}(\mathbb{S}^n)$ be a discrete subgroup. Then $x \in \mathbb{S}^n$ is a point of discontinuity for Γ if there is a neighborhood U of x such

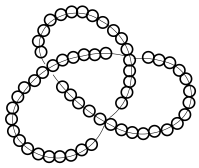


Figure 2: A pearl-necklace whose template is the trefoil knot.

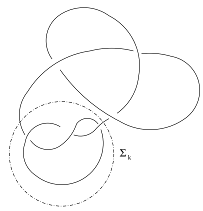


Figure 3: *Reflection with respect to* Σ_k.

that $U \cap gU \neq \emptyset$ only for finitely many $g \in \Gamma$. The *domain of discontinuity* $\Omega(\Gamma)$ consists of all points of discontinuity.

Definition 1.1 ([6]). A Kleinian group is a subgroup of $\text{Möb}(\mathbb{S}^n)$ with non-empty domain of discontinuity. The complement $\mathbb{S}^n - \Omega(\Gamma) = \Lambda(\Gamma)$ is called *limit set* of Γ.

Next, we will give the construction of dynamically defined wild knots.

Definition 1.2. A necklace T_1 of n-pearls ($n \geq 3$), is a collection of n consecutive 2-spheres $\Sigma_1, \Sigma_2, \dots, \Sigma_n$ in $\mathbb{S}^3$, such that $\Sigma_i \cap \Sigma_j = \emptyset$ ($j \neq i+1, i-1 \mod n$), except that Σ_i and Σ_{i+1} are tangent ($i = 1, 2, \dots, n-1$) and Σ_1 and Σ_n are tangent. Each 2-sphere is called a *pearl*.

If the points of tangency are joined by spherical geodesic segments in $\mathbb{S}^3$, we obtain a polygonal knot K_1. It is called the *polygonal template* of T_1. We define the *filled-in T* as $|T_1| = \bigcup_{i=1}^n B_i$, where B_i is the round closed 3-ball whose boundary ∂B_i is Σ_i.

Example 1.3. K = Trefoil knot (see figure 2).

Let Γ be the group generated by reflections I_j, through Σ_j ($j = 1, \dots, n$). Then Γ is a Kleinian group. We will describe geometrically the action of Γ.

(i) *First stage:* Observe that when we reflect with respect to each Σ_k ($k = 1, 2, \dots, n$), a mirror image of K_1 is mapped into the ball B_k (see figure 3).

After reflecting with respect to each pearl, we obtain a new necklace T_2 of $n(n-1)$ pearls, subordinate to a new knot K_2; which is in turn isotopic to the connected sum of $n+1$ copies of K_1 (see figure 4).

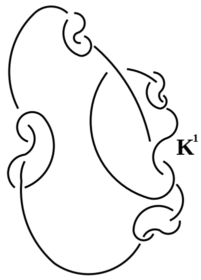


Figure 4: A schematic figure of the reflecting process first step.

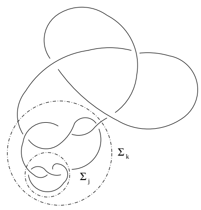


Figure 5: Reflection with respect to Σ_k after reflecting with respect to Σ_j.

(ii) *Second stage:* Now, we reflect with respect to each pearl of T_2. When we are finished, we obtain a new necklace T_3 of $n(n-1)^2$ pearls. Its template is a polygonal knot K_3; which is in turn isotopic to the connected sum of n^2-n+1 copies of K_1 and n copies of its mirror image (recall that composition of an even number of reflections is orientation-preserving). Observe that $|T_3| \subset |T_2|$ (see figure 5).

(iii) *k-th stage:* We reflect with respect of each pearl of T_k. At the end of this stage, we obtain a new necklace T_{k+1} of $n(n-1)^k$ pearls, subordinate to a polygonal knot K_{k+1}. By construction, $|T_{k+1}| \subset |T_k|$.

Then, the limit set is given by the inverse limit (see [6, 8])

$$\Lambda(\Gamma) = \varprojlim_k |T_k| = \bigcap_{k=1}^{\infty} |T_k|.$$

It has been proved (see [6, 8]) that the limit set $\Lambda(\Gamma)$ is a wild knot in the sense of Artin and Fox. It is called a *dynamically-defined wild knot.*

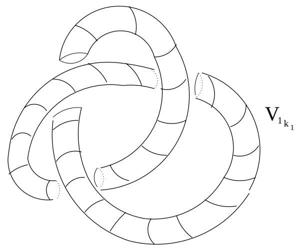


Figure 6: A tubular neighborhood as a union of "cylinders".

2. Homogeneity

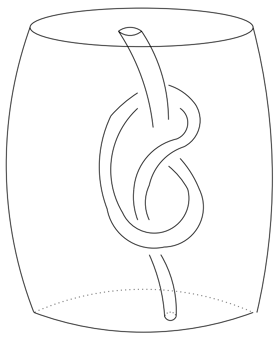
Let T_1 be a n_1-pearl necklace subordinate to the polygonal knot K_1. We can assume without loss of generality that $K_1 \subset \mathbb{R}^3 \subset \mathbb{S}^3 = \mathbb{R}^3 \cup \{\infty\}$.

Let V_1 be a closed tubular neighborhood of K_1 and $\pi_1 : V_1 \to K_1$ be the projection. We can assume that $\pi_1^{-1}(\{x\})$ is an Euclidean 2-disk of radius $r_1 > 0$ independent of x. If $\{p_{1_1}, \dots, p_{1_{n_1}}\}$ are the points of tangency of consecutive pearls of T_1, we can also assume that $\pi_1^{-1}(\{p_{1_j}\})$ is tangent to the consecutive pearls at p_{1_j} $(1 \leq j \leq n_1)$. The tubular neighborhood V_1 is the union of n_1 "solid cylinders" $V_{1_1}, \dots, V_{1_{n_1}}$ where $V_{1_j} = \pi_1^{-1}(\{[p_{1_j}, p_{1_{j+1}}]\})$ is called solid cylinder, since it is homeomorphic to a solid cylinder $C = \mathbb{D}^2 \times [0,1]$ (see figure 6).

For the second stage, we have a pearl necklace T_2 with n_2 pearls subordinate to the polygonal knot K_2. Let $V_2 \subset \operatorname{Int}(V_1)$ be a closed tubular neighborhood of K_2 and $\pi_2 : V_2 \to K_2$ be the projection. We again assume that $\pi_2^{-1}(\{x\})$ is an Euclidean 2-disk of radius $r_2 > 0$ independent of x. Notice that the points $\{\, p_{1_1}, \dots, p_{1_{n_1}} \}$ are also points of tangency of consecutive pearls of T_2. We will denote by $\{\, p_{1_i,2_1}, \dots, p_{1_i,2_{n-1}} \}$ $\subset T_2$ the corresponding points of tangency of consecutive pearls of $V_{1_i} \cap V_2$, $1 \leq i \leq n$. We can again assume that $\pi_2^{-1}(\{p_{1_i,2_j}\})$ is tangent to the consecutive pearls at $p_{1_i,2_j}$ $(1 \leq i \leq n,\ 1 \leq j \leq n-1)$. The tubular neighborhood V_2 is the union of n_2 solid cylinders $V_{1_i 2_j}$ $(1 \leq i \leq n,\ 1 \leq j \leq n-1)$ where $V_{1_i 2_j} = \pi_2^{-1}(\{\, [p_{1_i 2_j}, p_{1_i 2_{j+1}}] \,\})$.

We continue inductively, so at the end of the k-th stage of the reflecting process, we have the pearl necklace T_k with n_k pearls subordinate to the polygonal knot K_k. Let V_k be a closed tubular neighborhood of K_k such that $V_k \subset \operatorname{Int}(V_{k-1})$. Let $\pi_k : V_k \to K_k$ be the projection. We assume that $\pi_k^{-1}(\{x\})$ is an Euclidean 2-disk of radius $r_k > 0$ independent of x. We will denote by $\{\, p_{1_{i_1},2_{i_2},\dots,k_1}, \dots, p_{1_{i_1},2_{i_2},\dots,k_{n-1}} \}$ $\subset T_k$ the corresponding points of tangency of consecutive pearls of $(V_{1_{i_1},2_{i_2},\dots,(k-1)_{i_{k-1}}})$ $\cap V_k$, $1 \leq i_1 \leq n$ and $1 \leq i_2, \dots i_{k-1} \leq n-1$. The tubular neighborhood V_k is the union of n_k solid cylinders $V_{1_{i_1},2_{i_2},\dots,(k-1)_{i_{k-1}},k_{i_k}}$. Notice that $\lim_{k\to\infty} r_k = 0$ and $\Lambda = \cap_{k=1}^{\infty} V_k$.

Let p, $q \in \Lambda$. There exist two sequences of solid cylinders $\{V_{1_{i_1},2_{i_2},\dots,n_{i_n}}\}$, and $\{V_{1_{j_1},2_{j_2},\dots,n_{j_n}}\}$ where $V_{1_{i_1},2_{i_2},\dots,n_{i_n}}$ and $V_{1_{j_1},2_{j_2},\dots,n_{j_n}} \in V_n$, such that $p =$



1_{i_1}

Figure 7: A solid tangle.

$\bigcap_{n=1}^{\infty} V_{1_{i_1},2_{i_2},\ldots,n_{i_n}}$ and $q = \bigcap_{n=1}^{\infty} V_{1_{j_1},2_{j_2},\ldots,n_{j_n}}$. In fact, these sequences converge to p and q respectively, with respect to the Hausdorff metric of closed sets on $\mathbb{S}^3$.

We define the homeomorphism $F_0 : (\mathbb{S}^3 - \mathrm{Int}(V_1)) \to (\mathbb{S}^3 - \mathrm{Int}(V_1))$ such that it sends $\partial V_{1_{i_1}} \cap \partial V_1$ into $\partial V_{1_{j_1}} \cap \partial V_1$, $\partial V_{1_{i_1+1}} \cap \partial V_1$ into $\partial V_{1_{j_1+1}} \cap \partial V_1$ and so on. This map also sends $\pi_1^{-1}(p_{1_{i_1}}) \cap V_1$ into $\pi_1^{-1}(p_{1_{j_1}}) \cap V_1$, $\pi_1^{-1}(p_{1_{i_1+1}}) \cap V_1$ into $\pi_1^{-1}(p_{1_{j_1+1}}) \cap V_1$ and so on.

Next, we will define a homeomorphism $f_1 : (\partial V_1 \cup \partial V_2) \to (\partial V_1 \cup \partial V_2)$ such that $f_1|_{\partial V_1} = F_0$. Let $B_{1_{k_1}} = \cup_{k_2=1}^{n-1} V_{1_{k_1} 2_{k_2}}$. The pair $(V_{1_{k_1}}, B_{1_{k_1}} \cap V_2)$ is a solid tangle; i.e. a solid cylinder with a knotted hole (see Figure 7), and it is homeomorphic to the solid tangle $(C, \tilde{K})$, where C is a solid cylinder and $\tilde{K}$ is the mirror image of the knot K via the homeomorphism $h_{1_{k_1}}$, which will be used below.

Since, F_0 sends $\partial V_{1_{k_1}} \cap V_1$ into $\partial V_{1_{l_1}} \cap V_1$, we define the map f_1 in the following way. If $k_1 \neq i_1$, then f_1 sends the pair $(\partial V_{1_{k_1}}, \partial B_{1_{k_1}} \cap V_2)$ into $(\partial V_{1_{l_1}}, \partial B_{1_{l_1}} \cap V_2)$, where $\partial V_{1_{k_1} 2_{k_2}} \cap V_2$ is sent to $\partial V_{1_{l_1} 2_{k_2}} \cap V_2$ and $(\pi_1^{-1}(p_{1_{k_1}}) - \mathrm{Int}(\pi_2^{-1}(p_{1_{k_1} 2_{k_2}})))$ is sent to $(\pi_1^{-1}(p_{1_{l_1}}) - \mathrm{Int}(\pi_2^{-1}(p_{1_{l_1} 2_{k_2}})))$. If $k_1 = i_1$, then f_1 sends the pair $(\partial V_{1_{i_1}}, \partial B_{1_{i_1}} \cap V_2)$ into $(\partial V_{1_{j_1}}, \partial B_{1_{j_1}} \cap V_2)$ such that $\partial V_{1_{i_1} 2_{i_2}} \cap V_2$ goes into $\partial V_{1_{j_1} 2_{j_2}} \cap V_2$ and $(\pi_1^{-1}(p_{1_{i_1}}) - \mathrm{Int}(\pi_2^{-1}(p_{1_{i_1} 2_{i_2}})))$ is sent to $(\pi_1^{-1}(p_{1_{j_1}}) - \mathrm{Int}(\pi_2^{-1}(p_{1_{j_1} 2_{j_2}})))$ (see figure 8).

Notice that the composition map $h_{1_{l_1}} \circ f_1 \circ h_{1_{k_1}}^{-1}$ is isotopic to the identity map $I_{(C,\tilde{K})}$ and this fact will be used to extend the map f_1 to a map $F_1 : (V_1 - \mathrm{Int}(V_2)) \to (V_1 - \mathrm{Int}(V_2))$ via the following Lemma.

Lemma 2.1. *Let M be a compact 3-manifold with $\partial M \neq \emptyset$ (no necessarily connected). Let $g : \partial M \to \partial M$ be a homeomorphism which is isotopic to the identity map $I_{\partial M}$. Then g admits an extension $G : M \to M$. Furthermore, suppose in addition that there exists a locally trivial fibration $\pi : M \to \mathbb{S}^1$ such that its restriction to ∂M is also a locally trivial fibration and that g leaves invariant the fibers in the boundary. Then g can be extended to a homeomorphism which preserves the fibers of π.*

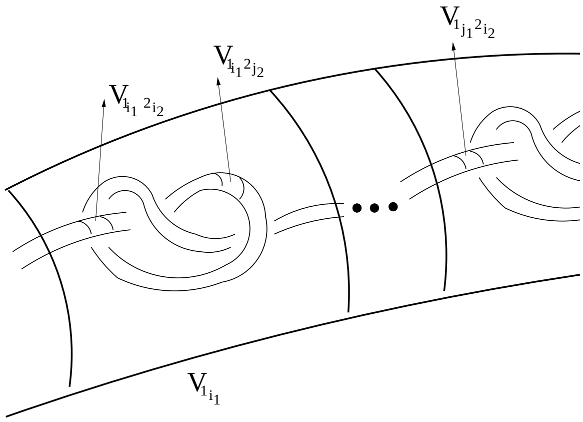


Figure 8: Geometric description of the map f_1.

Proof. Let $\psi : \partial M \times [0,1] \to M$ be a collaring of the boundary, i.e., ψ is a homeomorphism such that $\psi(x,0) = x$. Let $N = \psi(\partial M \times [0,1]) \subset M$. Let $\{g_t\},\ t \in [0,1]$, be an isotopy of g to the identity, i.e., $g_o = g$, $g_1 = I_{\partial M}$. Let $H : \partial M \times [0,1] \to \partial M \times [0,1]$ be given by the formula $H(x,t) = (g_t(x), t)$. Let $G_0 : N \to N = \psi \circ H \circ \psi^{-1}$. Then we define $G : M \to M$ as $G(y) = G_0(y)$ if $y \in N$ and $G(y) = y$ if $y \notin N$. For the rest we simply observe that the fibers of π are surfaces that meet transversally the boundary and therefore the collaring can be chosen in such a way that $\psi(\{(x,t)|t \in [0,1]\})$ is contained in a fiber for each fixed $x \in \partial M$. $\square$

We continue inductively, so at the k-stage, we have a homeomorphism $F_k : (V_k - \mathrm{Int}(V_{k+1})) \to (V_k - \mathrm{Int}(V_{k+1}))$ such that the solid cylinder $(\partial V_{1_{i_1}2_{i_2}\dots k_{i_k}}, \partial B_{1_{i_1}2_{i_2}\dots k_{i_k}} \cap V_k)$ is sent into $(\partial V_{1_{j_1}2_{j_2}\dots k_{j_k}}, \partial B_{1_{j_1}2_{j_2}\dots k_{j_k}} \cap V_k)$, in such a way that the cylinder $\partial V_{1_{i_1}2_{i_2}\dots(k+1)_{i_{k+1}}} \cap V_{k+1}$ goes into the cylinder $\partial V_{1_{j_1}2_{j_2}\dots(k+1)_{j_{k+1}}} \cap V_{k+1}$ and $(\pi_k^{-1}(p_{1_{i_1}2_{i_2}\dots k_{i_k}}) - \mathrm{Int}(\pi_{k+1}^{-1}(p_{1_{i_1}2_{i_2}\dots(k+1)_{i_{k+1}}})))$ is sent to $(\pi_k^{-1}(p_{1_{j_1}2_{j_2}\dots k_{j_k}}) - \mathrm{Int}(\pi_{k+1}^{-1}(p_{1_{j_1}2_{j_2}\dots(k+1)_{j_{k+1}}})))$.

This construction allows us to define a map $F : (\mathbb{S}^3 - \Lambda) \to (\mathbb{S}^3 - \Lambda)$ as $F(x) = F_k(x)$ if $x \in (V_k - \mathrm{Int}(V_{k+1}))$. Notice that F is a homeomorphism, since each F_k is a homeomorphism and $F_k(x) = F_{k+1}(x)$ for $x \in \partial V_{k+1}$ and for all k. We extend F to a map $\tilde{F} : \mathbb{S}^3 \to \mathbb{S}^3$ in the following way. Let $x \in \Lambda$. Then, there exists a sequence of cylinders $\{V_{1_{j_1},2_{j_2},\dots,n_{j_n}}\}$, where $V_{1_{j_1},2_{j_2},\dots,n_{j_n}} \subset V_n$ such that $x = \bigcap_{n=1}^{\infty} V_{1_{j_1},2_{j_2},\dots,n_{j_n}}$. We define $\tilde{F}(x) = \bigcap F(V_{1_{j_1},2_{j_2},\dots,n_{j_n}})$. Notice that $\tilde{F}$ is well-defined and is continuous. In fact, since F is a homeomorphism, we just need to prove that $\tilde{F}$ is continuous in Λ. Given $x \in \Lambda$ and let $\{x_n\}$ be a sequence that converges to x. We can assume, without loss of generality, that $x_n \in V_{1_{j_1},2_{j_2},\dots,n_{j_n}} \subset V_n$, hence $\tilde{F}(x_n) = F(x_n) \in F(V_{j_1,j_2,\dots,j_n})$, so $\lim_{n\to\infty} F(x_n) = \tilde{F}(x)$. Therefore, $\tilde{F}$ is continuous.

Theorem 2.2. *The map* $\tilde{F} : \mathbb{S}^3 \to \mathbb{S}^3$ *is a homeomorphism such that* $\tilde{F}|_\Lambda = \Lambda$ *and* $\tilde{F}(p) = \tilde{F}(q)$.

Proof. Since $\Lambda = \bigcap_{k=1}^{\infty} V_k$, and $\tilde{F}(V_k) = V_k$, we have that $\tilde{F}(\Lambda) = \Lambda$.

Next, we will prove that $\tilde{F}$ is a bijection. Since $\tilde{F}|_{\mathbb{S}^3-\Lambda}$ is a bijection, it is enough to prove that $\tilde{F}|_\Lambda$ is it. Let a, $b \in \Lambda$. Then $a = \cap V_{1_{l_1},2_{l_2},\dots,n_{l_n}}$ and $b = \cap V_{1_{r_1},2_{r_2},\dots,n_{r_n}}$, where $V_{1_{l_1},2_{l_2},\dots,n_{l_n}}$, $V_{1_{r_1},2_{r_2},\dots,n_{r_n}} \subset V_n$. If $\tilde{F}(a) = \tilde{F}(b)$, this implies that $\tilde{F}(V_{1_{l_1},2_{l_2},\dots,n_{l_n}}) \cap \tilde{F}(V_{1_{r_1},2_{r_2},\dots,n_{r_n}}) \neq \emptyset$. Since, $\tilde{F}|_{\mathbb{S}^3-\Lambda}$ is a homeomorphism, we have that $V_{1_{l_1},2_{l_2},\dots,n_{l_n}} \cap V_{1_{r_1},2_{r_2},\dots,n_{r_n}} \neq \emptyset$, but this is a contradiction. Hence $a = b$. For each $x = \cap V_{1_{j_1},2_{j_2},\dots,n_{j_n}} \in \Lambda$, let $x' = \cap \tilde{F}^{-1}(V_{1_{j_1},2_{j_2},\dots,n_{j_n}})$. By the above, $\tilde{F}(x') = x$. It follows that, $\tilde{F}|_\Lambda : \Lambda \to \Lambda$ is a bijection, hence $\tilde{F}$ is a bijection. Therefore $\tilde{F} : \mathbb{S}^3 \to \mathbb{S}^3$ is a continuous bijection, hence $\tilde{F}$ is a homeomorphism. □

Corollary 2.3. *Dynamically defined wild knots are homogeneous.*

Proof. Let p, $q \in \Lambda$. Then $p = \bigcap V_{1_{l_1},2_{l_2},\dots,n_{l_n}}$ and $q = \bigcap V_{1_{r_1},2_{r_2},\dots,n_{r_n}}$, where $V_{1_{l_1},2_{l_2},\dots,n_{l_n}}$, $V_{1_{r_1},2_{r_2},\dots,n_{r_n}} \in V_n$. By the above, we can construct a homeomorphism $\tilde{F} : \mathbb{S}^3 \to \mathbb{S}^3$ such that $\tilde{F}|_\Lambda = \Lambda$ and $\tilde{F}(p) = \tilde{F}(q)$. Therefore, Λ is homogeneous. □

Remark 2.4. The same method applies to prove the following theorem.

Theorem 2.5. *Let* $T_k \subset \mathbb{S}^3$ *be a nested decreasing sequence of smooth solid tori* ($k \in \mathbb{N}$)*, i.e.,* $T_{k+1} \subset \mathrm{Int}(T_k)$. *Suppose that* $\bigcap_{k=1}^{\infty} T_k := K$ *is a knot (wild or not). Then* K *is homogeneous.*

Remark 2.6. Let $\gamma \subset \Lambda$ be an orbit under the Kleinian group Γ. Notice that the action of Γ is minimal, i.e., the orbits are dense. Let x, $y \in \gamma$. Then, using the action of Γ, we can construct a homeomorphism $H : \mathbb{S}^3 \to \mathbb{S}^3$ such that $H|_\Lambda = \Lambda$ and $H(p) = H(q)$. However, the above theorem holds for any couple of points p, $q \in \Lambda$.

3. Dynamically-defined fibered wild knots

We recall that a knot or link L in $\mathbb{S}^3$ is *fibered* if there exists a locally trivial fibration $f : (\mathbb{S}^3 - L) \to \mathbb{S}^1$. We require that f be well-behaved near L, that is, each component L_i is to have a neighborhood framed as $\mathbb{D}^2 \times \mathbb{S}^1$, with $L_i \cong \{0\} \times \mathbb{S}^1$, in such a way that the restriction of f to $(\mathbb{D}^2 - \{0\}) \times \mathbb{S}^1$ is the map into $\mathbb{S}^1$ given by $(x, y) \to \frac{y}{|y|}$. It follows that each $f^{-1}(x) \cup L$, $x \in \mathbb{S}^1$, is a 2-manifold with boundary L: in fact a Seifert surface for L (see [9, page 323]).

Examples 3.1. The right-handed trefoil knot and the figure-eight knot are fibered knots with fiber the punctured torus.

For dynamically-defined wild knots we have the following result.

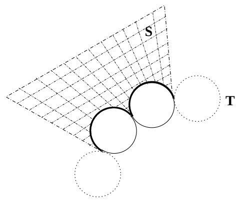


Figure 9: The fiber intersects each pearl in arcs.

Theorem 3.2. *Let Σ_1, $\Sigma_2, \dots, \Sigma_n$ be round 2-spheres in $\mathbb{S}^3$ which form a necklace whose template is a non-trivial tame fibered knot K. Let Γ be the group generated by reflections I_j on Σ_j $(j = 1, 2, \dots, n)$ and let $\widetilde{\Gamma}$ be the orientation-preserving index two subgroup of Γ. Let $\Lambda(\Gamma) = \Lambda(\widetilde{\Gamma})$ be the corresponding limit set. Then:*

(i) *There exists a locally trivial fibration $\psi : (\mathbb{S}^3 - \Lambda(\Gamma)) \to \mathbb{S}^1$, where the fiber $\Sigma_\theta = \psi^{-1}(\theta)$ is an orientable infinite-genus surface with one end.*

(ii) *$\overline{\Sigma_\theta} - \Sigma_\theta = \Lambda(\Gamma)$, where $\overline{\Sigma_\theta}$ is the closure of Σ_θ in $\mathbb{S}^3$.*

Next, we will briefly describe the fiber Σ_θ. For a proof of the above theorem, see [4].

Let T_1 be a pearl-necklace subordinate to the fibered tame knot K_1 with fiber S_1. Let $\tilde{P} : (\mathbb{S}^3 - K_1) \to \mathbb{S}^1$ be the given fibration. Observe that $\tilde{P}|_{\mathbb{S}^3 - |T_1|} \equiv P$ is a fibration and, after modifying $\widetilde{P}$ by isotopy if necessary, we can consider that the fiber S cuts each pearl $\Sigma_i \in T_1$ in arcs a_i, whose end-points are $\Sigma_{i-1} \cap \Sigma_i$ and $\Sigma_i \cap \Sigma_{i+1}$. These two points belong to the limit set (see figure 9).

The fiber $\widetilde{P}^{-1}(\theta) = P^{-1}(\theta)$ is a Seifert surface S_1^* of K_1, for each $\theta \in \mathbb{S}^1$. We suppose S_1^* is oriented.

The reflection I_j maps both a copy of $T_1 - \Sigma_j$ (called T_1^j) and a copy of S_1^* (called S_2^{*j}) into the ball B_j, for $j = 1, 2, \dots, n$. Observe that both T_1^j and S_2^{*j} have opposite orientation and that S_1^* and S_2^{*j} are joined by the arc a_j (see figure 10) which, in both surfaces, has the same orientation.

The necklaces T_1^j and T_1 are joined by the points of tangency of the pearl Σ_j and the orientation of these two points is preserved by the reflection I_j. Thus, we have obtained a new pearl-necklace isotopic to the connected sum $T_1 \# T_1^j$, whose complement also fibers over the circle with fiber the sum of S_1^* with S_2^{*j} along arc a_j, namely the fiber is $S_1^* \#_{a_j} S_2^{*j}$.

Now do this for each $j = 1, \dots, n$. At the end of the first stage, we have a new pearl-necklace T_2 whose template is the knot K_2 (see section 1). Its complement fibers over the circle with fiber the Seifert surface S_2^*, which is in turn homeomorphic to the sum of $n+1$ copies of S_1^* along the respective arcs.

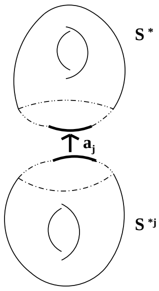


Figure 10: Sum of two surfaces S_1^* and S_2^{*j} along arc a_j.

Continuing this process, we have from the second step onwards, that $n-1$ copies of S_1^* are added along arcs to each surface S_k^{*i} (the surface corresponding to the k-th stage). Notice that in each step, the points of tangency are removed since they belong to the limit set, and the length of the arcs a_j tends to zero.

From the remarks above, we have that Σ_θ is homeomorphic to an orientable infinite-genus surface. In fact, it is the sum along arcs of an infinite number of copies of S^*.

Using the second part of Lemma 2.1 one has the following:

Theorem 3.3. *Let K_1 be a tame fibered knot with fiber S_1. Let Λ be the wild knot obtained from K_1 through a reflecting process. Then, given p, $q \in \Lambda$, there exists a homeomorphism $\tilde{F} : \mathbb{S}^3 \to \mathbb{S}^3$ such that $\tilde{F}|_\Lambda = \Lambda$, $\tilde{F}(p) = \tilde{F}(q)$ and preserves the fibers.*

Proof. Let $P : (\mathbb{S}^3 - K) \to \mathbb{S}^1$ be the given fibration. We can assume that the fiber cuts each pearl Σ_i of the pearl-necklace as in figure 9. Let p, $q \in \Lambda$. Then, there exist two sequences of solid cylinders $\{V_{1_{i_1},2_{i_2},\dots,n_{i_n}}\}$, and $\{V_{1_{j_1},2_{j_2},\dots,n_{j_n}}\}$ where $V_{1_{i_1},2_{i_2},\dots,n_{i_n}}$ and $V_{1_{j_1},2_{j_2},\dots,n_{j_n}} \in V_n$, such that $p = \bigcap_{n=1}^\infty V_{1_{i_1},2_{i_2},\dots,n_{i_n}}$ and $q = \bigcap_{n=1}^\infty V_{1_{j_1},2_{j_2},\dots,n_{j_n}}$. Let V_k be as in section 2. Consider the map $f_k : (\partial V_k \cap \partial V_{k+1}) \to (\partial V_k \cap \partial V_{k+1})$.

By the previous description, we know that $V_1 - \mathrm{Int}(V_2)$ fibers over the circle and that f_1 leaves invariant the fibers on the boundary. Then by Lemma 2.1, f_1 admits and extension $F_1 : (V_1 - \mathrm{Int}(V_2)) \to (V_1 - \mathrm{Int}(V_2))$ which preserves the fibers. We continue inductively, so at the end of the k-stage, we have that the homeomorphism f_k preserves the fibers, hence it can be extended to a map $F_k : (V_k - \mathrm{Int}(V_{k+1})) \to (V_k - \mathrm{Int}(V_{k+1}))$ which also preserves the fibers.

Let $\tilde{F} : \mathbb{S}^3 \to \mathbb{S}^3$ be as in section 2. Then, by Theorem 2.2 we have that $\tilde{F}|_\Lambda = \Lambda$, $\tilde{F}(p) = \tilde{F}(q)$. The map $F = \tilde{F}|_{\mathbb{S}^3 - \Lambda}$ defined by $F(x) = F_k(x)$ if $x \in V_k - \mathrm{Int}(V_{k+1})$ is a homeomorphism (see section 2) which preserves fibers. Therefore, the result follows. □

References

[1] E. Artin, *Zur Isotopie zweidimensionalen Flächen im* R_4, Abh. Math. Sem. Univ. Hamburg (1926), 174–177.

[2] R. H. Fox, *A quick trip through knot theory*, Topology of 3-manifolds and related topics, 1962, pp. 120–167.

[3] M. Gromov, H. B. Lawson Jr., and W. Thurston, *Hyperbolic* 4*-manifolds and conformally flat* 3*-manifolds*, Inst. Hautes Études Sci. Publ. Math. (1988), no. 68, 27–45 (1989).

[4] G. Hinojosa, *Wild knots as limit sets of Kleinian groups*, Contemp. Math., to appear.

[5] J. G. Hocking and G. S. Young, *Topology*, Addison-Wesley Publishing Co., Inc., Reading, Mass. — London **1961** (1961), ix+374.

[6] M. Kapovich, *Topological aspects of Kleinian groups in several dimensions* (1988), preprint.

[7] M. Kapovich, *Hyperbolic manifolds and discrete groups*, Progress in Mathematics, vol. 183, Birkhäuser Boston Inc., Boston, MA, 2001.

[8] B. Maskit, *Kleinian groups*, Grundlehren der Mathematischen Wissenschaften, vol. 287, Springer-Verlag, Berlin, 1988.

[9] D. Rolfsen, *Knots and links*, Publish or Perish Inc., Berkeley, Calif., 1976.

[10] T. B. Rushing, *Topological embeddings*, Academic Press, New York, 1973.