On Conjugacy of p-gonal Automorphisms of Riemann Surfaces

Grzegorz GROMADZKI

Institute of Mathematics
University of Gdańsk
Wita Stwosza 57
80-952 Gdańsk — Poland
greggrom@math.univ.gda.pl

Received: January 8, 2007
Accepted: August 30, 2007

ABSTRACT

The classical Castelnuovo-Severi theorem implies that for $g > (p-1)^2$, a p-gonal automorphism group of a cyclic p-gonal Riemann surface X of genus g is unique. Here we deal with the case $g \leq (p-1)^2$; we give a new and short proof of a result of González-Diez that a cyclic p-gonal Riemann surface of such genus has one conjugacy class of p-gonal automorphism groups in the group of automorphisms of X.

Key words: automorphisms of Riemann surfaces, fixed points, ramified coverings of Riemann surfaces, hyperellipticity.

2000 Mathematics Subject Classification: Primary 30F10; Secondary 14H37, 14H55.

Introduction

A compact Riemann surface X of genus $g \geq 2$ is said to be *cyclic p-gonal* if there is an automorphism φ of X of order p such that the orbit space X/φ is the Riemann sphere. Such automorphism is called *p-gonal automorphism* and it gives rise to a ramified covering of the Riemann sphere by X with p sheets. So Castelnuovo-Severi theorem [4] asserts that for $g > (p-1)^2$, the group generated by a p-gonal automorphism of a Riemann surface of genus g is unique as was mentioned by Accola in [1]. Here we

Supported by the grant SAB2005-0049 of the Spanish Ministry of Education and Sciences and by MTM2005-01637.

prove that for $g \leq (p-1)^2$, a cyclic p-gonal Riemann surface has one conjugacy class of p-gonal automorphism groups in the group $\mathrm{Aut}(X)$ of automorphisms of X. This result has been proved using different techniques by González-Diez in [5].

We shall use combinatorial methods based on the Riemann uniformization theorem and combinatorial theory of Fuchsian groups as in [6] (see also [3]), where the reader can find necessary notions and facts.

1. On fixed points of automorphisms of Riemann surfaces

By the Riemann uniformization theorem an arbitrary compact Riemann surface of genus g can be represented as the orbit space $\mathcal{H}/\Gamma$, where $\mathcal{H}$ is the upper half plane and Γ is a Fuchsian surface group with signature $(g; -)$. A group of automorphisms of a surface so given can be presented as Λ/Γ for some Fuchsian group Λ. So the Riemann Hurwitz formula gives at once the following easy but useful result.

Lemma 1.1. *A Riemann surface $X = \mathcal{H}/\Gamma$ is cyclic p-gonal for a prime p if and only if there exists a Fuchsian group with signature $(0; p, \overset{s}{\ldots}, p)$, where $s = 2(g+p-1)/(p-1)$, containing Γ as a normal subgroup of index p.*

Observe that a p-gonal automorphism of a Riemann surface of genus g has $2(g+p-1)/(p-1)$ fixed points. We shall use the following theorem of Macbeath [7] concerning fixed points of automorphisms of Riemann surfaces.

Theorem 1.2. *Let $X = \mathcal{H}/\Gamma$ be a Riemann surface with the automorphism group $G = \Lambda/\Gamma$ and let $x_1, \ldots, x_r$ be a set of elliptic canonical generators of Λ whose periods are $m_1, \ldots, m_r$ respectively. Let $\theta : \Lambda \to G$ be the canonical epimorphism. Then the number $F(\varphi)$ of points of X fixed by a nontrivial element φ of G is given by the formula*

$$F(\varphi) = |N_G(\langle\varphi\rangle)| \sum 1/m_i,$$

where N stands for the normalizer and the sum is taken over those i for which φ is conjugate to a power of $\theta(x_i)$.

Finally we shall use the following easy

Lemma 1.3. *Let G be a finite group of order bigger than p^2 generated by two elements a, b of prime order p. Then for the normalizer N of the group generated by a, $|N| \leq |G|/p$.*

Proof. Clearly no nontrivial power of b belongs to N since otherwise $|G| \leq p^2$. So $[G : N] \geq p$. □

2. On *p*-gonal automorphisms of Riemann surfaces

As we mentioned before for $g > (p-1)^2$, a p-gonal automorphism group of a cyclic p-gonal Riemann surface of genus g is unique. Here we deal with $g \leq (p-1)^2$.

Theorem 2.1. *A cyclic p-gonal Riemann surface of genus $g \leq (p-1)^2$, has one conjugacy class of p-gonal automorphism groups in the group* $\operatorname{Aut}(X)$ *of automorphisms of X.*

Proof. Let X be a cyclic p-gonal Riemann surface of genus $g \leq (p-1)^2$ and let $\langle a_1 \rangle, \dots, \langle a_m \rangle$ be representatives of all conjugacy classes of p-gonal automorphism groups. By the Riemann uniformization theorem $X = \mathcal{H}/\Gamma$ and by a Sylow theorem $a_1, \dots, a_m$ can be assumed to belong to a p-subgroup of $\operatorname{Aut}(X)$. Assume, to get a contradiction, that $m \geq 2$, denote $a_1 = a$, $a_2 = b$, and let $G = \langle a, b \rangle$. Then $G = \Lambda/\Gamma$, where Λ is a Fuchsian group with signature $(h; m_1, \dots, m_r)$. Let θ be the canonical projection of $\Lambda \to G$.

We shall show first that G has order p^2. In contrary assume that $|G| = n > p^2$. Then, by Lemma 1.3 and Theorem 1.2, every period of Λ produces at most n/p^2 fixed points of a or b and therefore in particular

$$4(g+p-1)/(p-1) \leq rn/p^2. \tag{1}$$

Now for $h \neq 0$, the area $\mu(\Lambda)$ of Λ satisfies $\mu(\Lambda) \geq 2\pi r(p-1)/p$ and so, by (1) and the Hurwitz-Riemann formula, $2g - 2 \geq 4p(g+p-1) \geq 12(g+2)$. Thus $h = 0$. But then $r \geq 3$.

First, let $r \geq 4$. Then $\mu(\Lambda) \geq 2\pi(-2+r(p-1)/p)$ and so, by the Hurwitz-Riemann formula, $g \geq nr(p-1)/2p - n + 1 \geq n(p-2)/p + 1 \geq n/3 + 1$. On the other hand the Hurwitz-Riemann formula and (1) gives also $2g-2 \geq -2n + 4p(g+p-1) \geq -2n + 12g$ and so $g < n/5$, a contradiction.

Now let $r = 3$. Since a and b can not be simultaneously conjugate to a power of some $\theta(x_i)$, only one proper period produces fixed points in a or in b by Theorem 1.2; assume that this is the case for a. Then since a and b have the same number of fixed points, the remaining two proper periods may produce at most n/p^2 fixed points in b. So (1) actually becomes

$$4(g+p-1)/(p-1) \leq 2n/p^2. \tag{2}$$

But for $p \geq 5$, $\mu(\Lambda) \geq 2\pi(-2 + 3(p-1)/p)$. Thus

$$\begin{aligned} 4\pi(g-1) &= n\mu(\Lambda) \\ &\geq 2\pi(-2n + 3n(p-1)/p) \\ &\geq 2\pi(-2n + 6p(g+p-1)) \end{aligned}$$

and so $g \leq n/14$. On the other hand, by the Hurwitz-Riemann formula $n = 4\pi(g-1)/\mu(\Lambda) \leq 2p(g-1)/(p-3)$ which gives $g \geq n/5$, a contradiction.

For $p = 3$, a period of Λ is at least 9 since $(0; 3, 3, 3)$ is not a signature of a Fuchsian group. But then $\mu(\Lambda) \geq 4\pi/9$ and therefore by the Hurwitz-Riemann formula $g \geq n/9$, while by (2) $g \leq n/9 - 2$, a contradiction.

So we can assume that G has order p^2 and therefore $G = \mathrm{Z}_p \oplus \mathrm{Z}_p$. Here Λ has signature $(h; p, \overset{r}{\ldots}, p)$, with $h = 0$ by the Hurwitz-Riemann formula. But then for $r \geq 5$, $\mu(\Lambda) \geq 2\pi(3p - 5)/p$ and so the Hurwitz-Riemann formula gives $g \geq p(3p - 5)/2 + 1$ which is bigger than $(p - 1)^2$. So $r \leq 4$.

However, for $r = 3$, there is a Fuchsian group Λ' with signature $(0; 3, 3, p)$ containing Λ as a subgroup and Γ as a normal subgroup by [8] and by Theorem 5.2 (i) of [2] respectively. Furthermore by N6 of [2], all canonical generators of Λ are conjugate in Λ' and so all p-gonal automorphism groups of our surface are conjugate in $\Lambda'/\Gamma \subseteq \mathrm{Aut}(X)$, a contradiction.

The case $r = 4$ is similar. Here each $\theta(x_i)$ is conjugate to a nontrivial power of a or b since otherwise a and b would have at most $3p$ fixed points in total, by Theorem 1.2, while on the other hand they should have $4p$ such points by Lemma 1.1, since by the Hurwitz-Riemann formula the genus of the corresponding surface equals $(p - 1)^2$. So, for some permutation σ,

$$\theta(x_{\sigma(1)}) = a^{\alpha}, \quad \theta(x_{\sigma(2)}) = a^{-\alpha}, \quad \theta(x_{\sigma(3)}) = b^{\beta}, \quad \theta(x_{\sigma(4)}) = b^{-\beta}.$$

But then the mappings

$$\theta(x_1) \longmapsto \theta(x_2), \quad \theta(x_2) \longmapsto \theta(x_1), \quad \theta(x_3) \longmapsto \theta(x_4), \quad \theta(x_4) \longmapsto \theta(x_3)$$

and

$$\theta(x_1) \longmapsto \theta(x_4), \quad \theta(x_2) \longmapsto \theta(x_3), \quad \theta(x_3) \longmapsto \theta(x_2), \quad \theta(x_4) \longmapsto \theta(x_1)$$

induce automorphisms of G. So by N4 of [2], we obtain that a nontrivial power of a is conjugated to a nontrivial power of b in $\mathrm{Aut}(X)$. This is a contradiction which completes the proof. $\square$

Acknowledgement. The author is grateful to Professor Antonio F. Costa for his helpful comment and to the anonymous referee for his/her useful suggestions and for pointing out some minor gaps in the proof of the main theorem.

References

[1] R. D. M. Accola, *On cyclic trigonal Riemann surfaces*, I, Trans. Amer. Math. Soc. **283** (1984), no. 2, 423–449.

[2] E. Bujalance, F. J. Cirre, and M. Conder, *On extendability of group actions on compact Riemann surfaces*, Trans. Amer. Math. Soc. **355** (2003), no. 4, 1537–1557.

[3] E. Bujalance, J. J. Etayo, J. M. Gamboa, and G. Gromadzki, *Automorphism groups of compact bordered Klein surfaces: A combinatorial approach*, Lecture Notes in Mathematics, vol. 1439, Springer-Verlag, Berlin, 1990.

[4] G. Castelnuovo, *Sulle serie algebriche di gruppi di punti appartenenti ad una curve algebraica*, Rend. Real Accad. Lincei (5) **15**. Memorie scelte, p. 509.

[5] G. González-Diez, *On prime Galois coverings of the Riemann sphere*, Ann. Mat. Pura Appl. (4) **168** (1995), 1–15.

[6] A. M. Macbeath, *Discontinuous groups and birational transformations*, Dundee Summer School (Univ. of St. Andrews, 1961).

[7] ______, *Action of automorphisms of a compact Riemann surface on the first homology group*, Bull. London Math. Soc. **5** (1973), 103–108.

[8] D. Singerman, *Finitely generated maximal Fuchsian groups*, J. London Math. Soc. (2) **6** (1972), no. 1, 29–38.