Handlebody Splittings of Compact 3-Manifolds with Boundary

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This paper is dedicated to Professor Kunio Murasugi for his 75th birthday.

ABSTRACT

The purpose of this paper is to relate several generalizations of the notion of the Heegaard splitting of a closed 3-manifold to compact, orientable 3-manifolds with nonempty boundary.

Key words: 3-manifolds with boundary, Heegaard splittings.

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1. Introduction

Throughout this paper we work in the piecewise-linear category, consisting of simplicial complexes and piecewise-linear maps.

We call a compact, connected, orientable 3-manifold M with nonempty boundary ∂M a *bordered* 3-manifold. A bordered 3-manifold H is said to be a *handlebody of genus* g iff H is the disk-sum (i.e., the boundary connected-sum) of g copies of the solid-torus $D^2 \times S^1$ (see Gross [3], Swarup [16], etc.). A handlebody of genus g is characterized as a regular neighborhood $N(P; \mathbb{R}^3)$ of a connected 1-polyhedron P with Euler characteristic $\chi(P) = 1 - g$ in the 3-dimensional Euclidean space $\mathbb{R}^3$ and as an irreducible bordered 3-manifold M with connected boundary whose fundamental group $\pi_1(M)$ is a free group of rank g (see Ochiai [10]).

It is well-known that a closed (i.e., compact, without boundary), connected, orientable 3-manifold M is decomposed into two homeomorphic handlebodies; that is,

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Proposition 1.1 (Heegaard Splittings; see Seifert-Threlfall [14], etc.)**.**

(i) *For every closed, connected, orientable 3-manifold M, there exist handlebodies H_1 and H_2 in M such that*

(a) $H_1 \cong H_2$*, that is,* $\mathrm{genus}(H_1) = \mathrm{genus}(H_2) = g$*,*

(b) $M = H_1 \cup H_2$*, and*

(c) $H_1 \cap H_2 = \partial H_1 \cap \partial H_2 = \partial H_1 = \partial H_2 = F$*, the Heegaard surface.*

(ii) *For every bordered 3-manifold M, there exist a handlebody H_1 and a disjoint union of 2-handles (i.e., 3-balls) $H_2 = h_1 \cup \cdots \cup h_s$ such that*

(a) $\mathrm{genus}(H_1) = g$*,*

(b) $M = H_1 \cup H_2$*, and*

(c) *each h_i attached to H_1 at $\partial H_1 = F$, the Heegaard surface.*

We call such a $(M; H_1, H_2; F)$ a *Heegaard splitting* (or *H-splitting*) for M of genus g, and call the minimum genus of such splittings for M the *Heegaard genus* (or *H-genus*) of M and denote it by $\mathrm{Hg}(M)$.

For an H-splitting for a closed orientable 3-manifold, Haken [4] proved the following fundamental theorem (see Hempel [5], Jaco [6], and also Ochiai [11]):

Proposition 1.2 (Haken [4])**.** *If a closed orientable 3-manifold M with a given Heegaard splitting $(M; H_1, H_2; F)$ contains an essential 2-sphere, then M contains a 2-sphere which meets F in a single circle.*

Since H_2 of an H-splitting for a bordered 3-manifold M is a disjoint union of 3-balls and so $\partial H_2 \neq F$, a Haken type theorem cannot be formulated for a H-splitting for M. Casson-Gordon [1] introduced the concept of compression bodies as a generalization of handlebodies, and for a bordered 3-manifold they defined a new Heegaard splitting using compression bodies, and formulated and proved a generalization of Haken's theorem.

On the other hand, in 1970 Downing [2] proved that every bordered 3-manifold can be decomposed into two homeomorphic handlebodies, and Roeling[13] discussed on these decompositions for bordered 3-manifolds with connected boundary. The purpose of the paper is to report the Downing's results [2] and Roeling's results [13] in slightly modified and generalized forms, and formulate a Haken type theorem for these decompositions in the way of Casson-Gordon [1].

2. Handlebody-splittings for bordered 3-manifolds

For a bordered 3-manifold M, let $\partial M = B_1 \cup B_2 \cup \cdots \cup B_m$, here B_i is a connected component for $i = 1, 2, \ldots, m$, and let $g_i = \mathrm{genus}(B_i)$.

Theorem 2.1 (Downing [2])**.** *For every bordered 3-manifold M, there exist handlebodies H_1 and H_2 in M which satisfy the following:*

(i) $H_1 \cong H_2$, *that is,* $\mathrm{genus}(H_1) = \mathrm{genus}(H_2) = g$,

(ii) $M = H_1 \cup H_2$,

(iii) $H_1 \cap H_2 = \partial H_1 \cap \partial H_2 = F_0$ *is a connected surface, the splitting-surface,*

(iv) $H_j \cap B_i = \partial H_j \cap B_i = K_{ji}$ *is a disk with g_i holes, and* $K_{1i} \cong K_{2i}$ *($j = 1, 2$, $i = 1, 2, \ldots, m$),*

(v) *the homomorphism induced from the inclusion*

$$\iota : \pi_1(K_{ji}; x_i) \to \pi_1(H_j; x_i), \quad x_i \in \partial K_{ji} \qquad (j = 1, 2,\ i = 1, 2, \ldots, m)$$

is injective.

We call such a $(M; H_1, H_2; F_0)$ a *Downing splitting* (or *D-splitting*) *for* M of genus g, and call the minimum genus of such splittings for M the *Downing genus* (or *D-genus*) of M and denote it by $\mathrm{Dg}(M)$. By the way, Roeling [13] has pointed out that $\pi_1(K_{ji}; x_i)$ in Theorem 2.1 (v) injects not only into $\pi_1(H_j; x_i)$ but also onto a free factor of $\pi_1(H_j; x_i)$, when the boundary ∂M is connected. In fact, it holds the following:

Theorem 2.2. *For every bordered 3-manifold M, there exists a D-splitting $(M; H_1, H_2; F_0)$ which satisfies the following:*

(v) *the homomorphism induced from the inclusion*

$$\iota : \pi_1(K_{ji}; x_i) \to \pi_1(H_j; x_i), \quad x_i \in \partial K_{1i} = \partial K_{2i} \qquad (j = 1, 2,\ i = 1, 2, \ldots, m)$$

is injective, and every image $\iota\pi_1(K_{ji}; x_i)$ is a free factor of the free group $\pi_1(H_j; x_i)$ of rank g ,

(vi) *there exists a tree T in F_0 connecting $x_1, x_2, \ldots, x_m$ such that the homomorphism induced from inclusion*

$$\iota : \pi_1(K_{j1} \cup \cdots \cup K_{jm} \cup T; x) \to \pi_1(H_j; x), \quad x \in T \qquad (j = 1, 2)$$

is injective, and the image is a free factor of $\pi_1(H_j; x)$.

By Zieschang [18, §3, Satz 2 and Korrollar], the above conditions (v) and (vi) are equivalent to the following geometric condition:

Theorem 2.3. *For every bordered 3-manifold M, there exists a D-splitting $(M; H_1, H_2; F_0)$ which satisfies the following (see figure 1):*

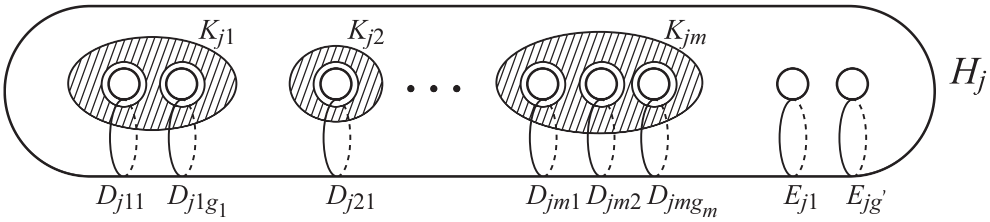

Figure 1

(vi*) *there exist systems of meridian-disks* $\mathcal{D}_{ji} = \{D_{ji1}, \dots, D_{jig_i}\}$ *(*$j = 1, 2$*,* $i = 1, 2, \dots, m$*) and* $\mathcal{E}_j = \{E_{j1}, \dots, E_{jg'}\}$*, where* $g' = g - (g_1 + \cdots + g_m)$ *of* H_j *satisfying the following:*

(a) $\mathcal{D}_{j1} \cup \cdots \cup \mathcal{D}_{jm} \cup \mathcal{E}_j$ *forms a complete system of meridian-disks of* H_j *,*

(b) $D_{jik} \cap (K_{j1} \cup \cdots \cup K_{jm}) = \partial D_{jik} \cap K_{ji}$ *consists of a single simple arc, and* $E_{jk} \cap K_{ji} = \emptyset$ *(*$j = 1, 2$*,* $i = 1, 2, \dots, m$*,* $k = 1, 2 \dots, g_i$*), and*

(c) $\mathrm{Cl}(K_{ji} - N(D_{ji1} \cup \cdots \cup D_{jig_i}; H_j))$ *is a disk (*$j = 1, 2$*).*

According to Roeling [13], we call a D-splitting for M satisfying the conditions (v) and (vi) in Theorem 2.2 or the condition (vi*) in Theorem 2.3 a *special Downing splitting* (or *SD-splitting*) for M of genus g, and call the minimum genus of such splittings for M the *special Downing genus* (or *SD-genus*) of M and denote it by $\mathrm{SDg}(M)$.

It will be noticed that for a closed, connected, orientable 3-manifold, the three splittings, an H-splitting, a D-splitting and an SD-splitting, are considered as the same one.

In order to prove Theorems 2.1 and 2.2, we need a lemma which is a generalization of Lemma 1 of Downing [2]. In proving the lemma, the notation and definitions of Downing [2] will be helpful. If g is a nonnegative integer, let $Y(g)$ be the set of all points (x, y) in the plane $\mathbb{R}^2$ which satisfy

$$x \in \{0, 1, \dots, g\} \quad \text{and} \quad -1 \le y \le 1$$

or

$$0 \le x \le g \quad \text{and} \quad |y| = 1.$$

We put

$$\begin{aligned} X(g) &= \{(x, y) \in Y(g) \mid y \ge 0\}, \\ \partial X(g) &= \{(x, 0) \in X(g)\}, \\ Z(g) &= \{(x, y) \in R^2 \mid 0 \le x \le g,\ 0 \le y \le 1\}. \end{aligned}$$

Let H be a handlebody with X a copy of $X(g)$ embedded as a PL subspace of H. X is said to be *proper* in H if $X \cap \partial H = \partial X$, and X is said to be *unknotted* if X is proper in H and the embedding of $X(g)$ can be extended to an embedding of $Z(g)$. Let $X_1 \cup \cdots \cup X_m$ be a copy of $X(g_1) \cup \cdots \cup X(g_m)$ properly embedded as a PL subspace of H. We say that $X_1 \cup \cdots \cup X_m$ is *unknotted* if the embedding of $X(g_1) \cup \cdots \cup X(g_m)$ can be extended to an embedding of $Z(g_1) \cup \cdots \cup Z(g_m)$.

Lemma 2.4 (Downing [2])**.** *Let M' be a closed, connected orientable 3-manifold, and $(M'; W_1, W_2; F)$ be an H-splitting for M'. Let S be a 1-dimensional spine of W_1. We suppose that $Y_1 \cup \cdots \cup Y_m$ is a copy of $Y(g_1) \cup \cdots \cup Y(g_m)$ embedded in S. Then there exists an ambient isotopy $\{\eta_t\}$ of M' satisfying the following:*

> $\eta_1(Y_1 \cup \cdots \cup Y_m) \cap W_j = X_{j1} \cup \cdots \cup X_{jm}$ *is a copy of* $X(g_1) \cup \cdots \cup X(g_m)$ *which is proper and unknotted in* W_j *for* $j = 1, 2$.

Proof. The case $m = 1$ is Lemma 1 of Downing [2], and the proof of the case $m \geq 2$, which is omitted here, is the same as that of the case $m = 1$. □

Proof of Theorems 2.1 and 2.2. The proof of Theorems 2.1 and 2.2 is almost similar to that of Theorem 1 of Downing [2] except for obvious modifications, but for future reference, we record it here.

Let V_i be a handlebody of genus g_i $(i = 1, 2, \ldots, m)$. We sew V_i into the boundary component B_i of M to form a closed, connected, orientable 3-manifold $M' = M \cup V_1 \cup \cdots \cup V_m$. Let Y_i be a copy of $Y(g_i)$ which is embedded as a 1-dimensional spine of V_i and we triangulate M' so that $Y_1 \cup \cdots \cup Y_m$ is contained in the 1-skeleton S.

Let $W_1 = N(S; M')$, a regular neighborhood of S in M', and let $W_2 = \mathrm{Cl}(M' - W_1; M')$. Then these form an H-splitting $(M'; W_1, W_2; F)$ for M', where $F = \partial W_1 = \partial W_2$. By Lemma 2.4, there exists an ambient isotopy $\{\eta_t\}$ of M' so that

> $\eta_1(Y_1 \cup \cdots \cup Y_m) \cap W_j = X_{j1} \cup \cdots \cup X_{jm}$ is a copy of $X(g_1) \cup \cdots \cup X(g_m)$ which is proper and unknotted in W_j $(j = 1, 2)$. (*)

We put

$$N = N(\eta_1(Y_1 \cup \cdots \cup Y_m); M'),$$
$$N_1 = N(X_{11} \cup \cdots \cup X_{1m}; W_1), \quad N_2 = N(X_{21} \cup \cdots \cup X_{2m}; W_2).$$

Then, $N = N_1 \cup N_2$, and $\mathrm{Cl}(M' - N)$ is homeomorphic to M because $\{\eta_t\}$ is an ambient isotopy. From the unknotted condition (*),

$$H_1 = \mathrm{Cl}(W_1 - N_1), \quad H_2 = \mathrm{Cl}(W_2 - N_2)$$

are homeomorphic handlebodies decomposing $\mathrm{Cl}(M' - N) = M$, and it is easily checked that this splitting satisfies the conditions (ii)–(vi) in Theorems 2.1 and 2.2, completing the proof. □

Proof of Theorem 2.3. Let $\partial K_{ji} = J_{ji0} \cup J_{ji1} \cup J_{ji2} \cup \cdots \cup J_{jig_i}$, and we assume $x_i \in J_{ji0}$, $j = 1, 2$, $i = 1, 2, \ldots, m$. Now, we can choose points $x_{jik} \in J_{jik}$ $(k = 1, 2, \ldots, g_i)$ and mutually disjoint simple proper arcs d_{jik} in K_{ji} which span x_i and x_{ik} so that $J_{ji1} \cup d_{ji1} \cup J_{ji2} \cup d_{ji2} \cup \cdots \cup J_{jig_i} \cup d_{jig_i}$ is a strong deformation retract of K_{ji}. Then, from the conditions (v) and (vi) in Theorem 2.2, the system of simple loops

$$\bigcup_{i=1}^{m} \{J_{ji1}, J_{ji2}, \ldots, J_{jig_i}\}$$

satisfies the condition of Satz 2 in Zieschang [18, §3], and we have the required systems of meridian-disks $\mathcal{D}_{ji} = \{D_{ji1}, \cdots, D_{jig_i}\}$ $(j = 1, 2,\ i = 1, 2, \ldots, m)$ and $\mathcal{E}_j = \{E_{j1}, \cdots, E_{jg'}\}$, where $g' = g - (g_1 + \cdots + g_m)$ of H_j of the condition (vi*) in Theorem 2.3.

It is easy to check that the condition (vi*) implies the conditions (v) and (vi) in Theorem 2.2, and we complete the proof. □

3. Genera of bordered 3-manifolds

From the definitions and the proofs of Theorems 2.1 and 2.2, we know:

Proposition 3.1. *For every bordered 3-manifold M, it holds the following:*

(i) $\mathrm{SDg}(M) \geq \mathrm{Dg}(M)$.

(ii) $\mathrm{SDg}(M) \geq g_1 + \cdots + g_m =$ *the total genus of* ∂M.

The following two theorems were proved by Roeling [13] when $m = 1$, and the proofs of the general case are almost the same as that of $m = 1$ under the condition (vi*).

Theorem 3.2 (Roeling [13, Theorem 1])**.** *If a bordered 3-manifold M has an SD-splitting $(M; H_1, H_2; F_0)$ of genus g , then M has an H-splitting of genus g.*

Proof. To make our notation consistent with Roeling [13], we will use the following notation in this proof and the proof of Theorem 3.4. If D is a disk, then $N(D)$ will denote a space homeomorphic to $D \times [-1, 1]$ where D corresponds to $D \times \{0\}$. We denote the 2-handles $h_1, \ldots, h_s$ by $N(D_1), \ldots, N(D_s)$, where D_k is a disk for each k, $N(D_k) \cap N(D_h) = \emptyset$ if $k \neq h$, and $N(D_k) \cap H_1 = \partial D_k \cap \partial H_1$ corresponds to $\partial D_k \times [-1, 1]$ in $N(D_k)$.

From the condition (vi*), we can choose a complete system of meridian-disks

$$\mathcal{D}_{21} \cup \cdots \cup \mathcal{D}_{2m} \cup \mathcal{E}_2$$

of H_2 also satisfying the conditions (b) and (c). Then,

$$\mathrm{Cl}\biggl(H_2 - \bigcup_{i=1}^{m} N(D_{2i1} \cup \cdots \cup D_{2ig_i}; H_2)\biggr)$$

is a handlebody of genus $g' = g - (g_1 + \cdots + g_m)$, and

$$X = \mathrm{Cl}\biggl(H_2 - \bigcup_{i=1}^{m} N(D_{2i1} \cup \cdots \cup D_{2ig_i}) - N(E_{21} \cup \cdots \cup E_{2g'})\biggr)$$

is a ball. We choose a system of properly embedded disks $\mathcal{D}' = \{D'_1, \ldots, D'_{m-1}\}$ of H_2 so that $\mathcal{D}'$ is disjoint from the complete system of meridian-disks and

$$Y = \mathrm{Cl}(X - N(D'_1 \cup \cdots \cup D'_{m-1}))$$

consists of $m-1$ balls. Now,

$$H_1^* = H_1 \cup \bigcup_{i=1}^{m} (N(D_{2i1}) \cup \cdots \cup N(D_{2ig_i}))$$

is a handlebody of genus g by the condition (b). Now we conclude that

$$M \cong H_1^* \cup N(E_{21}) \cup \cdots \cup N(E_{2g'}) \cup N(D'_1) \cup \cdots \cup N(D'_{m-1}),$$

because each ball of Y meets this in a disk on their common boundary. □

Corollary 3.3. *For every bordered 3-manifold M, it holds that*

$$\mathrm{Hg}(M) \le \mathrm{SDg}(M).$$

Theorem 3.4 (Roeling [13, Theorem 2])**.** *If a bordered 3-manifold M has an H-splitting $(M; H_1, H_2; F)$ of genus g, then M has a D-splitting of genus g .*

Proof. If $m = 1$, the result has been proved in Roeling [12, Theorem 2], so we assume $m \ge 2$. We suppose that $H_2 = N(D_1) \cup \cdots \cup N(D_s)$. Then,

$$\mathrm{Cl}\biggl(\partial H_1 - \bigcup_{k=1}^{s} N(D_k)\biggr)$$

consists of m connected orientable surfaces, say, $S_1, \ldots, S_m$, where

$$S_i = B_i \cap \mathrm{Cl}\biggl(\partial H_1 - \bigcup_{k=1}^{s} N(D_k)\biggr).$$

Here, B_i is a connected component of ∂M. Let $\alpha_k \cup \beta_k = \partial D_k \times \{-1\} \cup \partial D_k \times \{1\}$ be simple closed curves for $k = 1, \ldots, s$. Then, 2-handles can be classified into two types as follows:

(type I) α_k and β_k are contained in some S_i, or

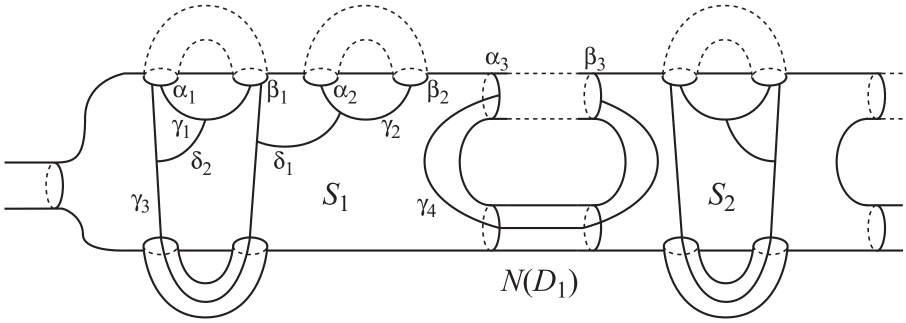

Figure 2

(type II) α_k is contained in a S_i and β_k is contained in a S_j with $i \neq j$.

We can choose $m-1$ handles, say, $N(D_1) = D_1 \times [-1,1], \dots, N(D_{m-1}) = D_{m-1} \times [-1,1]$, so that

$$S = S_1 \cup \dots \cup S_m \cup \partial D_1 \times [-1,1] \cup \dots \cup \partial D_{m-1} \times [-1,1]$$

is connected, because $\partial H_1 = F$ is connected.

Now we choose simple, properly embedded, pairwise disjoint arcs $\gamma_m, \gamma_{m+1}, \dots, \gamma_s$ in S so that

(i) each γ_k joins α_k to β_k,

(ii) if the 2-handle $N(D_k)$ is of type I, and α_k and β_k are contained in S_i, then $\gamma_k \subset S_i$, and if $N(D_k)$ is of type II, then γ_k crosses some of $\partial D_1, \dots, \partial D_{m-1}$ transversally, and

(iii) $T' = \mathrm{Cl}\big(S - \bigcup_{k=m}^{s} N(\gamma_k; S)\big)$ is connected orientable surface of genus $g-s+m-1$. See figure 2.

Now, we can check that

$$\chi(S) = 2-2g, \quad \chi(T') = 2-2g+(s-m+1).$$

As indicated in figure 2, we choose properly embedded, pairwise disjoint simple arcs $\delta_1, \delta_2, \dots, \delta_{s-m}$ in T' so that

(iv) each δ_k joins some γ_j to γ_r $(j \neq r)$,

(v) $T = \mathrm{Cl}(T' - \bigcup_{k=1}^{s-m} N(\delta_k; T'))$ is connected.

Then, we know that

(vi) $S_i^* = Cl(S_i - \bigcup_{k=1}^{s} N(\gamma_k; S_i) - \bigcup_{k=1}^{s-m} N(\delta_k; S_i))$ is a disk with g_i holes for $i = 1, \cdots, m$.

(vii) the inclusion induced homomorphism $\mu_* : \pi_1(S_i^*) \to \pi_1(S_i)$ is an injection.

[type I]: We assume that the inclusion induced homomorphism

$$\nu_i : \pi_1(S_i^*) \to \pi_1(H_1)$$

is not injective for some $i \in \{1, \dots, m\}$. Then, $\nu_i \mu_* : \pi_1(S_i^*) \to \pi_1(H_1)$ is not injective. We find by Dehn's lemma (see [5, 6]) a simple closed curve J in S_i^* that does not contract in S_i^* but bounds a disk E in H_1. Cutting along E, either we separate M into manifolds M_1 and M_2 with H-splittings of genuses $g(1) > 0$ and $g(2) > 0$ so that $g(1) + g(2) = g$, or we remove an 1-handle from M to get a manifold M_1 with an H-splitting of genus $g - 1$. Hence, by induction on g and the fact the theorem is trivial if $g = 1$, we are finished.

[type II]: We assume that the inclusion induced homomorphism

$$\nu_i : \pi_1(S_i^*) \to \pi_1(H_1)$$

is an injection for every $i = 1, \dots, m$. Then, $\nu_i \mu_* : \pi_1(S_i^*) \to \pi_1(H_1)$ is an injection. Let

$$H_2^* = \Big(\bigcup_{k=1}^{s} N(D_k)\Big) \cup \Big(\bigcup_{k=m}^{s} N(\gamma_k; H_1)\Big) \cup \Big(\bigcup_{k=1}^{s-m} N(\delta_k; H_1)\Big),$$

where

$$\Big[\Big(\bigcup_{k=m}^{s} N(\gamma_k; H_1)\Big) \cup \Big(\bigcup_{k=1}^{s-m} N(\delta_k; H_1)\Big)\Big] \cap S_i = \Big(\bigcup_{k=m}^{s} N(\gamma_k; S_i)\Big) \cup \Big(\bigcup_{k=1}^{s-m} N(\delta_k; S_i)\Big).$$

Let $H_1^* = \mathrm{Cl}(H_1 - H_2^*)$. Then, H_1^* and H_2^* are handlebodies of genus g and $M = H_1^* \cup H_2^*$. Since the pair $(H_1^*, H_1^* \cap B_i)$ is homeomorphic to (H_1, S_i^*), we have that $\pi_1(H_1^* \cap B_i)$ injects to $\pi_1(H_1^*)$. On the other hand, from our construction, we know that

$$H_2^* \cap B_i = \Big(\bigcup_{k=1}^{s} (D_k \times \{-1, 1\})\Big) \cup \Big(\bigcup_{k=m}^{s} N(\gamma_k; S_i)\Big) \cup \Big(\bigcup_{k=1}^{s-m} N(\delta_k; S_i)\Big)$$

is a disk with g_i holes for every $i = 1, \dots, m$, and the inclusion induced homomorphism $\pi_1(H_2^* \cap B_i) \to \pi_1(H_2^*)$ is injective. Hence, M has a D-splitting of genus g. This completes the proof. □

Corollary 3.5. *For every bordered 3-manifold M, it holds that*

$$\mathrm{Dg}(M) \le \mathrm{Hg}(M) \le \mathrm{SDg}(M).$$

Closed 3-manifolds of H-genus 0 are characterized as the 3-dimensional sphere $\mathbb{S}^3$. Corresponding to this fact, it holds the following:

Proposition 3.6. *Let M be a bordered 3-manifold with m boundary components.*

$$\begin{aligned} \mathrm{SDg}(M) = 0 &\iff \mathrm{Hg}(M) = 0 \\ &\iff M = \mathbb{S}^3 \textit{ with } m \textit{ holes} \\ &\iff M \textit{ is the connected sum of } m \textit{ copies of the 3-ball } D^3. \end{aligned}$$

Two H-splittings $(M; H_1, H_2; F)$ and $(M; H_1', H_2'; F')$ for a 3-manifold M are said to be *equivalent*, if there exists a homeomorphism $\psi : M \to M$ with $\psi(F) = F'$. Let $(M; H_1, H_2; F)$ be an H-splitting for M of genus g, and let $(\mathbb{S}^3; U_1, U_2; T^2)$ be an H-splitting for the 3-sphere $\mathbb{S}^3$ of genus 1. Remove a 3-ball from M and a 3-ball from $\mathbb{S}^3$, choosing these 3-balls so that they meet the respective Heegaard surfaces in disks. Then, if we use these 3-balls to form the connected sum $M\#\mathbb{S}^3 \cong M$ of M and $\mathbb{S}^3$, we shall obtain a new H-splitting for M with Heegaard surface $F\#T^2$ of genus $g+1$, and we denote this splitting by $(M; H_1, H_2; F)\#(\mathbb{S}^3; U_1, U_2; T^2)$. This process is called stabilizing; it may be iterated to obtain H-splittings $(M; H_1, H_2; F)\#n(\mathbb{S}^3; U_1, U_2; T^2)$ of any genus $g+n > g$. Two H-splittings $(M; H_1, H_2; F)$ and $(M; H_1', H_2'; F')$ are said to be *stably equivalent*, if there exist integers n, n' with $h = g+n = g'+n'$ so that the stabilizations $(M; H_1, H_2; F)\#n(\mathbb{S}^3; U_1, U_2; T^2)$ and $(M; H_1', H_2'; F')\#n'(\mathbb{S}^3; U_1, U_2; T^2)$ of genus h are equivalent as H-splittings. The following is known as the stabilization theorem:

Proposition 3.7 (Reidemeister [12], Singer [15])**.** *Arbitrary H-splittings for a 3-manifold M are stably equivalent.*

Similarly, we define, on D-splittings for a bordered 3-manifold, equivalence and stable equivalence relations. Two D-splittings $(M; H_1, H_2; F_0)$ and $(M; H_1', H_2'; F_0')$ for a bordered 3-manifold M are said to be *equivalent*, if there exists a homeomorphism $\psi : M \to M$ with $\psi(F_0) = F_0'$. Let $(\mathbb{D}^3; A)$ be a pair of the 3-ball $\mathbb{D}^3$ and a properly embedded, boundary parallel annulus A in $\mathbb{D}^3$, see figure 3. The boundary ∂A divides $\partial\mathbb{D}^3$ into two disks, say, D_+, D_-, and an annulus, say, A_0. We choose a disk $D_0^2 \subset \partial\mathbb{D}^3$ so that $D_0^2 \cap D_+$ is a disk, $D_0^2 \cap D_-$ is a disk, and $D_0^2 \cap A_0$ is also a disk. Let $(M; H_1, H_2; F_0)$ be a D-splitting for a bordered 3-manifold M, and we choose a disk $D^2 \subset \partial M$ so that $D^2 \cap \partial H_1$ is two disks and $D^2 \cap \partial H_2$ is a disk (or $D^2 \cap \partial H_2$ is two disks and $D^2 \cap \partial H_1$ is a disk). Then, if we use these disks $D^2 \subset \partial M$ and $D_0^2 \subset \partial\mathbb{D}^3$ to form the disk-sum $M\triangle\mathbb{D}^3 \cong M$ of M and $\mathbb{D}^3$, we shall obtain a new D-splitting for M with the splitting-surface $F_0 \cup A$ of genus $g(F_0)+1$, and we denote this splitting by $(M; H_1, H_2; F_0)\triangle(\mathbb{D}^3; A)$. This process is called stabilization; it may be iterated to obtain D-splittings $(M; H_1, H_2; F_0)\triangle n(\mathbb{D}^3; A)$ of any genus $g(F_0) + n$.

It will be noticed that

(i) $(M; H_1, H_2; F_0)\triangle(\mathbb{D}^3; A)$ depends on a disk $D^2 \subset \partial M$,

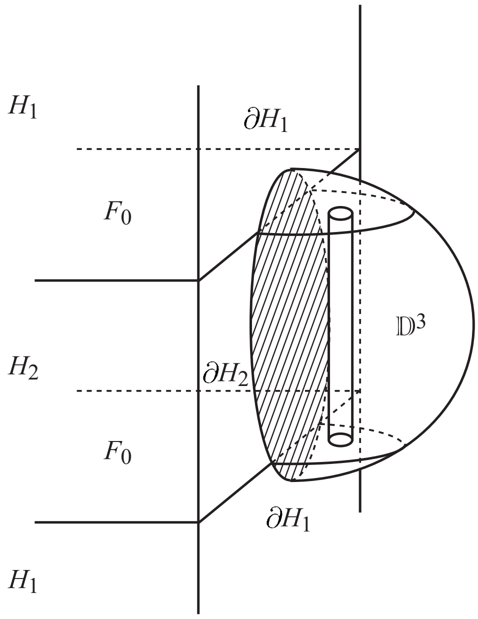

Figure 3

(ii) $(M; H_1, H_2; F_0)\#(\mathbb{S}^3; U_1, U_2; T^2)$ is equivalent to some $(M; H_1, H_2; F_0)\triangle(\mathbb{D}^3; A)$; however $(M; H_1, H_2; F_0)\triangle(\mathbb{D}^3; A)$ is not always equivalent to any $(M; H_1, H_2; F_0)$ $\#(\mathbb{S}^3; U_1, U_2; T^2)$.

Two D-splittings $(M; H_1, H_2; F_0)$ and $(M; H_1', H_2'; F_0')$ are said to be *stably equivalent*, if there exist integers n, n' with $h = g(F_0) + n = g(F_0') + n'$ and stabilizations $(M; H_1, H_2; F_0)\triangle n(\mathbb{D}^3; A)$ and $(M; H_1', H_2'; F_0')\triangle n'(\mathbb{D}^3; A)$ so that these stabilizations are equivalent.

The following is a corollary to Theorem 3.4 and Proposition 3.6:

Corollary 3.8. *Arbitrary D-splittings for a bordered 3-manifold M are stably equivalent.*

4. Haken Type Theorem (1)

A 2-sphere in a 3-manifold M is *essential* if it does not bound a 3-ball in M. A 3-manifold M is *irreducible* if it contains no essential 2-sphere.

The following corresponds to the Haken Theorem 1.2.

Theorem 4.1. *Let $(M; H_1, H_2; F_0)$ be an SD-splitting for a bordered 3-manifold M. If there exists an essential 2-sphere in M, then there exists an essential 2-sphere Σ in M such that $\Sigma \cap F_0$ consists of a single loop.*

Proof. We will give a mild generalization of this theorem in Theorem 4.3 below, and so we will not include a proof of Theorem 4.1, but simply refer the reader to Jaco's account of Haken's proof [4, chapter II] or the proof of Theorem 4.3 below. □

Corollary 4.2. *Suppose that a bordered 3-manifold M has a decomposition*

$$M = M_1 \# \cdots \# M_u$$

as a connected sum. Then it holds that

$$\mathrm{SDg}(M) = \mathrm{SDg}(M_1) + \cdots + \mathrm{SDg}(M_u).$$

Let F_0 be a compact orientable surface, and let $\mathcal{J}_1$ and $\mathcal{J}_2$ be proper 1-dimensional submanifolds in F_0. We shall say that $\mathcal{J}_1$ and $\mathcal{J}_2$ are in *reduced position*, if $\mathcal{J}_1 \cap \mathcal{J}_2$ consists of a finite number of points in which $\mathcal{J}_1$ and $\mathcal{J}_2$ cross one another, and there is no disk on F_0 whose boundary consists of an arc in $\mathcal{J}_1$ and an arc in $\mathcal{J}_2$.

Let M be a bordered 3-manifold and let $(M; H_1, H_2; F_0)$ be an SD-splitting for M. We call the complete systems of meridian-disks $\mathcal{D}_1$ of H_1 and $\mathcal{D}_2$ of H_2 which satisfy the condition (vi*) a *special* complete systems of meridian-disks. These special complete systems of meridian-disks $\mathcal{D}_1$ of H_1 and $\mathcal{D}_2$ of H_2 are said to be *irreducible* if $\mathcal{J}_1 = \mathcal{D}_1 \cap F_0$ and $\mathcal{J}_2 = \mathcal{D}_2 \cap F_0$ are in reduced position in F_0.

Theorem 4.3. *Let $(M; H_1, H_2; F_0)$ be an SD-splitting for a bordered 3-manifold M, and let $\mathcal{D}_j = \{D_{j1}, \ldots, D_{jg}\}$ be a special complete system of meridian-disks of H_j ($j = 1, 2$), and we suppose that $\mathcal{D}_1$ and $\mathcal{D}_2$ are irreducible. Let Σ be a disjoint union of essential 2-spheres in M. Then there exist a disjoint union of essential 2-spheres Σ^* and a complete system of meridian-disks $\mathcal{D}_2^*$ of H_2 such that*

- (i) Σ^* *is obtained from Σ by ambient 1-surgery and isotopy,*
- (ii) *each component of Σ^* meets F_0 in a single loop,*
- (iii) $\mathcal{D}_1 \cap \Sigma^* = \emptyset$, $\mathcal{D}_2^* \cap \Sigma^* = \emptyset$, *and* $\mathcal{D}_2^* \cap (F_{j1} \cup \cdots \cup F_{jm}) = \mathcal{D}_2 \cap (F_{j1} \cup \cdots \cup F_{jm})$, *where F_{ji} is the planar surface $\partial H_j \cap B_i$, B_i a connected component of ∂M.*

Proof. We choose a 1-dimensional spine S_{2i} of the planar surface F_{2i} so that S_{2i} consists of simple loops based at the point x_i and each loop intersected with D_2 at a single point ($i = 1, 2, \ldots, m$). Then we can choose a 1-dimensional spine S_2 of H_2 so that $S_2 \cap D_{2i}$ consists of a single point ($i = 1, 2, \ldots, m$) and $S_2 \cap \partial H_2 = S_{21} \cup \cdots \cup S_{2m}$. We may suppose that S_2 intersects transversally with Σ at a finite number of points. Since H_2 is a regular neighborhood of S_2 , we may assume that Σ intersects with H_2 at a finite number of disks, say, $\sigma_1, \cdots, \sigma_n$.

Let $\Sigma_0 = \mathrm{Cl}(\Sigma - (\sigma_1 \cup \cdots \cup \sigma_n); \Sigma)$. Then $\Sigma_0 \cap (D_{11} \cup \cdots \cup D_{1g})$ consists of a finite number of simple loops and proper arcs. Since H_1 is irreducible, we can remove all simple loops by cut-and-paste, and so we may assume that $\Sigma_0 \cap (D_{11} \cup \cdots \cup D_{1g})$ consists of a finite number of proper arcs, say, $\alpha_1, \dots, \alpha_k$. Since $\Sigma_0 \cap F_{1i} = \emptyset$ for $i = 1, 2, \dots, m$, we can choose an innermost arc, say, α_1, on one of $D_{11}, \dots, D_{1g}$, say, D_{11}, if $\Sigma_0 \cap (D_{11} \cup \cdots \cup D_{1g}) \neq \emptyset$. Let $\Delta \subset D_{11}$ be the disk cut off by α_1 so that

$$\Delta \cap \Sigma_0 = \partial\Delta \cap \Sigma_0 = \alpha_1,\ \Delta \cap (F_{11} \cup \cdots \cup F_{1m}) = \emptyset.$$

Now, we may apply the same argument as that of Jaco [6, II7–II9]; that is, we can deform Σ along Δ (by isotopy of type A) so that the new Σ^* does not meet at α_1. By the repetition of the procedure, we can get rid of all intersections $\alpha_1, \dots, \alpha_k$ of $\Sigma^* \cap \mathcal{D}_1$. Now, it is easy to see that the new Σ^* satisfies the conditions (i) and (ii), and the condition $\mathcal{D}_1 \cap \Sigma^* = \emptyset$ from (iii).

Since $H_2 \cap \Sigma^*$ consists of a finite number of disks and $\Sigma^* \cap (F_{21} \cup \cdots \cup F_{2m}) = \emptyset$, we can choose, if necessary, a complete system of meridian-disks $\mathcal{D}_2^*$ of H_2 so that $\mathcal{D}_2^*$ satisfies the other conditions in (iii), completing the proof. □

5. Haken type theorem (2)

A proper disk in a bordered 3-manifold M is said to be *essential* if it does not cut off a 3-ball from M. Using essential disks, Gross[3] and Swarup [16] have formulated another prime decomposition theorem under disk-sum (i.e., boundary connected sum) for a bordered 3-manifold.

Now the following question immediately comes to mind:

Question and Example 5.1. *Let $(M; H_1, H_2; F_0)$ be an SD-splitting for a bordered 3-manifold M. If there exists an essential proper disk in M, then does there exist an essential proper disk Δ in M such that $\Delta \cap F_0$ consists of a single arc?*

The answer is NO in general. The following counterexample is due to Dr. Kanji Morimoto. Let K be a simple loop on the boundary $S^1 \times S^1$ of the solid torus $D^2 \times S^1$ such that $K \cap D^2 = K \cap \partial D^2$ consists of two crossing points, where D^2 is a standard meridian-disk of $D^2 \times S^1$. Let $J \subset D^2$ be a simple proper arc joining the two points. Let $H_1 = N(K \cup J; D^2 \times S^1)$, and $H_2 = \mathrm{Cl}(D^2 \times S^1 - H_1; D^2 \times S^1)$. Then we have an SD-splitting $(D^2 \times S^1; H_1, H_2; F_0)$ for $D^2 \times S^1$ of genus 2, where F_0 is the surface $\mathrm{Cl}(\partial H_1 \cap \mathrm{Int}(D^2 \times S^1); D^2 \times S^1)$. The meridian-disk D^2 is an essential proper disk in $D^2 \times S^1$ which is unique up to ambient isotopy of $D^2 \times S^1$, and $D^2 \cap F_0$ consists of two arcs. It will be noticed that $D^2 \times S^1$ has an SD-splitting of genus 1, and the above splitting is of genus 2. □

Proposition 5.2. *Let $(M; H_1, H_2; F_0)$ be an SD-splitting for a bordered 3-manifold M. If there exists an essential 2-sphere in M which is not boundary parallel, then there exists an essential proper disk Δ in M such that $\Delta \cap F_0$ consists of a single arc.*

Proof. By Theorem 4.1 (or 4.3), we have an essential 2-sphere Σ in M such that $\Sigma \cap F_0$ consists of a single loop. Using this Σ, we can easily obtain a required essential disk Δ. □

The following lemma corresponds to Theorem 4.3.

Lemma 5.3. *Let $(M; H_1, H_2; F_0)$ be an SD-splitting for an irreducible bordered 3-manifold M. If there exists an essential proper disk in M, then there exist an essential proper disk Δ in M and a special complete system of meridian-disks $\mathcal{D}_j = \{D_{j1}, \dots, D_{jg}\}$ of H_j $(j = 1, 2)$ satisfying the following:*

(i) $\Delta \cap F_0$ *consists of a finite number of proper arcs,*

(ii) $\Delta \cap H_j$ *consists of a finite number of proper disks, and each component is essential in H_j $(j = 1, 2)$, and*

(iii) $\Delta \cap D_2 = \emptyset$.

Proof. We choose a 1-dimensional spine S_2 of H_2 in the same way as that of the proof of Theorem 4.3. Then, we may consider H_2 as a regular neighborhood of S_2.

Let $\square$ be an essential proper disk in M. We may assume that $\square$ intersects with S_2 transversally in a finite number of points, and so $\square \cap H_2$ consists of a finite number of proper disks, which are regular neighborhoods of $\square \cap S_2$ in $\square$. Now $\square \cap F_0$ consists of a finite number of proper arcs and loops. We can remove the loops in the same way as in the proof of Theorem 4.3 (see Jaco [6]), and let Δ be the new disk. It is easy to see that Δ satisfies the conditions (i) and (ii). If we cut H_2 along Δ then we have some handlebodies, and so we can choose a complete system of meridian-disks D_2 of H_2 with the condition (iii). This completes the proof. □

Using this Lemma, we can prove the following:

Proposition 5.4. *Let $(M; H_1, H_2; F_0)$ be an SD-splitting for an irreducible bordered 3-manifold M with connected boundary B of genus g. If there exists an essential proper disk in M and $\mathrm{SDg}(M) = g$, then there exists an essential proper disk Δ in M such that $\Delta \cap F_0$ consists of a single arc.*

Proof. Let $\Delta \subset M$ be an essential proper disk, and $\mathcal{D}_j$ be a special complete system of meridian-disks of H_j $(j = 1, 2)$ that satisfy the conditions of Lemma 5.3. We cut H_j along $\mathcal{D}_j$; we have a 3-ball D^3_j. On the boundary ∂D^3_j, F_{j1} appears as a disk from the condition (vi*)-(b). Using Δ we construct the required disk by the condition (iii). The proof is not so hard but fairly complicated, and we omit it here. □

As a corollary to this Proposition, we have the following characterization of handlebodies by SD-splittings.

Corollary 5.5. *Let M be an irreducible bordered 3-manifold with connected boundary B of genus g, and we suppose that M contains an essential proper disk. Then it holds that*

$$\mathrm{SDg}(M) = g \iff M \text{ is a handlebody of genus } g.$$

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