A New Proof of the Jawerth-Franke Embedding

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Received: May 28, 2007
Accepted: July 2, 2007

ABSTRACT

We present an alternative proof of the Jawerth embedding

$$F^{s_0}_{p_0q}(\mathbb{R}^n) \hookrightarrow B^{s_1}_{p_1p_0}(\mathbb{R}^n),$$

where

$$-\infty < s_1 < s_0 < \infty, \quad 0 < p_0 < p_1 \le \infty, \quad 0 < q \le \infty$$

and

$$s_0 - \frac{n}{p_0} = s_1 - \frac{n}{p_1}.$$

The original proof given in [3] uses interpolation theory. Our proof relies on wavelet decompositions and transfers the problem from function spaces to sequence spaces. Using similar techniques, we also recover the embedding of Franke [2].

Key words: Besov spaces, Triebel-Lizorkin spaces, Sobolev embedding, Jawerth-Franke embedding.

2000 Mathematics Subject Classification: 46E35.

Introduction

Let $B^s_{pq}(\mathbb{R}^n)$ and $F^s_{pq}(\mathbb{R}^n)$ denote the Besov and Triebel-Lizorkin function spaces, respectively. The classical Sobolev embedding theorem can be extended to these two scales.

Theorem 0.1. *Let* $-\infty < s_1 < s_0 < \infty$ *and* $0 < p_0 < p_1 \le \infty$ *with*

$$s_0 - \frac{n}{p_0} = s_1 - \frac{n}{p_1}. \tag{1}$$

(i) *If* $0 < q_0 \le q_1 \le \infty$, *then*

$$B^{s_0}_{p_0 q_0}(\mathbb{R}^n) \hookrightarrow B^{s_1}_{p_1 q_1}(\mathbb{R}^n).$$

(ii) *If* $0 < q_0, q_1 \le \infty$ *and* $p_1 < \infty$, *then*

$$F^{s_0}_{p_0 q_0}(\mathbb{R}^n) \hookrightarrow F^{s_1}_{p_1 q_1}(\mathbb{R}^n). \tag{2}$$

We observe that there is no condition on the fine paramters q_0, q_1 in (2). This surprising effect was first observed in full generality by Jawerth, [3]. Using (2), we may prove

$$F^{s_0}_{p_0 q}(\mathbb{R}^n) \hookrightarrow F^{s_1}_{p_1 p_1}(\mathbb{R}^n) = B^{s_1}_{p_1 p_1}(\mathbb{R}^n)$$

and

$$B^{s_0}_{p_0 p_0}(\mathbb{R}^n) = F^{s_0}_{p_0 p_0}(\mathbb{R}^n) \hookrightarrow F^{s_1}_{p_1 q}(\mathbb{R}^n)$$

for every $0 < q \le \infty$. But Jawerth [3] and Franke [2] showed that these embeddings are not optimal and may be improved.

Theorem 0.2. *Let* $-\infty < s_1 < s_0 < \infty$, $0 < p_0 < p_1 \le \infty$, *and* $0 < q \le \infty$ *with* (1).

(i) *Then*

$$F^{s_0}_{p_0 q}(\mathbb{R}^n) \hookrightarrow B^{s_1}_{p_1 p_0}(\mathbb{R}^n). \tag{3}$$

.

(ii) *If* $p_1 < \infty$, *then*

$$B^{s_0}_{p_0 p_1}(\mathbb{R}^n) \hookrightarrow F^{s_1}_{p_1 q}(\mathbb{R}^n). \tag{4}$$

The original proofs (see [2, 3]) use interpolation techniques. We rely on a different method. First, we observe that using (for example) the wavelet decomposition method, (3) and (4) are equivalent to

$$f^{s_0}_{p_0 q} \hookrightarrow b^{s_1}_{p_1 p_0} \quad \text{and} \quad b^{s_0}_{p_0 p_1} \hookrightarrow f^{s_1}_{p_1 q} \tag{5}$$

under the same restrictions on parameters s_0, s_1, p_0, p_1, q as in Theorem 0.2. Here, b^s_{pq} and f^s_{pq} stands for the sequence spaces of Besov and Triebel-Lizorkin type. We prove (5) directly using the technique of non-increasing rearrangement on a rather elementary level.

All the unimportant constants are denoted by the letter c, whose meaning may differ from one occurrence to another. If $\{a_n\}_{n=1}^\infty$ and $\{b_n\}_{n=1}^\infty$ are two sequences of positive real numbers, we write $a_n \lesssim b_n$ if, and only if, there is a positive real number $c > 0$ such that $a_n \le c\, b_n, n \in \mathbb{N}$. Furthermore, $a_n \approx b_n$ means that $a_n \lesssim b_n$ and simultaneously $b_n \lesssim a_n$.

1. Notation and definitions

We introduce the sequence spaces associated with the Besov and Triebel-Lizrokin spaces. Let $m \in \mathbb{Z}^n$ and $\nu \in \mathbb{N}_0$. Then $Q_{\nu\, m}$ denotes the closed cube in $\mathbb{R}^n$ with sides parallel to the coordinate axes, centred at $2^{-\nu}m$, and with side length $2^{-\nu}$. By $\chi_{\nu\, m} = \chi_{Q_{\nu\, m}}$ we denote the characteristic function of $Q_{\nu\, m}$. If

$$\lambda = \{\, \lambda_{\nu\, m} : \nu \in \mathbb{N}_0, m \in \mathbb{Z}^n \,\},$$

$-\infty < s < \infty$, and $0 < p, q \le \infty$, we set

$$\|\lambda \mid b^s_{pq}\| = \biggl(\sum_{\nu=0}^\infty 2^{\nu(s-\frac{n}{p})q}\Bigl(\sum_{m\in\mathbb{Z}^n} |\lambda_{\nu\, m}|^p\Bigr)^{\frac{q}{p}}\biggr)^{\frac{1}{q}},$$

appropriately modified if $p = \infty$ and/or $q = \infty$. If $p < \infty$, we define also

$$\|\lambda|f^s_{pq}\| = \biggl\|\biggl(\sum_{\nu=0}^\infty \sum_{m\in\mathbb{Z}^n} |2^{\nu s}\lambda_{\nu\, m}\chi_{\nu\, m}(\cdot)|^q\biggr)^{1/q} \biggm| L_p(\mathbb{R}^n)\biggr\|.$$

The connection between the function spaces $B^s_{pq}(\mathbb{R}^n)$, $F^s_{pq}(\mathbb{R}^n)$ and the sequence spaces b^s_{pq}, f^s_{pq} may be given by various decomposition techniques, we refer to [7, chapters 2 and 3] for details and further references.

As a result of these characterizations, (3) is equivalent to (5).

We use the technique of non-increasing rearrangement. We refer to [1, chapter 2] for details.

Definition 1.1. Let μ be the Lebesgue measure in $\mathbb{R}^n$. If h is a measurable function on $\mathbb{R}^n$, we define the non-increasing rearrangement of h through

$$h^*(t) = \sup\{\, \lambda > 0 : \mu\{x \in \mathbb{R}^n : |h(x)| > \lambda\} > t \,\}, \qquad t \in (0, \infty).$$

We denote its averages by

$$h^{**}(t) = \frac{1}{t}\int_0^t h^*(s)\, ds, \quad t > 0.$$

We shall use the following properties. The first two are very well known and their proofs may be found in [1, Proposition 1.8 in chapter 2, Theorem 3.10 in chapter 3].

Lemma 1.2. *If* $0 < p \le \infty$*, then*

$$\|h \mid L_p(\mathbb{R}^n)\| = \|h^* \mid L_p(0,\infty)\|$$

for every measurable function h*.*

Lemma 1.3. *If* $1 < p \le \infty$*, then there is a constant* c_p *such that*

$$\|h^{**} \mid L_p(0,\infty)\| \le c_p \|h^* \mid L_p(0,\infty)\|$$

for every measurable function h*.*

Lemma 1.4. *Let* h_1 *and* h_2 *be two non-negative measurable functions on* $\mathbb{R}^n$*. If* $1 \le p \le \infty$*, then*

$$\|h_1 + h_2 \mid L_p(\mathbb{R}^n)\| \le \|h_1^* + h_2^* \mid L_p(0,\infty)\|.$$

Proof. The proof follows from Theorems 3.4 and 4.6 in [1, chapter2]. □

2. Main results

In this part, we present a direct proof of the discrete versions of Jawerth and Franke embedding. We start with the Jawerth embedding.

Theorem 2.1. *Let* $-\infty < s_1 < s_0 < \infty$*,* $0 < p_0 < p_1 \le \infty$*, and* $0 < q \le \infty$*. Then*

$$f^{s_0}_{p_0 q} \hookrightarrow b^{s_1}_{p_1 p_0} \quad \textit{if} \quad s_0 - \frac{n}{p_0} = s_1 - \frac{n}{p_1}.$$

Proof. Using the elementary embedding

$$f^s_{pq_0} \hookrightarrow f^s_{pq_1} \quad \text{if} \quad 0 < q_0 \le q_1 \le \infty \tag{6}$$

and the lifting property of Besov and Triebel-Lizorkin spaces (which is even simpler in the language of sequence spaces), we may restrict ourselves to the proof of

$$f^s_{p_0\infty} \hookrightarrow b^0_{p_1 p_0}, \quad \text{where} \quad s = n\Big(\frac{1}{p_0} - \frac{1}{p_1}\Big).$$

Let $\lambda \in f^s_{p_0\infty}$ and set

$$h(x) = \sup_{\nu \in \mathbb{N}_0} 2^{\nu s} \sum_{m \in \mathbb{Z}^n} |\lambda_{\nu m}|\, \chi_{\nu m}(x).$$

Hence

$$|\lambda_{\nu\, m}| \le 2^{-\nu s} \inf_{x\in Q_{\nu\, m}} h(x), \quad \nu \in \mathbb{N}_0, \quad m \in \mathbb{Z}^n.$$

Using this notation,

$$\|\lambda \mid f^s_{p_0\infty}\| = \|h \mid L_{p_0}(\mathbb{R}^n)\|$$

and

$$\begin{aligned}\|\lambda \mid b^0_{p_1 p_0}\|^{p_0} &\le \sum_{\nu=0}^{\infty} 2^{-\nu n}\Big(\sum_{m\in\mathbb{Z}^n} \inf_{x\in Q_{\nu m}} h(x)^{p_1}\Big)^{p_0/p_1}\\ &\le \sum_{\nu=0}^{\infty} 2^{-\nu n}\Big(\sum_{k=1}^{\infty} h^*(2^{-\nu n}k)^{p_1}\Big)^{p_0/p_1}.\end{aligned}$$

Using the monotonicity of h^* and $p_0 < p_1$ we get

$$\begin{aligned}\|\lambda \mid b^0_{p_1 p_0}\|^{p_0} &\lesssim \sum_{\nu=0}^{\infty} 2^{-\nu n}\Big(\sum_{l=0}^{\infty} 2^{nl}\cdot(2^n-1)\cdot h^*(2^{-\nu n}2^{nl})^{p_1}\Big)^{p_0/p_1}\\ &\lesssim \sum_{\nu=0}^{\infty} 2^{-\nu n}\sum_{l=0}^{\infty} 2^{nl\frac{p_0}{p_1}} h^*(2^{-\nu n}2^{nl})^{p_0}.\end{aligned}$$

We substitute $j = l - \nu$ and obtain

$$\begin{aligned}\|\lambda \mid b^0_{p_1 p_0}\|^{p_0} &\lesssim \sum_{j=-\infty}^{\infty}\sum_{\nu=-j}^{\infty} 2^{-\nu n}2^{n(\nu+j)\frac{p_0}{p_1}} h^*(2^{jn})^{p_0}\\ &= \sum_{j=-\infty}^{\infty} 2^{nj\frac{p_0}{p_1}} h^*(2^{jn})^{p_0}\sum_{\nu=-j}^{\infty} 2^{n\nu\left(\frac{p_0}{p_1}-1\right)}\\ &\approx \sum_{j=-\infty}^{\infty} 2^{nj} h^*(2^{nj})^{p_0} \approx \|h^* \mid L_{p_0}(0,\infty)\|^{p_0} = \|h \mid L_{p_0}(\mathbb{R}^n)\|^{p_0}.\end{aligned}$$

If $p_1 = \infty$, only notational changes are necessary. □

Theorem 2.2. *Let* $-\infty < s_1 < s_0 < \infty, 0 < p_0 < p_1 < \infty$, *and* $0 < q \le \infty$. *Then*

$$b^{s_0}_{p_0 p_1} \hookrightarrow f^{s_1}_{p_1 q} \quad \textit{if} \quad s_0 - \frac{n}{p_0} = s_1 - \frac{n}{p_1}.$$

Proof. Using the lifting property and (6), we may suppose that $s_1 = 0$ and $0 < q < p_0$.

By Lemma 1.4, we observe that

$$\|\lambda | f^0_{p_1 q}\| = \Big\|\Big(\sum_{\nu=0}^{\infty}\sum_{m\in\mathbb{Z}^n} |\lambda_{\nu m}|^q \chi_{\nu m}(x)\Big)^{1/q} \;\Big|\; L_{p_1}(\mathbb{R}^n)\Big\|$$

may be estimated from above by

$$\Big\|\sum_{\nu=0}^{\infty}\sum_{m=0}^{\infty}\tilde{\lambda}_{\nu m}^{q}\tilde{\chi}_{\nu m}(\cdot)\ \Big|\ L_{\frac{p_1}{q}}(0,\infty)\Big\|^{1/q}, \tag{7}$$

where $\tilde{\lambda}_\nu=\{\tilde{\lambda}_{\nu m}\}_{m=0}^{\infty}$ is a non-increasing rearrangement of $\lambda_\nu=\{\lambda_{\nu m}\}_{m\in\mathbb{Z}^n}$ and $\tilde{\chi}_{\nu m}$ is a characteristic function of the interval $(2^{-\nu n}m, 2^{-\nu n}(m+1))$.

Using duality, (7) may be rewritten as

$$\sup_g\bigg(\int_0^\infty g(x)\Big(\sum_{\nu=0}^{\infty}\sum_{m=0}^{\infty}\tilde{\lambda}_{\nu m}^{q}\tilde{\chi}_{\nu m}(x)\Big)\,dx\bigg)^{1/q}=\sup_g\bigg(\sum_{\nu=0}^{\infty}\sum_{m=0}^{\infty}2^{-\nu n}\tilde{\lambda}_{\nu m}^{q}g_{\nu m}\bigg)^{1/q}, \tag{8}$$

where the supremum is taken over all non-increasing non-negative measurable functions g with $\|g\mid L_\beta(0,\infty)\|\le 1$ and $g_{\nu m}=2^{\nu n}\int g(x)\tilde{\chi}_{\nu m}(x)\,dx$. Here, β is the conjugated index to $\frac{p_1}{q}$. Similarly, α stands for the conjugated index to $\frac{p_0}{q}$.

We use twice Hölder's inequality and estimate (8) from above by

$$\bigg(\sum_{\nu=0}^{\infty}2^{-\nu n}\Big(\sum_{m=0}^{\infty}\tilde{\lambda}_{\nu m}^{p_0}\Big)^{\frac{p_1}{p_0}}\bigg)^{1/p_1}\cdot\sup_g\bigg(\sum_{\nu=0}^{\infty}2^{-\nu n}\Big(\sum_{m=0}^{\infty}g_{\nu m}^{\alpha}\Big)^{\frac{\beta}{\alpha}}\bigg)^{\frac{1}{\beta q}} \tag{9}$$

Since $s_0=n\big(\frac{1}{p_0}-\frac{1}{p_1}\big)$ and $p_1\big(s_0-\frac{n}{p_0}\big)=-n$, the first factor in (9) is equal to $\|\lambda\mid b^{s_0}_{p_0p_1}\|$. To finish the proof, we have to show that there is a number $c>0$ such that

$$\bigg(\sum_{\nu=0}^{\infty}2^{-\nu n}\Big(\sum_{m=0}^{\infty}g_{\nu m}^{\alpha}\Big)^{\frac{\beta}{\alpha}}\bigg)^{\frac{1}{\beta q}}\le c \tag{10}$$

holds for every non-increasing non-negative measurable functions g with $\|g\mid L_\beta(0,\infty)\|\le 1$. We fix such a function g. Using the monotonicity of g, we get

$$\begin{aligned}\sum_{m=0}^{\infty}g_{\nu m}^{\alpha}&=\sum_{l=0}^{\infty}\sum_{m=2^{ln}-1}^{2^{(l+1)n}}\bigg(2^{\nu n}\int_{2^{-\nu n}m}^{2^{-\nu n}(m+1)}g(x)\,dx\bigg)^{\alpha}\\&\lesssim\sum_{l=0}^{\infty}2^{ln}\bigg(2^{\nu n}\int_{2^{-\nu n}(2^{ln}-1)}^{2^{-\nu n}2^{ln}}g(x)\,dx\bigg)^{\alpha}\le\sum_{l=0}^{\infty}2^{ln}(g^{**})^{\alpha}(2^{(l-\nu)n}).\end{aligned}$$

We use $1<\beta<\alpha$, Lemma 1.3 and obtain

$$\begin{aligned}\left(\sum_{\nu=0}^{\infty}2^{-\nu n}\left(\sum_{m=0}^{\infty}g_{\nu m}^{\alpha}\right)^{\frac{\beta}{\alpha}}\right)^{1/\beta} &\le \left(\sum_{\nu=0}^{\infty}2^{-\nu n}\left(\sum_{l=0}^{\infty}2^{ln}(g^{**})^{\alpha}(2^{(l-\nu)n})\right)^{\frac{\beta}{\alpha}}\right)^{1/\beta}\\ &\le \left(\sum_{\nu=0}^{\infty}2^{-\nu n}\sum_{l=0}^{\infty}2^{ln\frac{\beta}{\alpha}}(g^{**})^{\beta}(2^{(l-\nu)n})\right)^{1/\beta}\\ &\le \left(\sum_{k=-\infty}^{\infty}2^{kn\frac{\beta}{\alpha}}\sum_{\nu=-k}^{\infty}2^{\nu n(\frac{\beta}{\alpha}-1)}(g^{**})^{\beta}(2^{kn})\right)^{1/\beta}\\ &\lesssim \left(\sum_{k=-\infty}^{\infty}2^{kn}(g^{**})^{\beta}(2^{kn})\right)^{1/\beta}\\ &\lesssim \|g^{**}\mid L_{\beta}(0,\infty)\|\le c\,\|g\mid L_{\beta}(0,\infty)\|\le c.\end{aligned}$$

Taking the $\frac{1}{q}$-power of this estimate, we finish the proof of (10). □

The Theorems 2.1 and 2.2 are sharp in the following sense.

Theorem 2.3. *Let* $-\infty<s_1<s_0<\infty$, $0<p_0<p_1\le\infty$, *and* $0<q_0,q_1\le\infty$ *with*

$$s_0-\frac{n}{p_0}=s_1-\frac{n}{p_1}.$$

(i) *If*

$$f^{s_0}_{p_0q_0}\hookrightarrow b^{s_1}_{p_1q_1}, \tag{11}$$

then $q_1\ge p_0$.

(ii) *If* $p_1<\infty$ *and*

$$b^{s_0}_{p_0q_0}\hookrightarrow f^{s_1}_{p_1q_1}, \tag{12}$$

then $q_0\le p_1$.

Remark 2.4. Using (any of) the usual decomposition techniques, the same statements hold true also for the function spaces. These results were first proved in [4].

Proof. (i) Suppose that $0<q_1<p_0<\infty$ and set

$$\lambda_{\nu m}=\begin{cases}\nu^{-\frac{1}{q_1}}2^{\nu(\frac{n}{p_1}-s_1)} & \text{if}\quad \nu\in\mathbb{N}\quad\text{and}\quad m=0,\\ 0, & \text{otherwise.}\end{cases}$$

A simple calculation shows that $\|\lambda\mid f^{s_0}_{p_0q_0}\|<\infty$ and $\|\lambda\mid b^{s_1}_{p_1q_1}\|=\infty$. Hence, (11) does not hold.

(ii) Suppose that $0<p_1<q_0\le\infty$ and set

$$\lambda_{\nu m} = \begin{cases} \nu^{-\frac{1}{p_1}} 2^{\nu(\frac{n}{p_1}-s_1)} & \text{if} \quad \nu \in \mathbb{N} \quad \text{and} \quad m = 0, \\ 0, & \text{otherwise.} \end{cases}$$

Again, it is a matter of simple calculation to show, that $\|\lambda \mid b^{s_0}_{p_0 q_0}\| < \infty$ and $\|\lambda \mid f^{s_1}_{p_1 q_1}\| = \infty$. Hence, (12) is not true. □

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