Entropy and Approximation Numbers of Embeddings of Function Spaces with Muckenhoupt Weights, I

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ABSTRACT

We study compact embeddings for weighted spaces of Besov and Triebel-Lizorkin type where the weight belongs to some Muckenhoupt $\mathcal{A}_p$ class. For weights of purely polynomial growth, both near some singular point and at infinity, we obtain sharp asymptotic estimates for the entropy numbers and approximation numbers of this embedding. The main tool is a discretization in terms of wavelet bases.

Key words: wavelet bases, Muckenhoupt weighted function spaces, compact embeddings, entropy numbers, approximation numbers.

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Introduction

In recent years, some attention has been paid to compactness of embeddings of function spaces of Sobolev type as well as to analytic and geometric quantities describing this compactness, in particular, corresponding approximation and entropy numbers. As an application D. E. Edmunds and H. Triebel [12] proposed a program to investigate the spectral properties of certain pseudo-differential operators based on the

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asymptotic behavior of entropy and approximation numbers, together with Carl's inequality and the Birman-Schwinger principle. Similar questions in the context of weighted function spaces of this type were studied by the first named author and H. Triebel, see [19], and were continued and extended by Th. Kühn, H.-G. Leopold, W. Sickel and the second author in the series of papers [23–25]. In all the above papers the authors considered the class of so-called "admissible" weights. These are smooth weights with no singular points. One can take $w(x) = (1 + |x|^2)^{\alpha/2}$, $\alpha \in \mathbb{R}$, $x \in \mathbb{R}^n$, as a prominent example.

In this paper we follow a different approach and consider weights from the Muckenhoupt class $\mathcal{A}_\infty$. In contrast to "admissible" weights the $\mathcal{A}_\infty$ weights may have local singularities, which can influence properties of the embeddings of function spaces. Now the weight $w(x) = |x|^\alpha$, $\alpha > -n$, may serve as a classical example. Weighted Besov and Triebel-Lizorkin spaces with Muckenhoupt weights are well known concepts, see [3–6, 15, 31, 32] and, more recently, [1, 2, 18]. But (the compactness of) Sobolev embeddings of such spaces were not yet studied in detail. The present paper fills in this gap. First we give a necessary and sufficient condition on the parameters and weights of the Besov spaces which guarantees compactness of the corresponding embeddings. Then we determine the exact asymptotic behavior of entropy and approximation numbers of the embeddings of the spaces with weights that have purely polynomial growth, both near some singular point and at infinity. In both cases we use the technique of discretization, i.e., we reduce the problem to the corresponding problem for some suitable sequence spaces. This can be done in terms of wavelet bases. Then one obtains estimates of the following type: if the weight is of type

$$w(x) \sim \begin{cases} |x|^\alpha & \text{if} \quad |x| \le 1, \\ |x|^\beta & \text{if} \quad |x| > 1, \end{cases} \quad \text{with} \quad \alpha > -n, \quad \beta > 0,$$

and $A^s_{p,q}$ stands for either Besov spaces $B^s_{p,q}$ or Triebel-Lizorkin spaces $F^s_{p,q}$ with $s_2 \le s_1$, $0 < p_1, p_2 < \infty$, $0 < q_1, q_2 \le \infty$, then

$$\mathrm{id} : A^{s_1}_{p_1,q_1}(\mathbb{R}^n, w) \hookrightarrow A^{s_2}_{p_2,q_2}(\mathbb{R}^n)$$

is compact if, and only if,

$$\frac{\beta}{p_1} > \frac{n}{p^*} \quad \text{and} \quad \delta > \max\left(\frac{\alpha}{p_1}, \frac{n}{p^*}\right),$$

where $\delta = s_1 - s_2 - \frac{n}{p_1} + \frac{n}{p_2}$ and $\frac{1}{p^*} = \max\left(\frac{1}{p_2} - \frac{1}{p_1}, 0\right)$, as usual. For the entropy numbers of this embedding we can prove that for $k \in \mathbb{N}$,

$$e_k\big(\mathrm{id} : A^{s_1}_{p_1,q_1}(\mathbb{R}^n, w) \hookrightarrow A^{s_2}_{p_2,q_2}(\mathbb{R}^n)\big) \sim k^{-(\frac{\beta}{np_1} + \frac{1}{p_1} - \frac{1}{p_2})} \qquad \text{if} \quad \frac{\beta}{p_1} < \delta,$$

and

$$e_k\big(\mathrm{id} : A^{s_1}_{p_1,q_1}(\mathbb{R}^n, w) \hookrightarrow A^{s_2}_{p_2,q_2}(\mathbb{R}^n)\big) \sim k^{-\frac{s_1 - s_2}{n}} \qquad \text{if} \quad \frac{\beta}{p_1} > \delta.$$

There are parallel results for the limiting case $\frac{\beta}{p_1} = \delta$ and for approximation numbers. It is remarkable that — apart from the criterion for compactness of the embedding — the parameter α, connected with the "local" singularity has no further influence on the "degree" of compactness (measured in terms of entropy or approximation numbers, respectively).

The study of entropy and approximation numbers of embeddings in the context of more general weights is postponed; likewise applications are out of the scope of the present paper.

The paper is organized as follows. In section 1 we recall basic facts about Muckenhoupt weights and weighted spaces needed later on. We also prove the wavelet characterization of Besov spaces via compactly supported wavelets. Section 2 is devoted to the continuity and compactness of the embeddings. For weights of purely polynomial growth we find simpler conditions. In the last two sections we determine exact asymptotic behavior of the entropy and approximation numbers for purely polynomial weights.

1. Weighted function spaces

First of all we need to fix some notation. By $\mathbb{N}$ we denote the set of natural numbers, by $\mathbb{N}_0$ the set $\mathbb{N} \cup \{0\}$, by $\mathbb{C}$ the complex plane, by $\mathbb{R}^n$ euclidean n-space, $n \in \mathbb{N}$, and by $\mathbb{Z}^n$ the set of all lattice points in $\mathbb{R}^n$ having integer components.

The positive part of a real function f is given by $f_+(x) = \max(f(x), 0)$. For two positive real sequences $\{a_k\}_{k\in\mathbb{N}}$ and $\{b_k\}_{k\in\mathbb{N}}$ we mean by $a_k \sim b_k$ that there exist constants $c_1, c_2 > 0$ such that $c_1\ a_k \le b_k \le c_2\ a_k$ for all $k \in \mathbb{N}$; similarly for positive functions.

Given two (quasi-)Banach spaces X and Y, we write $X \hookrightarrow Y$ if $X \subset Y$ and the natural embedding of X in Y is continuous.

All unimportant positive constants will be denoted by c, occasionally with subscripts. For convenience, let both $\mathrm{d}x$ and $|\cdot|$ stand for the (n-dimensional) Lebesgue measure in the sequel. If not otherwise indicated, log is always taken with respect to base 2.

1.1. Muckenhoupt weights

We briefly recall some fundamentals on Muckenhoupt classes $\mathcal{A}_p$.

Definition 1.1. Let w be a positive, locally integrable function on $\mathbb{R}^n$, and $1 < p < \infty$. Then w belongs to the Muckenhoupt class $\mathcal{A}_p$, if there exists a constant $0 < A < \infty$ such that for all balls B the following inequality holds:

$$\left(\frac{1}{|B|}\int_B w(x)\,\mathrm{d}x\right)^{1/p} \cdot \left(\frac{1}{|B|}\int_B w(x)^{-p'/p}\,\mathrm{d}x\right)^{1/p'} \le A,$$

where p' is the dual exponent to p given by $1/p' + 1/p = 1$ and $|B|$ stands for the Lebesgue measure of the ball B.

The limiting cases $p = 1$ and $p = \infty$ can be incorporated as follows. By a weight w we shall always mean a locally integrable function $w \in L_1^{\mathrm{loc}}(\mathbb{R}^n)$, positive a.e. in the sequel. Let M stand for the Hardy-Littlewood maximal operator given by

$$Mf(x) = \sup_{B(x,r)\in\mathcal{B}} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)| \,\mathrm{d}y, \quad x \in \mathbb{R}^n,$$

where $\mathcal{B}$ is the collection of all open balls

$$B(x,r) = \{\, y \in \mathbb{R}^n : |y - x| < r \,\}, \quad r > 0.$$

Definition 1.2. A weight w belongs to the Muckenhoupt class $\mathcal{A}_1$ if there exists a constant $0 < A < \infty$ such that the inequality

$$Mw(x) \le Aw(x)$$

holds for almost all $x \in \mathbb{R}^n$. The Muckenhoupt class $\mathcal{A}_\infty$ is given by

$$\mathcal{A}_\infty = \bigcup_{p>1} \mathcal{A}_p.$$

Since the pioneering work of Muckenhoupt [27–29], these classes of weight functions have been studied in great detail, we refer, in particular, to the monographs [16; 36, chap. 5; 37; 38, chap. 9], for a complete account on the theory of Muckenhoupt weights. We use the abbreviation

$$w(\Omega) = \int_\Omega w(x) \,\mathrm{d}x, \tag{1}$$

where $\Omega \subset \mathbb{R}^n$ is some bounded, measurable set. For convenience, we recall a few basic properties only; in particular, the class $\mathcal{A}_p$ is stable with respect to translation, dilation and multiplication by a positive scalar. Moreover, it is known:

Lemma 1.3. *Let* $1 < p < \infty$.

(i) *If* $w \in \mathcal{A}_p$, *then we have* $w^{-p'/p} \in \mathcal{A}_{p'}$, *where* $1/p + 1/p' = 1$.

(ii) $w \in \mathcal{A}_p$ *possesses the* doubling property, *i.e., there exists a constant* $c > 0$ *such that*

$$w(B_2) \le c w(B_1)$$

holds for arbitrary balls $B_1 = B(x,r)$ *and* $B_2 = B(x,2r)$ *with* $x \in \mathbb{R}^n$, $r > 0$.

(iii) *Let* $1 \le p_1 < p_2 \le \infty$. *Then we have* $\mathcal{A}_{p_1} \subset \mathcal{A}_{p_2}$.

(iv) *If $w \in \mathcal{A}_p$, then there exists some number $r < p$ such that $w \in \mathcal{A}_r$.*

Note that the somehow surprising property (iv) is closely connected with the so-called "reverse Hölder inequality," a fundamental feature of $\mathcal{A}_p$ weights, see [36, chap. 5, Prop. 3, Cor.]. In our case this fact will re-emerge in the number

$$r_w := \inf\{r \geq 1 : w \in \mathcal{A}_r\}, \quad w \in \mathcal{A}_\infty, \tag{2}$$

that plays an essential role later on. Obviously, $1 \leq r_w < \infty$, and $w \in \mathcal{A}_{r_w}$ if, and only if, $r_w = 1$ in view of (iv).

In the sequel we shall use decompositions of $\mathcal{A}_p$ weights into $\mathcal{A}_1$ weights several times, for that reason we collect a few related facts, see [13, Lemma 2.3; 36, chap. 5, §§1.4, 1.9, 5.3, 6.1], and also [38, chap. 9, Thm. 2.1, secs. 4, 5].

Lemma 1.4.

(i) *Let $1 \leq p_1, p_2 < \infty$, $w_1 \in \mathcal{A}_{p_1}$, $w_2 \in \mathcal{A}_{p_2}$, and $\theta \in [0,1]$. Let*

$$w^{\frac{1}{p}} = w_1^{\frac{1-\theta}{p_1}} w_2^{\frac{\theta}{p_2}}, \qquad \frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}.$$

Then $w \in \mathcal{A}_p$.

(ii) *The minimum, maximum, and the sum of finitely many $\mathcal{A}_1$ weights yields again an $\mathcal{A}_1$ weight.*

(iii) *Let w_1 and w_2 be $\mathcal{A}_1$ weights, and $1 \leq p < \infty$. Then $w = w_1 w_2^{1-p} \in \mathcal{A}_p$. Conversely, suppose that $w \in \mathcal{A}_p$, then there exist $v_1, v_2 \in \mathcal{A}_1$ such that $w = v_1 v_2^{1-p}$.*

(iv) *A positive, locally integrable function w on $\mathbb{R}^n$ belongs to $\mathcal{A}_p$, $1 \leq p < \infty$, if, and only if,*

$$\frac{1}{|B|} \int_B f(y)\, \mathrm{d}y \leq \left(\frac{c}{w(B)} \int_B f^p(x) w(x)\, \mathrm{d}x \right)^{1/p} \tag{3}$$

holds for all nonnegative f and all balls B.

Of course, Lemma 1.3 (i) can be understood as a special case of Lemma 1.4 (i). Moreover, let $E \subset B$ and $f = \chi_E$, then (3) implies that

$$\frac{|E|}{|B|} \leq c' \left(\frac{w(E)}{w(B)} \right)^{1/p}, \qquad E \subset B, \tag{4}$$

whenever $w \in \mathcal{A}_p$, $1 \leq p < \infty$.

Examples 1.5.

(i) One of the most prominent examples of a Muckenhoupt weight $w \in \mathcal{A}_p$, $1 \le p < \infty$, is given by $w(x) = |x|^\varrho$, with

$$w(x) = |x|^\varrho \in \mathcal{A}_p \quad \text{if, and only if,} \quad \begin{cases} -n < \varrho < n(p-1), & \text{if } 1 < p < \infty, \\ -n < \varrho \le 0, & \text{if } p = 1. \end{cases}$$

Thus $r_w = 1 + \frac{\varrho_+}{n}$ and $w \in \mathcal{A}_{r_w}$ if $\varrho \le 0$, whereas $w \notin \mathcal{A}_{r_w}$ for $\varrho > 0$.

(ii) Let

$$v(x) = |x|^\alpha \log^{-\beta}(2+|x|) \quad \text{or} \quad v(x) = |x|^\alpha \log^{\beta}(2+|x|^{-1}).$$

Then, also in view of Lemma 1.4 one verifies that

$$v \in \mathcal{A}_1 \qquad \text{if} \qquad \begin{cases} \beta \in \mathbb{R}, & \text{if } -n < \alpha < 0, \\ \beta \ge 0, & \text{if } \alpha = 0, \end{cases}$$

whereas the counterpart for $1 < p < \infty$ reads as

$$v \in \mathcal{A}_p \qquad \text{if} \qquad -n < \alpha < n(p-1), \quad \beta \in \mathbb{R},$$

see also [13, Lemma 2.3]. Similarly as above, $r_v = 1 + \frac{\alpha_+}{n}$.

(iii) Finally,

$$w(x) = |x_n|^\alpha \in \mathcal{A}_p \quad \text{if, and only if,} \quad -1 < \alpha < p-1,$$

where $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ and $1 \le p < \infty$. This is a special case of [13, Lemma 2.3], see also [18, Prop. 2.8]. We return to these examples (and combinations of them) in the sequel.

1.2. Function spaces of type $B^s_{p,q}(\mathbb{R}^n, w)$ and $F^s_{p,q}(\mathbb{R}^n, w)$ with $w \in \mathcal{A}_\infty$

Let $w \in \mathcal{A}_\infty$ be a Muckenhoupt weight, and $0 < p < \infty$. Then the weighted Lebesgue space $L_p(\mathbb{R}^n, w)$ contains all measurable functions such that

$$\|f \mid L_p(\mathbb{R}^n, w)\| = \left(\int_{\mathbb{R}^n} |f(x)|^p w(x) \,\mathrm{d}x \right)^{1/p}$$

is finite. Note that for $p = \infty$ one obtains the classical (unweighted) Lebesgue space,

$$L_\infty(\mathbb{R}^n, w) = L_\infty(\mathbb{R}^n), \quad w \in \mathcal{A}_\infty. \tag{5}$$

We thus restrict ourselves to $p < \infty$ in what follows.

The Schwartz space $\mathcal{S}(\mathbb{R}^n)$ and its dual $\mathcal{S}'(\mathbb{R}^n)$ of all complex-valued tempered distributions have their usual meaning here. Let $\varphi_0 = \varphi \in \mathcal{S}(\mathbb{R}^n)$ be such that

$$\operatorname{supp} \varphi \subset \{y \in \mathbb{R}^n : |y| < 2\} \quad \text{and} \quad \varphi(x) = 1 \quad \text{if} \quad |x| \le 1,$$

and for each $j \in \mathbb{N}$ let $\varphi_j(x) = \varphi(2^{-j}x) - \varphi(2^{-j+1}x)$. Then $\{\varphi_j\}_{j=0}^{\infty}$ forms a *smooth dyadic resolution of unity.* Given any $f \in \mathcal{S}'(\mathbb{R}^n)$, we denote by $\mathcal{F}f$ or $\hat{f}$, and $\mathcal{F}^{-1}f$ or $f^{\vee}$, its Fourier transform and its inverse Fourier transform, respectively. Let $f \in \mathcal{S}'(\mathbb{R}^n)$, then the compact support of $\varphi_j \hat{f}$ implies by the Paley-Wiener-Schwartz theorem that $(\varphi_j \hat{f})^{\vee}$ is an entire analytic function on $\mathbb{R}^n$.

Definition 1.6. Let $0 < q \le \infty$, $0 < p < \infty$, $s \in \mathbb{R}$, and $\{\varphi_j\}_j$ a smooth dyadic resolution of unity. Assume $w \in \mathcal{A}_\infty$.

(i) *The weighted Besov space* $B^s_{p,q}(\mathbb{R}^n, w)$ is the set of all distributions $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|f \mid B^s_{p,q}(\mathbb{R}^n, w)\| = \left(\sum_{j=0}^{\infty} 2^{jsq} \|\mathcal{F}^{-1}(\varphi_j \mathcal{F}f) \mid L_p(\mathbb{R}^n, w)\|^q\right)^{1/q} \tag{6}$$

is finite. In the limiting case $q = \infty$ the usual modification is required.

(ii) *The weighted Triebel-Lizorkin space* $F^s_{p,q}(\mathbb{R}^n, w)$ is the set of all distributions $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|f \mid F^s_{p,q}(\mathbb{R}^n, w)\| = \left\| \left(\sum_{j=0}^{\infty} 2^{jsq} |\mathcal{F}^{-1}(\varphi_j \mathcal{F}f)(\cdot)|^q\right)^{1/q} \,\middle|\, L_p(\mathbb{R}^n, w)\right\|$$

is finite. In the limiting case $q = \infty$ the usual modification is required.

Remark 1.7. The spaces $B^s_{p,q}(\mathbb{R}^n, w)$ and $F^s_{p,q}(\mathbb{R}^n, w)$ are independent of the particular choice of the smooth dyadic resolution of unity $\{\varphi_j\}_j$ appearing in their definitions. They are quasi-Banach spaces (Banach spaces for $p, q \ge 1$), and $\mathcal{S}(\mathbb{R}^n) \hookrightarrow B^s_{p,q}(\mathbb{R}^n, w) \hookrightarrow \mathcal{S}'(\mathbb{R}^n)$, similarly for the F-case, where the first embedding is dense if $q < \infty$; see [3]. Moreover, for $w_0 \equiv 1 \in \mathcal{A}_\infty$ we re-obtain the usual (unweighted) Besov and Triebel-Lizorkin spaces; we refer, in particular, to the series of monographs by Triebel [39–42], for a comprehensive treatment of the unweighted spaces.

The above spaces with weights of type $w \in \mathcal{A}_\infty$ have been studied systematically by Bui et al. in [3–6]. It turned out that many of the results from the unweighted situation have weighted counterparts: e.g., we have $F^0_{p,2}(\mathbb{R}^n, w) = h_p(\mathbb{R}^n, w)$, $0 < p < \infty$, where the latter are Hardy spaces, see [3, Thm. 1.4], and, in particular, $h_p(\mathbb{R}^n, w) = L_p(\mathbb{R}^n, w) = F^0_{p,2}(\mathbb{R}^n, w)$, $1 < p < \infty$, $w \in \mathcal{A}_p$, see [37, chap. VI, Thm. 1]. Concerning (classical) Sobolev spaces $W^k_p(\mathbb{R}^n, w)$ (built upon $L_p(\mathbb{R}^n, w)$ in the usual way) it holds

$$W^k_p(\mathbb{R}^n, w) = F^k_{p,2}(\mathbb{R}^n, w), \quad k \in \mathbb{N}_0, \quad 1 < p < \infty, \quad w \in \mathcal{A}_p,$$

see [3, Thm. 2.8]. Further results, concerning, for instance, embeddings, (real) interpolation, extrapolation, lift operators, duality assertions can be found in [3, 4, 16, 32].

Rychkov extended in [33] the above class of weights in order to incorporate locally regular weights, creating in that way the class $\mathcal{A}_p^{\rm loc}$. Recent works are due to Roudenko [15, 31, 32] and Bownik [1, 2]. We partly rely on our approach [18].

We collect some more or less immediate embedding results for weighted spaces of the above type that will be used later. For that purpose we adopt the nowadays usual custom to write $A^s_{p,q}$ instead of $B^s_{p,q}$ or $F^s_{p,q}$, respectively, when both scales of spaces are meant simultaneously in some context.

Proposition 1.8. *Let* $0 < q \le \infty$, $0 < p < \infty$, $s \in \mathbb{R}$, *and* $w \in \mathcal{A}_\infty$.

(i) *Let* $-\infty < s_1 \le s_0 < \infty$ *and* $0 < q_0 \le q_1 \le \infty$. *Then*

$$A^{s_0}_{p,q}(\mathbb{R}^n, w) \hookrightarrow A^{s_1}_{p,q}(\mathbb{R}^n, w) \quad \text{and} \quad A^{s}_{p,q_0}(\mathbb{R}^n, w) \hookrightarrow A^{s}_{p,q_1}(\mathbb{R}^n, w).$$

(ii) *We have*

$$B^s_{p,\min(p,q)}(\mathbb{R}^n, w) \hookrightarrow F^s_{p,q}(\mathbb{R}^n, w) \hookrightarrow B^s_{p,\max(p,q)}(\mathbb{R}^n, w).$$

(iii) *Assume that there are numbers* $c > 0$, $d > 0$, *such that, for all balls,*

$$w(B(x,r)) \ge \; cr^d, \quad 0 < r \le 1, \quad x \in \mathbb{R}^n. \tag{7}$$

Let $0 < p_0 < p_1 < \infty$, $-\infty < s_1 < s_0 < \infty$, *with*

$$s_0 - \frac{d}{p_0} = s_1 - \frac{d}{p_1}. \tag{8}$$

Then

$$B^{s_0}_{p_0,q}(\mathbb{R}^n, w) \hookrightarrow B^{s_1}_{p_1,q}(\mathbb{R}^n, w), \tag{9}$$

and

$$F^{s_0}_{p_0,\infty}(\mathbb{R}^n, w) \hookrightarrow F^{s_1}_{p_1,q}(\mathbb{R}^n, w). \tag{10}$$

(iv) *Let* w *satisfy* (7) *and let* $0 < p_0 < p < p_1 < \infty$, $-\infty < s_1 < s < s_0 < \infty$ *satisfy*

$$s_0 - \frac{d}{p_0} = s - \frac{d}{p} = s_1 - \frac{d}{p_1}.$$

Then

$$B^{s_0}_{p_0,p}(\mathbb{R}^n, w) \hookrightarrow F^{s}_{p,q}(\mathbb{R}^n, w) \hookrightarrow B^{s_1}_{p_1,p}(\mathbb{R}^n, w). \tag{11}$$

Proof. Parts (i)–(iii) coincide with [3, Thm. 2.6] where, in particular, assumption (7) is denoted by $w \in \mathcal{M}_d$. As for (iv) we use the partial result

$$F^{s_0}_{p_0,\infty}(\mathbb{R}^n, w) \hookrightarrow B^{s_1}_{p_1,p_0}(\mathbb{R}^n, w) \tag{12}$$

from [3, Thm. 2.6] together with (10) and interpolation results for weighted F-spaces in [3, Thm. 3.5]. Obviously, the right-hand side of (11) is a consequence of (12) together with (i) (for $A = F$). As for the left-hand side, choose, for given s_0, p_0, and s, suitable numbers σ_1, σ_2, and $0 < \theta < 1$ such that for appropriate r_1, r_2,

$$\sigma_1 > s_0 > \sigma_2 > s, \quad s_0 = (1-\theta)\sigma_1 + \theta\sigma_2 \qquad \text{and} \qquad \sigma_i - \frac{d}{p_0} = s - \frac{d}{r_i}, \quad i = 1,2.$$

According to (10),

$$F^{\sigma_i}_{p_0,\infty}(\mathbb{R}^n, w) \hookrightarrow F^s_{r_i,q}(\mathbb{R}^n, w), \quad i = 1,2.$$

On the other hand, the interpolation results [3, Thm. 3.5] read in our case as

$$\big(F^{\sigma_1}_{p_0,\infty}(\mathbb{R}^n, w), F^{\sigma_2}_{p_0,\infty}(\mathbb{R}^n, w)\big)_{\theta,p} = B^{s_0}_{p_0,p}(\mathbb{R}^n, w)$$

and

$$\big(F^{s}_{r_1,q}(\mathbb{R}^n, w), F^{s}_{r_2,q}(\mathbb{R}^n, w)\big)_{\theta,p} = F^{s}_{p,q}(\mathbb{R}^n, w),$$

which concludes the proof of (11). □

Remark 1.9. The above embeddings (i) and (ii) are natural extensions from the unweighted case $w \equiv 1$, see [39, Prop. 2.3.2/2], whereas (iii) and (iv) with $d = n$ have their unweighted counterparts in [35, Thm. 3.2.1; 39, Thm. 2.7.1].

Remark 1.10. We shall need an extension of the (unweighted) Besov spaces $B^s_{p,q}(\mathbb{R}^n)$ to so-called *weak*-$B^s_{p,q}$ spaces. By this we mean the modification of (6) (with $w \equiv 1$) when $L_p(\mathbb{R}^n)$ is replaced by the Lorentz space $L_{p,\infty}(\mathbb{R}^n)$,

$$\|f \mid \textit{weak-}B^s_{p,q}(\mathbb{R}^n)\| = \bigg(\sum_{j=0}^{\infty} 2^{jsq} \|\mathcal{F}^{-1}(\varphi_j \mathcal{F} f) \mid L_{p,\infty}(\mathbb{R}^n)\|^q\bigg)^{1/q}. \tag{13}$$

Note that

$$\|f \mid L_{p,\infty}(\mathbb{R}^n)\| \sim \sup_{t>0} t^{\frac{1}{p}} f^*(t),$$

where f^* is the non-increasing rearrangement of f, as usual,

$$f^*(t) = \inf\big\{\, s \ge 0 : |\{\, x \in \mathbb{R}^n : |f(x)| > s \,\}| \le t \,\big\}, \quad t \ge 0.$$

We recall the definition of atoms. Let for $m \in \mathbb{Z}^n$ and $\nu \in \mathbb{N}_0$, $Q_{\nu,m}$ denote the n-dimensional cube with sides parallel to the axes of coordinates, centered at $2^{-\nu}m$ and with side length $2^{-\nu}$. For $0 < p < \infty$, $\nu \in \mathbb{N}_0$, and $m \in \mathbb{Z}^n$ we denote by $\chi^{(p)}_{\nu,m}$ the p-normalized characteristic function of the cube $Q_{\nu,m}$,

$$\chi^{(p)}_{\nu,m}(x) = 2^{\frac{\nu n}{p}} \chi_{\nu,m}(x) = \begin{cases} 2^{\frac{\nu n}{p}} & \text{for} \quad x \in Q_{\nu,m}, \\ 0 & \text{for} \quad x \notin Q_{\nu,m}, \end{cases}$$

hence $\|\chi^{(p)}_{\nu,m} \mid L_p(\mathbb{R}^n)\| = 1$.

Definition 1.11. Let $K \in \mathbb{N}_0$ and $b > 1$.

(i) The complex-valued function $a \in C^K(\mathbb{R}^n)$ is said to be an 1_K-*atom* if $\operatorname{supp} a \subset bQ_{0,m}$ for some $m \in \mathbb{Z}^n$, and $|\mathrm{D}^\alpha a(x)| \le 1$ for $|\alpha| \le K$, $x \in \mathbb{R}^n$.

(ii) Let $s \in \mathbb{R}$, $0 < p \le \infty$, and $L + 1 \in \mathbb{N}_0$. The complex-valued function $a \in C^K(\mathbb{R}^n)$ is said to be an $(s,p)_{K,L}$-*atom* if for some $\nu \in \mathbb{N}_0$,

$$\begin{aligned} \operatorname{supp} a &\subset bQ_{\nu,m} && \text{for some } m \in \mathbb{Z}^n, \\ |\mathrm{D}^\alpha a(x)| &\le 2^{-\nu(s-\frac{n}{p})+|\alpha|\nu} && \text{for } |\alpha| \le K, \quad x \in \mathbb{R}^n, \\ \int_{\mathbb{R}^n} x^\beta a(x)\, \mathrm{d}x &= 0 && \text{for } |\beta| \le L. \end{aligned}$$

We shall denote an atom $a(x)$ supported in some $Q_{\nu,m}$ by $a_{\nu,m}$ in the sequel. For $0 < p < \infty$, $0 < q \le \infty$, $w \in \mathcal{A}_\infty$, we introduce suitable sequence spaces $b_{pq}(w)$ by

$$b_{p,q}(w) = \Bigg\{ \lambda = \{\lambda_{\nu,m}\}_{\nu,m} : \quad \lambda_{\nu,m} \in \mathbb{C}, \\ \|\lambda \mid b_{p,q}(w)\| = \Bigg(\sum_{\nu=0}^{\infty} \Bigg\| \sum_{m \in \mathbb{Z}^n} \lambda_{\nu,m}\, \chi_{\nu,m}^{(p)} | L_p(\mathbb{R}^n, w) \Bigg\|^q \Bigg)^{1/q} < \infty \Bigg\}. \tag{14}$$

Then the atomic decomposition result used below reads as follows.

Proposition 1.12. *Let $0 < p < \infty$, $0 < q \le \infty$, $s \in \mathbb{R}$, and $w \in \mathcal{A}_\infty$ be a weight with r_w given by* (2)*. Let $K, L+1 \in \mathbb{N}_0$ with*

$$K \ge (1 + [s])_+ \quad \textit{and} \quad L \ge \max\left(-1, \left[n\left(\frac{r_w}{p} - 1\right)_+ - s \right]\right).$$

Then $f \in \mathcal{S}'(\mathbb{R}^n)$ belongs to $B^s_{pq}(\mathbb{R}^n, w)$ if, and only if, it can be written as a series

$$f = \sum_{\nu=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda_{\nu m} a_{\nu,m}(x), \quad \textit{converging in } \mathcal{S}'(\mathbb{R}^n), \tag{15}$$

where $a_{\nu,m}(x)$ are 1_K-atoms ($\nu = 0$) or $(s,p)_{K,L}$-atoms ($\nu \in \mathbb{N}$) and $\lambda \in b_{pq}(w)$. Furthermore

$$\inf \|\lambda \mid b_{pq}(w)\|$$

is an equivalent quasi-norm in $B^s_{pq}(\mathbb{R}^n, w)$, where the infimum ranges over all admissible representations (15)*.*

The above result coincides with [18, Thm. 3.10], see also [1, Theorem 5.10].

1.3. Wavelet characterizations of Besov spaces with $\mathcal{A}_\infty$ weights

Nowadays there is a variety of excellent (text)books on wavelet theory and hence we may assume that the reader is familiar with basic assertions. For general background material on wavelets we refer, in particular, to [10, 21, 26, 43].

Let $\widetilde{\phi}$ be an orthogonal scaling function on $\mathbb{R}$ with compact support and of sufficiently high regularity. Let $\widetilde{\psi}$ be an associated wavelet. Then the tensor-product approach yields a scaling function ϕ and associated wavelets $\psi_1, \dots, \psi_{2^n-1}$, all defined now on $\mathbb{R}^n$. We suppose

$$\widetilde{\phi} \in C^{N_1}(\mathbb{R}) \qquad \text{and} \qquad \operatorname{supp} \widetilde{\phi} \subset [-N_2, N_2]$$

for certain natural numbers N_1 and N_2. This implies

$$\phi, \psi_i \in C^{N_1}(\mathbb{R}^n) \quad \text{and} \quad \operatorname{supp} \phi, \operatorname{supp} \psi_i \subset [-N_3, N_3]^n, \;\; i = 1, \dots, 2^n - 1. \tag{16}$$

We shall use the standard abbreviations

$$\phi_{\nu,m}(x) = 2^{\nu n/2} \phi(2^\nu x - m) \quad \text{and} \quad \psi_{i,\nu,m}(x) = 2^{\nu n/2} \psi_i(2^\nu x - m).$$

Apart from function spaces with weights we introduce sequence spaces with weights. Let $\sigma \in \mathbb{R}$. We extend (14) by

$$b^\sigma_{p,q}(w) := \Bigg\{ \lambda = \{\lambda_{\nu,m}\}_{\nu,m} : \quad \lambda_{\nu,m} \in \mathbb{C}, \\ \|\lambda \mid b^\sigma_{p,q}(w)\| = \bigg(\sum_{\nu=0}^\infty 2^{\nu\sigma q} \bigg\| \sum_{m\in\mathbb{Z}^n} \lambda_{\nu,m}\, \chi^{(p)}_{\nu,m} \bigg| L_p(\mathbb{R}^n, w) \bigg\|^q \bigg)^{1/q} < \infty \Bigg\}$$

and

$$\ell_p(w) := \Bigg\{ \lambda = \{\lambda_m\}_m : \lambda_m \in \mathbb{C}, \\ \|\lambda \mid \ell_p(w)\| = \bigg\| \sum_{m\in\mathbb{Z}^n} \lambda_m \chi^{(p)}_{0,m} \bigg| L_p(\mathbb{R}^n, w) \bigg\| < \infty \Bigg\}.$$

If $\sigma = 0$ we write $b_{p,q}(w)$ instead of $b^\sigma_{p,q}(w)$; moreover, if $w \equiv 1$ we write $b^\sigma_{p,q}$ instead of $b^\sigma_{p,q}(w)$.

For smooth weights and compactly supported wavelets it makes sense to consider the Fourier-wavelet coefficients of tempered distributions $f \in \mathcal{S}'(\mathbb{R}^n)$ with respect to such an orthonormal basis.

Theorem 1.13. *Let $0 < p, q \le \infty$ and let $s \in \mathbb{R}$. Let ϕ be a scaling function and let ψ_i, $i = 1, \dots, 2^n - 1$, be the corresponding wavelets satisfying* (16)*. We assume* *that* $|s| < N_1$. *Then a distribution* $f \in \mathcal{S}'(\mathbb{R}^n)$ *belongs to* $B^s_{p,q}(\mathbb{R}^n, w)$, *if, and only if,*

$$\big\|f \mid B^s_{p,q}(\mathbb{R}^n, w)\big\|^* = \big\|\{\langle f, \phi_{0,m}\rangle\}_{m\in\mathbb{Z}^n} \,\big|\, \ell_p(w)\big\| + \sum_{i=1}^{2^n-1} \big\|\{\langle f, \psi_{i,\nu,m}\rangle\}_{\nu\in\mathbb{N}_0, m\in\mathbb{Z}^n} \,\big|\, b^\sigma_{p,q}(w)\big\| < \infty, \quad (17)$$

where $\sigma = s + \frac{n}{2} - \frac{n}{p}$. *Furthermore,* $\|f \mid B^s_{p,q}(\mathbb{R}^n, w)\|^*$ *may be used as an equivalent (quasi-)norm in* $B^s_{p,q}(\mathbb{R}^n, w)$.

Proof. The idea of the proof is standard, see [31, Theorem 10.2].

First we assume that (17) holds. There exists a positive constant c such that functions $a_{i,\nu,m}(x) = c^{-1}2^{-\nu(s+\frac{n}{2}-\frac{n}{p})}\psi_{i,\nu,m}$ are (s,p)-atoms and $a_{0,m}(x) = c^{-1}\phi_{0,m}$ are 1-atoms. The distribution f can be represented in the following way:

$$\begin{aligned} f &= \sum_{m\in\mathbb{Z}^n} \langle f, \phi_{0,m}\rangle \phi_{0,m} + \sum_{i=0}^{2^n-1}\sum_{\nu\in\mathbb{N}_0}\sum_{m\in\mathbb{Z}^n} \langle f, \psi_{i,\nu,m}\rangle\, \psi_{i,\nu,m} \\ &= \sum_{m\in\mathbb{Z}^n} c\langle f, \phi_{0,m}\rangle a_{0,m} + \sum_{i=1}^{2^n-1}\sum_{\nu\in\mathbb{N}_0}\sum_{m\in\mathbb{Z}^n} c2^{\nu(s+\frac{n}{2}-\frac{n}{p})}\langle f, \psi_{i,\nu,m}\rangle a_{i,\nu,m}, \end{aligned} \quad (18)$$

see also Remark 1.14 below. So Proposition 1.12 implies

$$\begin{aligned} &\|f \mid B^s_{p,q}(\mathbb{R}^n, w)\| \\ &\quad\le C\bigg\|\sum_{m\in\mathbb{Z}^n}\langle f, \phi_{0,m}\rangle \chi^{(p)}_{\nu,m} \,\bigg|\, L_p(\mathbb{R}^n, w)\bigg\| \\ &\qquad + \sum_{i=0}^{2^n-1}\bigg(\sum_{\nu=0}^{\infty}\bigg\|\sum_{m\in\mathbb{Z}^n} 2^{\nu(s+\frac{n}{2}-\frac{n}{p})}\langle f, \psi_{i,\nu,m}\rangle\, \chi^{(p)}_{\nu,m}\bigg| L_p(\mathbb{R}^n, w)\bigg\|^q\bigg)^{1/q} \\ &\quad\le C\big\|\big\{\langle f, \phi_{0,m}\rangle\big\}_{m\in\mathbb{Z}^n} \,\big|\, \ell_p(w)\big\| + \sum_{i=0}^{2^n-1}\big\|\big\{\langle f, \psi_{i,\nu,m}\rangle\big\}_{\nu\in\mathbb{N}_0, m\in\mathbb{Z}^n}\big| b^\sigma_{p,q}(w)\big\|. \end{aligned}$$

Now let $f \in B^s_{p,q}(\mathbb{R}^n, w)$. We can define an equivalent (quasi-)norm in the Besov spaces with $\mathcal{A}_\infty$ weights using the inhomogeneous φ-transform, see [1, 14]. Let $\Phi \in \mathcal{S}(\mathbb{R}^n)$ and $\Phi_0 \in \mathcal{S}(\mathbb{R}^n)$ satisfy $\operatorname{supp}\widehat{\Phi} \subset [-\pi,\pi] \setminus \{0\}$, $\operatorname{supp}\widehat{\Phi_0} \subset [-\pi,\pi]$, and $\sup_{j\in\mathbb{N}}\{\,|\widehat{\Phi}(2^{-j}\xi)|, |\widehat{\Phi_0}(\xi)|\,\} > 0$ for all $\xi \in \mathbb{R}^n$. Given a pair $\Phi_0, \Phi \in \mathcal{S}(\mathbb{R}^n)$ satisfying the above conditions one can find functions $\Psi_0, \Psi \in \mathcal{S}(\mathbb{R}^n)$ satisfying the same conditions and such that

$$\overline{\widehat{\Phi_0}(\xi)}\widehat{\Psi_0}(\xi) + \sum_{j=1}^{\infty}\overline{\widehat{\Phi}(2^{-j}\xi)}\widehat{\Psi}(2^{-j}\xi) = 1$$

for all $\xi \in \mathbb{R}^n$, see [14]. Let $\Phi_{j,m}(x) = 2^{nj/2}\Phi(2^j x - m)$, $j \in \mathbb{N}$, and $m \in \mathbb{Z}^n$. Moreover, we put $\Phi_{0,m}(x) = \Phi_0(x-m)$, $m \in \mathbb{Z}^n$. In a similar way we define $\Psi_{j,m}$. The inhomogeneous φ-transform S_Φ is a map taking each $f \in \mathcal{S}'(\mathbb{R}^n)$ into the sequence $S_\Phi(f) = (S_\Phi f)_{j,m}$ defined by

$$(S_\Phi f)_{j,m} = \langle f, \Phi_{j,m}\rangle, \quad j \in \mathbb{N}_0, \quad \text{and} \quad m \in \mathbb{Z}^n.$$

The inhomogeneous inverse φ-transform T_Ψ is a map taking each sequence $\lambda = \{\lambda_{j,m}\}_{j\in\mathbb{N}_0, m\in\mathbb{Z}^n}$ to

$$T_\Psi(\lambda) = \sum_{j=0}^{\infty} \sum_{m\in\mathbb{Z}^n} \lambda_{j,m}\Psi_{j,m},$$

with convergence in $\mathcal{S}'(\mathbb{R}^n)$. We have the following representation formula,

$$f = T_\Psi \circ S_\Phi(f) \tag{19}$$

for any $f \in \mathcal{S}'(\mathbb{R}^n)$. The operator S_Φ is a bounded operator from $B^s_{p,q}(\mathbb{R}^n, w)$ into $b^\sigma_{p,q}(w)$ and T_Ψ is a bounded operator from $b^\sigma_{p,q}(w)$ onto $B^s_{p,q}(\mathbb{R}^n, w)$. Moreover, $T_\Psi \circ S_\Phi$ is the identity operator on $B^s_{p,q}(\mathbb{R}^n, w)$. Thus

$$\big\|\{\langle f, \Phi_{j,m}\rangle\}_{j,m} \,\big|\, b^\sigma_{p,q}(w)\big\| \le C\big\|f \,\big|\, B^s_{p,q}(\mathbb{R}^n, w)\big\|. \tag{20}$$

Applying the formula (19) to the function $\psi_{i,\nu,m}$ we get

$$\langle f, \psi_{i,\nu,m}\rangle = \sum_{j=0}^{\infty}\sum_{k\in\mathbb{Z}^n} \langle f, \Psi_{j,k}\rangle \overline{\langle \psi_{i,\nu,m}, \Phi_{j,k}\rangle}.$$

A similar formula holds for $\langle f, \phi_{0,m}\rangle$. The numbers $\lambda_{(\nu,m),(j,k)} = \overline{\langle \psi_{i,\nu,m}, \Phi_{j,k}\rangle}$ form the almost diagonal matrix in the sense of Frazier and Jawerth, see [14, Lemmas 3.6, 3.8 and Remarks in §12]. Thus the almost diagonal operator related to the matrix is bounded in $b^\sigma_{p,q}(w)$, see [1, §5.4]. Since the coefficients $\langle f, \Psi_{j,k}\rangle$ satisfy also the inequality (20) we get

$$\|f \mid B^s_{p,q}(\mathbb{R}^n, w)\|^* \le C\big\|\{\langle f, \Psi_{j,m}\rangle\}_{j,m} \,\big|\, b^\sigma_{p,q}(w)\big\| \le C\big\|f \,\big|\, B^s_{p,q}(\mathbb{R}^n, w)\big\|.$$

This finishes the proof. □

Remark 1.14. The representability (18) of $f \in \mathcal{S}'(\mathbb{R}^n)$ may not be clear immediately. However, using Hölder's inequality and the already mentioned "reverse Hölder inequality" of $\mathcal{A}_\infty$ weights ([36, chap. 5, Prop. 3, Cor.]) one can reduce the argument to the corresponding one for admissible weights, say, of type $\langle x\rangle^\alpha$, $\alpha \in \mathbb{R}$, since $B^s_{p,q}(\mathbb{R}^n, w)$ can be squeezed in between spaces of type $B^s_{p_i,q}(\mathbb{R}^n, \langle x\rangle^{\alpha_i})$, $i = 1, 2$ (for suitably chosen α_i, p_i, $i = 1, 2$). But then [20] implies the representability (18) (at the expense of some higher smoothness and cancellation needed for the atomic decomposition argument according to Proposition 1.12), see also [42].

2. Continuity and compactness of embeddings

We start with a general result on weighted embeddings and discuss its consequences in different settings afterwards.

Proposition 2.1. *Let w_1 and w_2 be two $\mathcal{A}_\infty$ weights and let $-\infty < s_2 \le s_1 < \infty$, $0 < p_1, p_2 \le \infty$, $0 < q_1, q_2 \le \infty$. We put*

$$\frac{1}{p^*} := \left(\frac{1}{p_2} - \frac{1}{p_1}\right)_+ \qquad \text{and} \qquad \frac{1}{q^*} := \left(\frac{1}{q_2} - \frac{1}{q_1}\right)_+ .$$

(i) *There is a continuous embedding $B^{s_1}_{p_1,q_1}(\mathbb{R}^n, w_1) \hookrightarrow B^{s_2}_{p_2,q_2}(\mathbb{R}^n, w_2)$ if, and only if,*

$$\left\{ 2^{-\nu(s_1-s_2)} \big\| \{ (w_2(Q_{\nu,m}))^{1/p_2} (w_1(Q_{\nu,m}))^{-1/p_1} \}_m \mid \ell_{p^*} \big\| \right\}_\nu \in \ell_{q^*}. \tag{21}$$

(ii) *The embedding $B^{s_1}_{p_1,q_1}(\mathbb{R}^n, w_1) \hookrightarrow B^{s_2}_{p_2,q_2}(\mathbb{R}^n, w_2)$ is compact if, and only if,* (21) *holds and, in addition,*

$$\lim_{\nu\to\infty} 2^{-\nu(s_1-s_2)} \|\{(w_2(Q_{\nu,m}))^{1/p_2} (w_1(Q_{\nu,m}))^{-1/p_1}\}_m \mid \ell_{p^*}\| = 0 \quad \text{if } q^* = \infty,$$

and

$$\lim_{|m|\to\infty} (w_2(Q_{\nu,m}))^{-1/p_2} (w_1(Q_{\nu,m}))^{1/p_1} = \infty \quad \text{for all } \nu \in \mathbb{N}_0 \quad \text{if } p^* = \infty. \tag{22}$$

Proof. It follows from the last theorem that the mapping

$$T : f \longmapsto \left(\{\langle f, \phi_{0,m}\rangle\}_{m\in\mathbb{Z}^n},\ \ \{\langle f, \psi_{i,\nu,m}\rangle\}_{\nu\in\mathbb{N}_0, m\in\mathbb{Z}^n, i=1,\ldots,2^n-1} \right)$$

is an isomorphism of $B^s_{p,q}(\mathbb{R}^n, w)$ onto $\ell_p(w) \oplus \left(\bigoplus_{i=1}^{2^n-1} b^\sigma_{p,q}(w)\right)$, $\sigma = s + \frac{n}{2} - \frac{n}{p}$. It can be easily seen that the last sequence space is isomorphic to $b^\sigma_{p,q}(w)$. Consequently we have the following commutative diagrams,

$$\begin{array}{ccc} B^{s_1}_{p_1,q_1}(\mathbb{R}^n, w_1) & \xrightarrow{T} & b^{\sigma_1}_{p_1,q_1}(w_1) \\ \Big\downarrow \mathrm{Id} & & \Big\downarrow \mathrm{id} \\ B^{s_2}_{p_2,q_2}(\mathbb{R}^n, w_2) & \xleftarrow{S} & b^{\sigma_2}_{p_2,q_2}(w_2) \end{array} \qquad \text{and} \qquad \begin{array}{ccc} b^{\sigma_1}_{p_1,q_1}(w_1) & \xrightarrow{T^{-1}} & B^{s_1}_{p_1,q_1}(\mathbb{R}^n, w_1) \\ \Big\downarrow \mathrm{id} & & \Big\downarrow \mathrm{Id} \\ b^{\sigma_2}_{p_2,q_2}(w_2) & \xleftarrow{S^{-1}} & B^{s_2}_{p_2,q_2}(\mathbb{R}^n, w_2), \end{array}$$

where T and S are the corresponding isomorphisms and $\sigma_i = s_i + \frac{n}{2} - \frac{n}{p_i}$, $i = 1, 2$. On the other hand, one can easily verify that the expression

$$\left(\sum_{\nu=0}^{\infty} 2^{\nu\sigma q} \left(\sum_{m\in\mathbb{Z}^n} |\lambda_{\nu,m}|^p 2^{\nu n} w(Q_{\nu,m}) \right)^{q/p} \right)^{1/q}$$

is an equivalent norm in $b^{\sigma}_{p,q}(w)$. But $w(Q_{\nu,m}) > 0$ for any ν and m. So we can reduce the investigation of the embeddings of two weighted sequence spaces to the study of embeddings of a weighted space into an unweighted one, using the following commutative diagrams,

$$\begin{array}{ccc} b^{\sigma_1}_{p_1,q_1}(w_1) & \xrightarrow{A} & b^{\sigma_1}_{p_1,q_1}(w_1/w_2) \\ {\scriptstyle \mathrm{Id}}\downarrow & & \downarrow{\scriptstyle \mathrm{id}} \\ b^{\sigma_2}_{p_2,q_2}(w_2) & \xleftarrow{A^{-1}} & b^{\sigma_2}_{p_2,q_2} \end{array} \qquad \text{and} \qquad \begin{array}{ccc} b^{\sigma_1}_{p_1,q_1}(w_1/w_2) & \xrightarrow{A^{-1}} & b^{\sigma_1}_{p_1,q_1}(w_1) \\ {\scriptstyle \mathrm{id}}\downarrow & & \downarrow{\scriptstyle \mathrm{Id}} \\ b^{\sigma_2}_{p_2,q_2} & \xleftarrow{A} & b^{\sigma_2}_{p_2,q_2}(w_2), \end{array}$$

where $b^{\sigma_2}_{p_2,q_2}$ denotes an unweighted space, i.e., with weight $w \equiv 1$. But the necessary and sufficient conditions for the boundedness and compactness of the embeddings $b^{\sigma_1}_{p_1,q_1}(w_1/w_2) \to b^{\sigma_2}_{p_2,q_2}$ are known, see [24, Theorem 1]. Taking $w_{\nu,m} = \big(2^{\nu n} w(Q_{\nu,m})\big)^{1/p}$ in the last mentioned theorem we get the result. □

Remark 2.2. In view of (5) it is clear that we obtain unweighted Besov spaces if $p_1 = p_2 = \infty$. Then by (1), $w_1(Q_{\nu,m}) = w_2(Q_{\nu,m}) = 2^{-\nu n}$ for all $\nu \in \mathbb{N}_0$ and $m \in \mathbb{Z}^n$, such that (21) leads to $p^* = \infty$, i.e., $p_1 \le p_2$, and

$$\delta := s_1 - \frac{n}{p_1} - s_2 + \frac{n}{p_2} > 0, \tag{23}$$

with the extension to $\delta = 0$ if $q_1 \le q_2$, i.e., $q^* = \infty$. Moreover, by (22), the embedding is never compact (as is well-known in this case).

Furthermore, (i) generalizes Proposition 1.8 (iii), in particular (9), since taking $w_1 = w_2 = w$ satisfying (7), we obtain that $w(Q_{\nu,m}) \ge c2^{-\nu d}$ for all $m \in \mathbb{Z}^n$ which immediately leads to $p^* = \infty$ in (21), i.e., $p_1 \le p_2$. Moreover, (8) then yields that $2^{-\nu(s_1 - s_2 + d(\frac{1}{p_2} - \frac{1}{p_1}))} = 1$ for all $\nu \in \mathbb{N}_0$. Hence $q^* = \infty$, that is, $q_1 \le q_2$. Thus Proposition 2.1 (i) implies (9).

Examples 2.3. We collect some elementary examples and explicate the proposition in their context.

(i) Let

$$w_\alpha(x) = |x|^\alpha, \quad x \in \mathbb{R}^n, \quad -n < \alpha < \infty. \tag{24}$$

It is well known that $w_\alpha \in \mathcal{A}_1$ if $\alpha \le 0$ and $w_\alpha \in \mathcal{A}_r$ provided that $\alpha < n(r-1)$. If $\alpha > 0$, then the embedding

$$B^{s_1}_{p_1,q_1}(\mathbb{R}^n, w_\alpha) \hookrightarrow B^{s_2}_{p_2,q_2}(\mathbb{R}^n)$$

is continuous if, and only if, $\frac{\alpha}{p_1} > \frac{n}{p^*}$ and $\delta \ge \frac{\alpha}{p_1}$ if $q^* = \infty$ or $\delta > \frac{\alpha}{p_1}$ if $q^* < \infty$. The embedding is compact if, and only if, $\frac{\alpha}{p_1} > \frac{n}{p^*}$ and $\delta > \frac{\alpha}{p_1}$. If $-n < \alpha < 0$, then the embedding is not continuous.

(ii) We consider (24) with $-n < \alpha < 0$; here one can deal with embeddings

$$B^{s_1}_{p_1,q_1}(\mathbb{R}^n) \hookrightarrow B^{s_2}_{p_2,q_2}(\mathbb{R}^n, w_\alpha).$$

The embedding is continuous if, and only if, $\frac{\alpha}{p_2} < -\frac{n}{p^*}$ and $\delta \geq -\frac{\alpha}{p_2}$ if $q^* = \infty$ or $\delta > -\frac{\alpha}{p_2}$ if $q^* < \infty$. The embedding is compact if, and only if, $\frac{\alpha}{p_2} < -\frac{n}{p^*}$ and $\delta > -\frac{\alpha}{p_2}$.

(iii) If

$$w_{\alpha,(n)}(x) = |x_n|^\alpha, \quad x = (x_1, \ldots, x_n) \in \mathbb{R}^n, \quad \alpha > 0,$$

then the embedding

$$B^{s_1}_{p_1,q_1}(\mathbb{R}^n, w_{\alpha,(n)}) \hookrightarrow B^{s_2}_{p_2,q_2}(\mathbb{R}^n)$$

is continuous if, and only if, $p^* = \infty$ and $\delta \geq \frac{\alpha}{p_1}$ if $q^* = \infty$ or $\delta > \frac{\alpha}{p_1}$ if $q^* < \infty$. The embedding is never compact.

Further examples are studied in detail in the next sections.

As already mentioned, we may restrict ourselves to the situation when only the source space is weighted, and the target space unweighted. However, for comparison with the unweighted case we shall study two types of embeddings; firstly and essentially we concentrate on

$$B^{s_1}_{p_1,q_1}(\mathbb{R}^n, w) \hookrightarrow B^{s_2}_{p_2,q_2}(\mathbb{R}^n), \tag{25}$$

where $w \in \mathcal{A}_\infty$. We shall assume in the sequel that $p_1 < \infty$ for convenience, since otherwise we have $B^{s_1}_{p_1,q_1}(\mathbb{R}^n, w) = B^{s_1}_{p_1,q_1}(\mathbb{R}^n)$, recall (5), and we arrive at the unweighted situation in (25) which is well-known already. Therefore we stick here to the general assumptions

$$-\infty < s_2 \leq s_1 < \infty, \quad 0 < p_1 < \infty, \quad 0 < p_2 \leq \infty, \quad 0 < q_1, q_2 \leq \infty. \tag{26}$$

Secondly, we shall occasionally formulate some results in the *"double-weighted"* situation, in particular, corresponding to the setting

$$B^{s_1}_{p_1,q_1}(\mathbb{R}^n, w) \hookrightarrow B^{s_2}_{p_2,q_2}(\mathbb{R}^n, w) \tag{27}$$

with

$$-\infty < s_2 \leq s_1 < \infty, \quad 0 < p_1, p_2 < \infty, \quad 0 < q_1, q_2 \leq \infty. \tag{28}$$

Example 2.4. Obviously all Examples 2.3 have their immediate counterparts for embeddings of type (27), e.g.,

$$B^{s_1}_{p_1,q_1}(\mathbb{R}^n, w_\alpha) \hookrightarrow B^{s_2}_{p_2,q_2}(\mathbb{R}^n, w_\alpha), \quad -n < \alpha < \infty, \tag{29}$$

is continuous if, and only if, $p^* = \infty$ (that is, $p_1 \leq p_2$) and $\delta \geq \alpha(\frac{1}{p_1} - \frac{1}{p_2}) \geq 0$ if $q^* = \infty$, or $\delta > \alpha(\frac{1}{p_1} - \frac{1}{p_2}) \geq 0$ if $q^* < \infty$. The embedding (29) is compact if, and only if, $\delta > \alpha(\frac{1}{p_1} - \frac{1}{p_2}) > 0$. If $-n < \alpha < 0$, then the embedding is never compact.

Remark 2.5. The continuity assertion of the above example is well-known in the unweighted case $\alpha = 0$. However, there is no compactness in unweighted situations possible, unlike in (29).

2.1. Weights of purely polynomial growth

We consider weights of polynomial growth both near zero and infinity of the form

$$w_{\alpha,\beta}(x) = \begin{cases} |x|^\alpha & \text{if} \quad |x| \le 1, \\ |x|^\beta & \text{if} \quad |x| > 1, \end{cases} \quad \text{with} \quad \alpha > -n, \quad \beta > -n. \tag{30}$$

Obviously this refines the approach (24), i.e., for $\alpha = \beta > -n$ we arrive at Example 2.3 (i), $w_{\alpha,\alpha} = w_\alpha$. Note that $r_{w_{\alpha,\beta}} = 1 + \frac{\max(\alpha,\beta,0)}{n}$ in this case.

Proposition 2.6. *Let w_α, w_β be given by* (24)*, respectively, and $w_{\alpha,\beta}$ by* (30)*.*

(i) *Let $w_\alpha \in \mathcal{A}_r$ and $w_\beta \in \mathcal{A}_r$, $1 \le r < \infty$. Then $w_{\alpha,\beta} \in \mathcal{A}_r$.*

(ii) *Let the parameters be given by* (26)*. The embedding $B^{s_1}_{p_1,q_1}(\mathbb{R}^n, w_{\alpha,\beta}) \hookrightarrow B^{s_2}_{p_2,q_2}(\mathbb{R}^n)$ is continuous if, and only if,*

$$\begin{cases} \textit{either} \quad \beta \ge 0 & \textit{if} \quad p^* = \infty, \\ \textit{or} \quad \dfrac{\beta}{p_1} > \dfrac{n}{p^*} & \textit{if} \quad p^* < \infty, \end{cases} \tag{31}$$

and one of the following conditions is satisfied:

$$\delta \ge \max\left(\frac{\alpha}{p_1}, 0\right) \qquad \textit{if} \quad q^* = \infty, \quad p^* = \infty,$$

$$\delta \ge \max\left(\frac{\alpha}{p_1}, \frac{n}{p^*}\right) \qquad \textit{if} \quad q^* = \infty, \quad p^* < \infty, \quad \frac{n}{p^*} \ne \frac{\alpha}{p_1},$$

$$\delta > \max\left(\frac{\alpha}{p_1}, \frac{n}{p^*}\right) \qquad \textit{otherwise.}$$

(iii) *The embedding $A^{s_1}_{p_1,q_1}(\mathbb{R}^n, w_{\alpha,\beta}) \hookrightarrow A^{s_2}_{p_2,q_2}(\mathbb{R}^n)$ is compact if, and only if,*

$$\frac{\beta}{p_1} > \frac{n}{p^*} \qquad \textit{and} \qquad \delta > \max\left(\frac{n}{p^*}, \frac{\alpha}{p_1}\right).$$

Proof. According to Lemma 1.4, the minimum and maximum of two $\mathcal{A}_1$ weights is also an $\mathcal{A}_1$ weight. Moreover, $w^{1-r} \in \mathcal{A}_r$ if $w \in \mathcal{A}_1$. Using the above facts one can prove part (i).

Parts (ii) and (iii) for Besov spaces follow easily from Proposition 2.1 since for any $\alpha > -n$ we have

$$w_\alpha(Q_{\nu,0}) \sim 2^{-\nu(n+\alpha)} \quad \text{and} \quad w_\alpha(Q_{\nu,m}) \sim 2^{-\nu(n+\alpha)}|m|^\alpha \quad \text{if} \quad m \neq 0.$$

Part (iii) for Triebel-Lizorkin spaces follows from the Besov case in combination with Proposition 1.8 (iii). □

Remark 2.7. When $p^* < \infty$, the restriction $\frac{\beta}{p_1} > \frac{n}{p^*}$ in (31) cannot be weakened; however, the limiting case $\frac{\beta}{p_1} = \frac{n}{p^*}$ can be included in the framework of *weak*-Besov spaces, recall their definition in (13). This will be contained as a special case in Proposition 2.10 below. Obviously, Proposition 1.8 and part (ii) of Proposition 2.6 imply continuity assertion in the F-case, too, where some care is needed whenever the q-parameters are involved. However, we are mainly interested in compact embeddings in the sequel and omit further discussion.

We turn to the double-weighted situation now.

Proposition 2.8. *Let w_{α_1,β_1} and w_{α_2,β_2} be given by* (30)*.*

(i) *Let the parameters be given by* (28)*. The embedding $B^{s_1}_{p_1,q_1}(\mathbb{R}^n, w_{\alpha_1,\beta_1}) \hookrightarrow B^{s_2}_{p_2,q_2}(\mathbb{R}^n, w_{\alpha_2,\beta_2})$ is continuous if, and only if,*

$$\begin{cases} \text{either} \quad \dfrac{\beta_1}{p_1} - \dfrac{\beta_2}{p_2} \geq 0 & \text{if} \quad p^* = \infty, \\ \text{or} \quad \dfrac{\beta_1}{p_1} - \dfrac{\beta_2}{p_2} > \dfrac{n}{p^*} & \text{if} \quad p^* < \infty, \end{cases}$$

and one of the following conditions is satisfied:

$$\begin{aligned} &\delta \geq \max\left(\frac{\alpha_1}{p_1} - \frac{\alpha_2}{p_2}, 0\right) && \text{if} \quad q^* = \infty, \quad p^* = \infty, \\ &\delta \geq \max\left(\frac{\alpha_1}{p_1} - \frac{\alpha_2}{p_2}, \frac{n}{p^*}\right) && \text{if} \quad q^* = \infty, \quad p^* < \infty, \quad \frac{n}{p^*} \neq \frac{\alpha_1}{p_1} - \frac{\alpha_2}{p_2}, \\ &\delta > \max\left(\frac{\alpha_1}{p_1} - \frac{\alpha_2}{p_2}, \frac{n}{p^*}\right) && \text{otherwise.} \end{aligned} \tag{32}$$

(ii) *The embedding $A^{s_1}_{p_1,q_1}(\mathbb{R}^n, w_{\alpha_1,\beta_1}) \hookrightarrow A^{s_2}_{p_2,q_2}(\mathbb{R}^n, w_{\alpha_2,\beta_2})$ is compact if, and only if,*

$$\frac{\beta_1}{p_1} - \frac{\beta_2}{p_2} > \frac{n}{p^*} \qquad \text{and} \qquad \delta > \max\left(\frac{\alpha_1}{p_1} - \frac{\alpha_2}{p_2}, \frac{n}{p^*}\right).$$

Proof. The argument is parallel to that one given for Proposition 2.6 above. □

Remark 2.9. If we take $\alpha_2 = \beta_2 = 0$ we get Proposition 2.6. In case of $\alpha_1 = \alpha_2 = \alpha$ and $\beta_1 = \beta_2 = \beta$ we obtain an embedding of type (27), in that way extending Example 2.4 (where we considered the special case $\alpha = \beta$). Then the embedding $B^{s_1}_{p_1,q_1}(\mathbb{R}^n, w_{\alpha,\beta}) \hookrightarrow B^{s_2}_{p_2,q_2}(\mathbb{R}^n, w_{\alpha,\beta})$ is continuous if, and only if, $p_1 \le p_2$, $\beta(\frac{1}{p_1} - \frac{1}{p_2}) \ge 0$, and one of the following conditions is satisfied:

$$\delta \ge \max(\alpha, 0)\left(\frac{1}{p_1} - \frac{1}{p_2}\right) \qquad \text{if} \quad q^* = \infty,$$

$$\delta > \max(\alpha, 0)\left(\frac{1}{p_1} - \frac{1}{p_2}\right) \qquad \text{if} \quad q^* < \infty.$$

The embedding $B^{s_1}_{p_1,q_1}(\mathbb{R}^n, w_{\alpha,\beta}) \hookrightarrow B^{s_2}_{p_2,q_2}(\mathbb{R}^n, w_{\alpha,\beta})$ is compact if, and only if,

$$\beta > 0, \qquad p_1 < p_2, \qquad \text{and} \qquad \delta > \max(\alpha, 0)\left(\frac{1}{p_1} - \frac{1}{p_2}\right).$$

As already mentioned, we deal with the limiting case $\frac{\beta}{p_1} = \frac{n}{p^*}$ separately; in view of (31) it is excluded in the context of target spaces $B^s_{p,q}(\mathbb{R}^n)$ (apart from $p^* = \infty$). But in the context of *weak*-Besov spaces we obtain the following extension.

Proposition 2.10. *Let* $-\infty < s_2 \le s_1 < \infty$, $0 < p_1 < \infty$, $0 < q_1, q_2 \le \infty$, $\alpha > -n$, $\beta > 0$, *and assume*

$$\frac{(\alpha - \beta)_+}{p_1} \le s_1 - s_2.$$

Let p_2 *be given by* $\frac{1}{p_2} = \frac{1}{p_1} + \frac{\beta}{np_1}$. *Then*

$$B^{s_1}_{p_1,q_1}(\mathbb{R}^n, w_{\alpha,\beta}) \longrightarrow \textit{weak-}B^{s_2}_{p_2,q_2}(\mathbb{R}^n) \tag{33}$$

if

$$\begin{cases} \textit{either} \quad s_2 < s_1 - \dfrac{(\alpha - \beta)_+}{p_1}, & 0 < q_2 \le \infty, \\ \textit{or} \quad s_2 = s_1 - \dfrac{(\alpha - \beta)_+}{p_1}, & q_1 \le q_2 \le \infty. \end{cases}$$

Proof. Step 1. By elementary embeddings (monotonicity of the spaces) it is sufficient to prove (33) for the case $s_2 = s_1 - \frac{(\alpha-\beta)_+}{p_1}$, and $q_1 = q_2 = q$ only, that is, using notation (30), we have to show that

$$B^{s_1}_{p_1,q}(\mathbb{R}^n, w_{\alpha,\beta}) \longrightarrow \textit{weak-}B^{s_2}_{p_2,q}(\mathbb{R}^n), \quad \frac{1}{p_2} = \frac{1}{p_1} + \frac{\beta}{np_1}, \quad s_2 = s_1 - \frac{(\alpha - \beta)_+}{p_1}, \tag{34}$$

with $s_1 \in \mathbb{R}$, $0 < p_1 < \infty$, $0 < q \le \infty$, $\alpha > -n$, $\beta > 0$.

Step 2. We first assume $\alpha \le \beta$ such that $s_2 = s_1 = s$ and (34) reads as

$$B^s_{p_1,q}(\mathbb{R}^n, w_{\alpha,\beta}) \longrightarrow \textit{weak-}B^s_{p_2,q}(\mathbb{R}^n), \quad \frac{1}{p_2} = \frac{1}{p_1} + \frac{\beta}{np_1}, \quad \alpha \le \beta.$$

The argument is based on the observation that

$$L_{p_1}(\mathbb{R}^n, w) \longrightarrow L_{p_2,\infty}(\mathbb{R}^n) \tag{35}$$

if $w^{-1/p_1} \in L_{r,\infty}(\mathbb{R}^n)$, $\frac{1}{r} = \frac{1}{p_2} - \frac{1}{p_1}$. Recall that real interpolation together with Hölder's inequality gives

$$L_{p_1}(\mathbb{R}^n) \cdot L_{r,\infty}(\mathbb{R}^n) \longrightarrow L_{p_2,p_1}(\mathbb{R}^n)$$

in the sense that for all $h \in L_{p_1}(\mathbb{R}^n)$, $g \in L_{r,\infty}(\mathbb{R}^n)$, then $hg \in L_{p_2,p_1}(\mathbb{R}^n)$ with

$$\|hg \mid L_{p_2,p_1}(\mathbb{R}^n)\| \le c\,\|h \mid L_{p_1}(\mathbb{R}^n)\|\,\|g \mid L_{r,\infty}(\mathbb{R}^n)\|. \tag{36}$$

Now let $f \in L_{p_1}(\mathbb{R}^n, w)$. Then $h = fw^{1/p_1} \in L_{p_1}(\mathbb{R}^n)$. Assume for the moment that $g = w^{-1/p_1} \in L_{r,\infty}(\mathbb{R}^n)$ with $r = \frac{np_1}{\beta}$. Then (36) implies (35).

$$\begin{aligned}\|f \mid L_{p_2,\infty}(\mathbb{R}^n)\| &\le c\,\|f \mid L_{p_2,p_1}(\mathbb{R}^n)\|\\ &\le c'\,\|fw^{1/p_1} \mid L_{p_1}(\mathbb{R}^n)\|\,\|w^{-1/p_1} \mid L_{r,\infty}(\mathbb{R}^n)\|\\ &\le c''\,\|f \mid L_{p_1}(\mathbb{R}^n, w)\|.\end{aligned}$$

It remains to show that $w^{-1/p_1}_{\alpha,\beta} \in L_{r,\infty}(\mathbb{R}^n)$ for $-n < \alpha \le \beta$, $\beta > 0$. Let

$$g(x) = w^{-1/p_1}_{\alpha,\beta}(x) = \begin{cases} |x|^{-\frac{\alpha}{p_1}} & \text{if} \quad |x| \le 1,\\ |x|^{-\frac{\beta}{p_1}} & \text{if} \quad |x| > 1.\end{cases}$$

Then, for $\alpha \ge 0$,

$$g^*(t) \sim \begin{cases} t^{-\frac{\alpha}{np_1}} & \text{if} \quad t \le 1,\\ t^{-\frac{\beta}{np_1}} & \text{if} \quad t > 1,\end{cases}$$

and so

$$\|g \mid L_{r,\infty}(\mathbb{R}^n)\| \sim \sup_{0<t<1} t^{\frac{1}{r}} g^*(t) + \sup_{t \ge 1} t^{\frac{1}{r}} g^*(t) \sim \sup_{0<t<1} t^{\frac{\beta-\alpha}{np_1}} + \sup_{t\ge 1} t^{\frac{\beta-\beta}{np_1}} \sim 1$$

for $0 \le \alpha \le \beta$. In case of $-n < \alpha < 0$ we get at least $g^*(t) \le ct^{-\frac{1}{r}}$, $t > 0$, leading to $g = w^{-1/p_1}_{\alpha,\beta} \in L_{r,\infty}(\mathbb{R}^n)$ again.

Step 3. Assume $\alpha > \beta > 0$ such that we have $s_2 = s_1 - \frac{\alpha-\beta}{p_1} < s_1$ in (34). In view of Proposition 2.8 (i) we have

$$B^{s_1}_{p_1,q}(\mathbb{R}^n, w_{\alpha,\beta}) \hookrightarrow B^{s_2}_{p_1,q}(\mathbb{R}^n, w_{\beta,\beta}), \quad \delta = s_1 - s_2 = \frac{\alpha-\beta}{p_1} > 0,$$

see (32). Moreover, Step 2 yields

$$B^{s_2}_{p_1,q}(\mathbb{R}^n, w_{\beta,\beta}) \longleftrightarrow \textit{weak-}B^{s_2}_{p_2,q}(\mathbb{R}^n), \quad \frac{1}{p_2} = \frac{1}{p_1} + \frac{\beta}{np_1},$$

and this concludes the proof. □

2.2. General weights

Now we deal with general weights $w \in \mathcal{A}_r$, $1 \le r < \infty$, and will check what Proposition 2.1 means in this context. Essentially, we concentrate on two types of embeddings: either the target space is unweighted,

$$\mathrm{id}_w : B^{s_1}_{p_1,q_1}(\mathbb{R}^n, w) \hookrightarrow B^{s_2}_{p_2,q_2}(\mathbb{R}^n), \tag{37}$$

with (26), or both source space and target space share the same weight,

$$\mathrm{id}_{ww} : B^{s_1}_{p_1,q_1}(\mathbb{R}^n, w) \hookrightarrow B^{s_2}_{p_2,q_2}(\mathbb{R}^n, w), \tag{38}$$

with (28), recall also Proposition 1.8.

We begin with a short preparation in order to apply Proposition 2.1 for such situations. Hence we have to consider expressions of type $w_2(Q_{\nu,m})^{1/p_2} w_1(Q_{\nu,m})^{-1/p_1}$ only, that is, $2^{-\nu n/p_2} w(Q_{\nu,m})^{-1/p_1}$ in case of (37), and $w(Q_{\nu,m})^{\frac{1}{p_2}-\frac{1}{p_1}}$ in case of (38). Let $w \in \mathcal{A}_r$, $1 \le r < \infty$. Then, by (4),

$$w(Q_{\nu,m}) \ge c 2^{-\nu n r} w(Q_{0,l}) \qquad \text{for all} \quad Q_{\nu,m} \subset Q_{0,l}, \quad \nu \in \mathbb{N}_0, \quad m, l \in \mathbb{Z}^n. \tag{39}$$

So, for any $\varkappa < 0$,

$$w(Q_{\nu,m})^{\varkappa} \le c' 2^{-\nu n r \varkappa} w(Q_{0,l})^{\varkappa}, \quad Q_{\nu,m} \subset Q_{0,l},$$

and thus

$$\|\{w(Q_{\nu,m})^{\varkappa}\}_m \mid \ell_\infty\| \le c 2^{-\nu r n \varkappa} \big(\inf_l w(Q_{0,l})\big)^{\varkappa}, \quad \varkappa < 0, \quad \nu \in \mathbb{N}_0, \tag{40}$$

with $w \in \mathcal{A}_r$, $1 \le r < \infty$. Moreover, for arbitrary $\gamma > 0$, (39) leads to

$$\lim_{|m|\to\infty} w(Q_{\nu,m})^{\gamma} = \infty \quad \text{for all } \nu \in \mathbb{N}_0 \quad \text{if, and only if,} \quad \lim_{|l|\to\infty} w(Q_{0,l}) = \infty. \tag{41}$$

Summarizing the above considerations, we find that for embeddings of type (37) or (38) the conditions

$$\inf_l w(Q_{0,l}) \ge c > 0, \tag{42}$$

and

$$\lim_{|l|\to\infty} w(Q_{0,l}) = \infty, \tag{43}$$

are essential when $p_1 \le p_2$, i.e., $p^* = \infty$. If $p^* < \infty$, then careful calculation leads to

$$\begin{aligned}\big\| \{w(Q_{\nu,m})^{\varkappa}\}_m \,|\, \ell_{p^*} \big\| &\le c\, 2^{-\nu r n \varkappa + \nu \frac{n}{p^*}} \Big(\sum_l w(Q_{0,l})^{\varkappa p^*} \Big)^{1/p^*} \\ &= c\, 2^{-\nu r n \varkappa + \nu \frac{n}{p^*}} \big\| \{w(Q_{0,l})^{\varkappa}\}_l \,|\, \ell_{p^*} \big\|\end{aligned}$$

for $\varkappa < 0$, $\nu \in \mathbb{N}_0$, as the counterpart of (40). This corresponds to (37) with $\varkappa = -1/p_1 < 0$. Therefore the adequate replacement of (42) for $p^* < \infty$ reads as

$$\big\| \big\{ w(Q_{0,l})^{-1/p_1} \big\}_l \,|\, \ell_{p^*} \big\| < \infty. \tag{44}$$

Note that there cannot be a continuous embedding of type (38) if $p_1 > p_2$, $w \in \mathcal{A}_\infty$, and (at least) (42) is assumed to hold, since (39) implies then for some r, $1 \le r < \infty$, that

$$w(Q_{\nu,m}) \ge c' 2^{-\nu n r} \quad \text{for all} \quad \nu \in \mathbb{N}_0, \;\; m \in \mathbb{Z}^n,$$

such that

$$\Big\| \Big\{ w(Q_{\nu,m})^{\frac{1}{p_2} - \frac{1}{p_1}} \Big\}_m \,\Big|\, \ell_{p^*} \Big\| = \Big(\sum_m w(Q_{\nu,m}) \Big)^{1/p^*}$$

diverges for any $\nu \in \mathbb{N}_0$. So we are left to consider the case $p_1 \le p_2$ as far as (38) is concerned. We begin with the case (37).

Corollary 2.11. *Let the parameters be given by* (26) *with* $p_1 \le p_2$. *Let* $w \in \mathcal{A}_\infty$ *with* r_w *given by* (2).

(i) *Let*

$$\delta > \frac{n}{p_1}(r_w - 1). \tag{45}$$

Then id_w *in* (37) *is continuous if, and only if,* (42) *is satisfied.*

The embedding id_w *in* (37) *is compact if, and only if,* (43) *holds.*

(ii) *Let* $\delta < 0$, *then* $B^{s_1}_{p_1,q_1}(\mathbb{R}^n, w)$ *is not embedded in* $B^{s_2}_{p_2,q_2}(\mathbb{R}^n)$.

(iii) *Let* $\delta = 0$. *When* $q^* < \infty$, *then* $B^{s_1}_{p_1,q_1}(\mathbb{R}^n, w)$ *is not embedded in* $B^{s_2}_{p_2,q_2}(\mathbb{R}^n)$.

When $q^* = \infty$ *and* $w \in \mathcal{A}_{r_w}$, *that is,* $r_w = 1$, *then* id_w *in* (37) *is continuous if, and only if,* (42) *is satisfied.*

(iv) *If* (45) *is not satisfied, then for every* $r > r_w$ *there exists an* $\mathcal{A}_r$ *weight* v *satisfying* (42) *such that the space* $B^{s_1}_{p_1,q_1}(\mathbb{R}^n, v)$ *is not embedded in* $B^{s_2}_{p_2,q_2}(\mathbb{R}^n)$.

Proof. Step 1. If (42) does not hold, then obviously there is no embedding (independent of δ) in view of (21) and our above discussion. Similarly, if (43) does not hold, then the embedding cannot be compact in view of (22) and (41) with $\gamma = 1/p_1$.

Step 2. Let (45) be satisfied and $r > r_w$ such that

$$\delta > \frac{n}{p_1}(r-1) > \frac{n}{p_1}(r_w - 1), \tag{46}$$

then $w \in \mathcal{A}_r$. Thus (40) with $\varkappa = -1/p_1 < 0$ leads for (21) (with $w_1 \equiv w$, $w_2 \equiv 1$) to the estimate

$$\begin{aligned} 2^{-\nu(s_1-s_2)} \big\| \big\{ (w_2(Q_{\nu,m}))^{1/p_2} (w_1(Q_{\nu,m}))^{-1/p_1} \big\}_m \,\big|\, \ell_\infty \big\| \\ &\leq c\, 2^{-\nu(s_1 - s_2 + \frac{n}{p_2} - \frac{nr}{p_1})} \big(\inf_l w(Q_{0,l}) \big)^{-1/p_1} \\ &= c\, 2^{-\nu(\delta - \frac{n}{p_1}(r-1))} \big(\inf_l w(Q_{0,l}) \big)^{-1/p_1}. \end{aligned} \tag{47}$$

So the continuity and compactness of the embedding follow from Proposition 2.1 in view of (42) and (46). Together with Step 1 this concludes the proof of (i). As far as (iii) with $q^* = \infty$ and $w \in \mathcal{A}_{r_w} = \mathcal{A}_1$ is concerned, we can adapt the above argument by taking $r = r_w = 1$ such that (47) and (21) (with $q^* = \infty$) give the result.

Step 3. Concerning (ii) and (iii) it remains to show that for $\delta < 0$ or $\delta = 0$ and $q^* < \infty$ there is no embedding (37) (even) if (42) is satisfied. Since

$$w(Q_{0,m}) \geq 2^{n\nu} \min_{l: Q_{\nu,l} \subset Q_{0,m}} w(Q_{\nu,l}),$$

one obtains $\|\{w(Q_{0,m})^{-1/p_1}\}_m \mid \ell_\infty\| \leq 2^{-n\nu/p_1} \|\{w(Q_{\nu,l})^{-1/p_1}\}_l \mid \ell_\infty\|$. Thus,

$$\begin{aligned} \Big\| \Big\{ 2^{-\nu(s_1-s_2)-\nu\frac{n}{p_2}} \big\| \big\{ w(Q_{\nu,l})^{-1/p_1} \big\}_l \,\big|\, \ell_\infty \big\| \Big\}_\nu \,\Big|\, \ell_{q^*} \Big\| \\ &\geq \Big\| \Big\{ 2^{-\nu\delta} \big\| \big\{ w(Q_{0,m})^{-1/p_1} \big\}_m \,\big|\, \ell_\infty \big\| \Big\}_\nu \,\Big|\, \ell_{q^*} \Big\| \\ &= \big\| \big\{ w(Q_{0,m})^{-1/p_1} \big\}_m \,\big|\, \ell_\infty \big\| \, \big\| \big\{ 2^{-\nu\delta} \big\}_\nu \,\big|\, \ell_{q^*} \big\| = \infty \end{aligned}$$

if $\delta < 0$ or $\delta \leq 0$ and $q^* < \infty$.

Step 4. If (45) is not satisfied, then for any $r > r_w$ there is some $0 < \alpha$ such that

$$\delta < \frac{\alpha}{p_1} < \frac{n}{p_1}(r-1).$$

Hence $v = w_\alpha \in \mathcal{A}_r$ and the embedding $B^{s_1}_{p_1,q_1}(\mathbb{R}^n, w_\alpha) \hookrightarrow B^{s_2}_{p_2,q_2}(\mathbb{R}^n)$ does not hold, see Example 2.3 (i). □

Remark 2.12. It is obvious that we obtain by (i)–(iii) a complete characterization (with respect to δ) in case of $w \in \mathcal{A}_1$ only, i.e., when $r_w = 1$. Otherwise there remains the gap

$$0 < \delta \leq \frac{n}{p_1}(r_w - 1)$$

(apart from the complementing assertion (iv), of course). However, it is not surprising that general features of w like r_w and (42) are not appropriately adapted for the interplay with the parameters (26) as required in Proposition 2.1. Reviewing, for instance, Proposition 2.6 and its proof one realizes that more information of the weight is used than reflected by r_w and (42) only. For example, Corollary 2.11 covers only the cases $\beta \geq 0$ and $\delta > \frac{\max(\alpha,\beta)}{p_1}$ in Proposition 2.6 (with $p_1 \leq p_2$), thus neglecting the (admitted) situations when $\frac{\max(\alpha,0)}{p_1} \leq \delta \leq \frac{\max(\alpha,\beta)}{p_1}$. But this requires further information on the weight, as already mentioned.

Corollary 2.13. *Let the parameters be given by* (26) *with* $p_1 > p_2$. *Let* $w \in \mathcal{A}_\infty$ *and* r_w *be given by* (2).

(i) *Let*

$$\delta > \frac{n}{p^*} + \frac{n}{p_1}(r_w - 1).$$

The embedding id_w *in* (37) *is compact if, and only if,* (44) *holds.*

(ii) *Let* $\delta < \frac{n}{p^*}$, *then* $B^{s_1}_{p_1,q_1}(\mathbb{R}^n, w)$ *is not embedded in* $B^{s_2}_{p_2,q_2}(\mathbb{R}^n)$.

(iii) *Let* $\delta = \frac{n}{p^*}$. *When* $q^* < \infty$, *then* $B^{s_1}_{p_1,q_1}(\mathbb{R}^n, w)$ *is not embedded in* $B^{s_2}_{p_2,q_2}(\mathbb{R}^n)$. *When* $q^* = \infty$ *and* $w \in \mathcal{A}_{r_w}$, *that is,* $r_w = 1$, *then* id_w *in* (37) *is continuous if, and only if,* (44) *is satisfied.*

(iv) *If* $\frac{n}{p^*} < \delta < \frac{n}{p_1}(r_w - 1)$, *then for every* $r > r_w$ *there exists an* $\mathcal{A}_r$ *weight* v *satisfying* (44) *such that the space* $B^{s_1}_{p_1,q_1}(\mathbb{R}^n, v)$ *is not embedded in* $B^{s_2}_{p_2,q_2}(\mathbb{R}^n)$.

Proof. This is a consequence of our above considerations and parallel arguments as presented for the case $p^* = \infty$. As for (iv), our assumption on r implies that we can always find some number α with

$$\frac{n}{p_1}(r-1) > \frac{\alpha}{p_1} > \delta > \frac{n}{p^*},$$

such that $v = w_\alpha \in \mathcal{A}_r$ serves as an example in view of Example 2.3 (i). □

We turn to the double-weighted situation now and restrict ourselves to the case $w_1 = w_2$, i.e., when both spaces are weighted in the same way.

Corollary 2.14. *Let the parameters be given by* (28) *with* $p_1 < p_2$. *Let* $w \in \mathcal{A}_\infty$ *and* r_w *be given by* (2).

(i) *Let*

$$\delta > (r_w - 1)\left(\frac{n}{p_1} - \frac{n}{p_2}\right). \tag{48}$$

The embedding

$$\mathrm{id}_{ww} : B^{s_1}_{p_1,q_1}(\mathbb{R}^n, w) \hookrightarrow B^{s_2}_{p_2,q_2}(\mathbb{R}^n, w) \tag{49}$$

is continuous if, and only if, (42) *is satisfied.*

The embedding (49) *is compact if, and only if,* (43) *is satisfied.*

(ii) *Let* $\delta < 0$*, then* $B^{s_1}_{p_1,q_1}(\mathbb{R}^n, w)$ *is not embedded in* $B^{s_2}_{p_2,q_2}(\mathbb{R}^n, w)$*.*

(iii) *Let* $\delta = 0$*. When* $q^* < \infty$*, then* $B^{s_1}_{p_1,q_1}(\mathbb{R}^n, w)$ *is not embedded in* $B^{s_2}_{p_2,q_2}(\mathbb{R}^n, w)$*. When* $q^* = \infty$ *and* $w \in \mathcal{A}_{r_w}$*, that is,* $r_w = 1$*, then* id_{ww} *in* (49) *is continuous if, and only if,* (42) *is satisfied.*

(iv) *If the condition* (48) *does not hold, then for every* $r > r_w$ *there exists an* $\mathcal{A}_r$ *weight* v *satisfying* (42) *such that the space* $B^{s_1}_{p_1,q_1}(\mathbb{R}^n, v)$ *is not embedded in* $B^{s_2}_{p_2,q_2}(\mathbb{R}^n, v)$*.*

Proof. The proof of parts (i)–(iii) is completely parallel to the proof of Corollary 2.11, where we apply (40) for $\varkappa = \frac{1}{p_2} - \frac{1}{p_1} < 0$, and (41) with $\gamma = \frac{1}{p_2} - \frac{1}{p_1}$. If (48) does not hold, then there is some $0 < \alpha$ such that

$$\delta < \alpha\left(\frac{1}{p_1} - \frac{1}{p_2}\right) < (r-1)\left(\frac{n}{p_1} - \frac{n}{p_2}\right).$$

Then $v = w_\alpha \in \mathcal{A}_r$ and the embedding $B^{s_1}_{p_1,q_1}(\mathbb{R}^n, w_\alpha) \hookrightarrow B^{s_2}_{p_2,q_2}(\mathbb{R}^n, w_\alpha)$ does not hold, see Example 2.4. □

Remark 2.15. Note that the compactness in (i) is in some sense surprising as it is different from the unweighted situation $w \equiv 1$ (where one cannot have a compact embedding as is well-known). Of course, there is no contradiction as (43) is not satisfied in this case.

Moreover, Corollary 2.14 refines Proposition 1.8 (iii) in some sense: Assume that (42) is satisfied; then since for an arbitrary ball $B(x, \varrho)$ with radius $0 < \varrho < 1$ there is some $m \in \mathbb{Z}^n$ such that $B(x, \varrho) \subset Q_{0,m}$ (apart from a universal constant) and (4) implies for $w \in \mathcal{A}_r$ that

$$w(B(x, \varrho)) \geq c\varrho^{nr} w(Q_{0,m}),$$

we obtain (7) with $d = rn$. The limiting case $\delta = (r-1)(\frac{n}{p_1} - \frac{n}{p_2})$ in (48) coincides with (8) for $d = nr$ such that (9) covers the continuity of the embedding (49).

We end this section with a somehow astonishing result dealing with the situation $p_1 = p_2$. It turns out that there is no direct influence of the weights on the continuity or compactness of the embedding (49).

Corollary 2.16. *Let the parameters be given by* (28) *with* $p_1 = p_2$*. Let* $w \in \mathcal{A}_\infty$*. Then the embedding* (49) *is continuous if, and only if,*

$$\begin{cases} s_1 - s_2 > 0 & \text{if} \quad q^* < \infty, \\ s_1 - s_2 \geq 0 & \text{if} \quad q^* = \infty. \end{cases}$$

The embedding (49) *is never compact.*

Proof. This is an immediate consequence of (21) and (22). □

3. Entropy numbers of compact embeddings

Let X, Y be two quasi-Banach spaces and let $T : X \to Y$ be a bounded linear operator. The k-th (dyadic) entropy number of T, $k \in \mathbb{N}$, is defined as

$$e_k(T) = \inf\{\varepsilon > 0 : T(B_X) \text{ can be covered by } 2^{k-1} \text{ balls of radius } \varepsilon \text{ in } Y\},$$

where B_X denotes the closed unit ball in X. Due to the well known fact that

$$T : X \longrightarrow Y \quad \text{is compact} \qquad \text{if, and only if,} \qquad \lim_{k\to\infty} e_k(T : X \to Y) = 0,$$

the entropy numbers can be viewed as a quantification of the notion of compactness. On the other hand, the k-th approximation number of T is defined as

$$a_k(T) = \inf\{\, \|T - L\| \,:\, \operatorname{rank} L < k \,\}.$$

If $a_k(T) \to 0$ for $k \to \infty$, then T is compact. So the asymptotic behavior of approximation numbers also gives us the quantitative analysis of compactness of the operator. Further properties like multiplicativity and additivity, as well as applications of entropy and approximation numbers can be found in [9, 11, 12, 30].

We consider the weights $w \in \mathcal{A}_\infty$ such that

$$w(x) \sim w_{\alpha,\beta}(x) = \begin{cases} |x|^\alpha & \text{if} \quad |x| \le 1, \\ |x|^\beta & \text{if} \quad |x| > 1, \end{cases} \quad \text{with} \quad \alpha > -n,\ \beta > 0. \tag{50}$$

It follows from Theorem 1.13 that the investigation of the asymptotic behavior of entropy numbers of the embedding

$$B^{s_1}_{p_1,q_1}(\mathbb{R}^n, w) \hookrightarrow B^{s_2}_{p_2,q_2}(\mathbb{R}^n) \tag{51}$$

can be reduced to the estimation of the asymptotic behavior of entropy numbers of embeddings of corresponding sequence spaces $b^{s_1}_{p_1,q_1}(w) \hookrightarrow b^{s_2}_{p_2,q_2}$, see the proof of Proposition 2.1. First we regard the part near zero. To make the notation more transparent we introduce the following spaces,

$$\ell_q(2^{j\theta}\ell_p(w)) = \Bigg\{ \lambda = \{\lambda_{j,m}\}_{j,m} : \lambda_{j,m} \in \mathbb{C},\ j, m \in \mathbb{N}_0,$$
$$\|\lambda \mid \ell_q(2^{j\theta}\ell_p(w))\| = \Bigg(\sum_{j=0}^{\infty} 2^{j\theta q} \Bigg(\sum_{m=0}^{\infty} |\lambda_{j,m}|^p w(j,m) \Bigg)^{\frac{q}{p}} \Bigg)^{\frac{1}{q}} < \infty \Bigg\}, \tag{52}$$
$$\ell_q(2^{j\theta}\ell_p^{\gamma 2^{nj}}(w)) = \Big\{ \{s_{j,l}\}_{j,l} \in \ell_q(2^{j\theta}\ell_p(w)) : s_{j,l} = 0 \text{ if } l > \gamma 2^{nj} \Big\},$$
$$\ell_q(2^{j\theta}\widetilde{\ell}_p^{\gamma 2^j}(w)) = \Big\{ \{s_{j,l}\}_{j,l} \in \ell_q(2^{j\theta}\ell_p(w)) : s_{j,l} = 0 \text{ if } l \le \gamma 2^{nj} \Big\},$$

(with the usual modification in (52) when $p = \infty$ and/or $q = \infty$), $\gamma \in \mathbb{N}$. We put $w_\xi(j, l) = l^\xi$ if $l \neq 0$ and $w_\xi(j, 0) = 1$, $j \in \mathbb{N}_0$.

Lemma 3.1. *Let $0 < p_1 < \infty$, $0 < p_2 \leq \infty$, $0 < q_1, q_2 \leq \infty$, $\xi \in \mathbb{R}$, $\theta > 0$, and $\gamma \in \mathbb{N}$. Assume $\frac{\xi}{p_1} > -\frac{\theta}{n} + \frac{1}{p^*}$. Then there are positive constants c and C such that for all $k \in \mathbb{N}$ the estimates*

$$ck^{-(\frac{\theta}{n}+\frac{\xi}{p_1}+\frac{1}{p_1}-\frac{1}{p_2})} \leq e_k(\mathrm{id} : \ell_{q_1}(2^{j\theta}\ell_{p_1}^{\gamma 2^{jn}}(w_\xi)) \to \ell_{q_2}(\ell_{p_2}^{\gamma 2^{nj}})) \leq Ck^{-(\frac{\theta}{n}+\frac{\xi}{p_1}+\frac{1}{p_1}-\frac{1}{p_2})}$$

hold.

Proof. Step 1. Preparations. For $\xi = 0$ (unweighted case) the result is known, see [41, Thm. 8.2]. So we assume that $\xi \neq 0$. Let

$$\Lambda := \{\lambda = \{\lambda_{j,l}\}_{j,l} : \lambda_{j,l} \in \mathbb{C}, \quad j \in \mathbb{N}_0, \quad 0 \leq l \leq \gamma 2^{nj}\}, \tag{53}$$

and

$$B_1 = \ell_{q_1}(2^{j\theta}\ell_{p_1}^{\gamma 2^{nj}}(w_\xi)), \qquad B_2 = \ell_{q_2}(\ell_{p_2}^{\gamma 2^{nj}}). \tag{54}$$

Let $P_j : \Lambda \mapsto \Lambda$ be the canonical projection onto j-level, i.e., for $\lambda = \{\lambda_{j,l}\}$ we put

$$(P_j\lambda)_l := \begin{cases} \lambda_{k,l} & \text{if} \quad k = j, \\ 0 & \text{otherwise,} \end{cases} \qquad l \in \mathbb{N}_0. \tag{55}$$

To shorten the notation we put $1/p = 1/p_1 - 1/p_2$. Elementary properties of the entropy numbers yield

$$e_k\big(P_j : B_1 \mapsto B_2\big) \leq 2^{-j\theta}\, e_k(\mathrm{Id} : \ell_{p_1}^{\gamma 2^{nj}}(w_\xi) \mapsto \ell_{p_2}^{\gamma 2^{nj}}), \tag{56}$$

and

$$e_k\big(\mathrm{Id} : \ell_{p_1}^{\gamma 2^{nj}}(w_\xi) \mapsto \ell_{p_2}^{\gamma 2^{nj}}\big) = e_k\big(D_\sigma : \ell_{p_1}^{\gamma 2^{nj}} \mapsto \ell_{p_2}^{\gamma 2^{nj}}\big), \tag{57}$$

where D_σ is the diagonal operator defined by the sequence $\sigma_l = l^{-\xi/p_1}$ if $l > 0$ and $\sigma_0 = 1$.

Step 2. The estimate from above. We use the notation of operator ideals, see [9,30] for details. Here we recall only what we need for the proof. For a given bounded linear operator $T \in \mathcal{L}(X,Y)$, where X and Y are Banach spaces, and a positive real number r we put

$$L_{r,\infty}^{(e)}(T) := \sup_{k\in\mathbb{N}} k^{1/r}\, e_k(T). \tag{58}$$

The last expression is an operator quasi-norm. Using (56) and (57) we find

$$L_{r,\infty}^{(e)}(P_j : B_1 \mapsto B_2) \leq c\, 2^{-j\theta}\, L_{r,\infty}^{(e)}(D_\sigma : \ell_{p_1}^{\gamma 2^{nj}} \mapsto \ell_{p_2}^{\gamma 2^{nj}}). \tag{59}$$

Substep 2.1. We fix j in this substep and put $N_j = \gamma 2^{nj}$. Let σ_l^* denote the non-increasing rearrangement of σ_l, $l = 0, \dots, N_j$. Thus

$$\sigma_l^* = \begin{cases} \sigma_l & \text{if } \xi \ge 0, \\ (N_j - l)^{-\xi/p_1} & \text{if } \xi < 0. \end{cases}$$

It should be clear that

$$a_k\big(D_\sigma : \ell_{p_1}^{N_j} \mapsto \ell_{p_1}^{N_j}\big) \le \sigma_k^* \qquad \text{if} \quad k \le N_j + 1, \tag{60}$$

and $a_k\big(D_\sigma : \ell_{p_1}^{N_j} \mapsto \ell_{p_1}^{N_j}\big) = 0$ if $k > N_j + 1$.

For any $\beta > 0$ there exists $C_\beta > 0$ such that

$$\sup_{l \le k} l^\beta e_l\big(D_\sigma : \ell_{p_1}^{N_j} \mapsto \ell_{p_1}^{N_j}\big) \le C_\beta \sup_{l \le k} l^\beta a_l\big(D_\sigma : \ell_{p_1}^{N_j} \mapsto \ell_{p_1}^{N_j}\big), \tag{61}$$

see [8, Thm. 1; 9, p. 96] and its extension to quasi-Banach spaces in [12, Thm. 1.3.3]. Let $\xi > 0$. Taking $\beta = \frac{\xi}{p_1}$ we get from (60) and (61) that

$$k^{\frac{\xi}{p_1}} e_k\big(D_\sigma : \ell_{p_1}^{N_j} \mapsto \ell_{p_1}^{N_j}\big) \le c \sup_{l \le k} l^{\frac{\xi}{p_1}} a_l(D_\sigma) \le C_{\xi, p_1}. \tag{62}$$

Now (62) and the multiplicativity of entropy numbers imply

$$k^{\frac{1}{r}} e_{2k-1}\big(D_\sigma : \ell_{p_1}^{N_j} \mapsto \ell_{p_2}^{N_j}\big) \le C\, k^{\frac{1}{r} - \frac{\xi}{p_1}} e_k\big(\mathrm{id} : \ell_{p_1}^{N_j} \mapsto \ell_{p_2}^{N_j}\big).$$

Consequently,

$$L_{r,\infty}^{(e)}\big(D_\sigma : \ell_{p_1}^{N_j} \mapsto \ell_{p_2}^{N_j}\big) \le c\, L_{s,\infty}^{(e)}\big(\mathrm{id} : \ell_{p_1}^{N_j} \mapsto \ell_{p_2}^{N_j}\big) \tag{63}$$

if $\frac{1}{s} = \frac{1}{r} - \frac{\xi}{p_1} > 0$.

If $0 < p_1 \le p_2 \le \infty$, then Schütt's characterization of the asymptotic behavior of the entropy numbers $e_k\big(\mathrm{id} : \ell_{p_1}^N \mapsto \ell_{p_2}^N\big)$, see [34], implies

$$L_{s,\infty}^{(e)}\big(\mathrm{id} : \ell_{p_1}^N \mapsto \ell_{p_2}^N\big) \le c \begin{cases} N^{\frac{1}{s} - \frac{1}{p}} & \text{if } \frac{1}{s} - \frac{1}{p} > 0, \\ (\log N)^{1/s} & \text{if } \frac{1}{s} - \frac{1}{p} \le 0. \end{cases} \tag{64}$$

Under the assumption $\frac{1}{r} > \frac{\xi}{p_1} + \frac{1}{p}$ we conclude from (63) and (64) that

$$L_{r,\infty}^{(e)}\big(D_\sigma : \ell_{p_1}^{N_j} \mapsto \ell_{p_2}^{N_j}\big) \le C\, 2^{jn(\frac{1}{r} - \frac{\xi}{p_1} - \frac{1}{p})}, \tag{65}$$

where the constant C depends on γ, but is independent of j.

If $0 < p_2 < p_1 < \infty$, then

$$L_{s,\infty}^{(e)}\big(\mathrm{id} : \ell_{p_1}^N \mapsto \ell_{p_2}^N\big) \le c\, N^{\frac{1}{s} - \frac{1}{p}}, \tag{66}$$

and (63) and (66) imply for $\frac{1}{r} > \max(\frac{\xi}{p_1} + \frac{1}{p}, 0)$ that

$$L^{(e)}_{r,\infty}\big(D_\sigma : \ell^{N_j}_{p_1} \mapsto \ell^{N_j}_{p_2}\big) \le C\, 2^{jn(\frac{1}{r} - \frac{\xi}{p_1} - \frac{1}{p})}. \tag{67}$$

Let $\xi < 0$. Then

$$\begin{aligned}\sup_{l\le k} l^\beta a_l\big(D_\sigma : \ell^{N_j}_{p_1} \mapsto \ell^{N_j}_{p_1}\big) &\le \sup_{l\le k} l^\beta (N_j - l)^{-\xi/p_1}\\ &\le \min(k, c_{\beta,\xi,p_1} N_j)^\beta (N_j - \min(k, c_{\beta,\xi,p_1} N_j))^{-\xi/p_1}\\ &\le C_{\beta,\xi,p_1} \begin{cases} k^\beta \big(N_j - k\big)^{-\xi/p_1} & \text{if} \quad k < c_{\beta,\xi,p_1} N_j,\\ N_j^{\beta-\xi/p_1} & \text{if} \quad k \ge c_{\beta,\xi,p_1} N_j.\end{cases}\end{aligned}$$

Consequently,

$$k^\beta e_k\big(D_\sigma : \ell^{N_j}_{p_1} \mapsto \ell^{N_j}_{p_1}\big) \le C_{\beta,\xi,p_1} N_j^{\beta-\xi/p_1}. \tag{68}$$

Now using (68) we obtain

$$k^{\frac{1}{r}} e_{2k-1}\big(D_\sigma : \ell^{N_j}_{p_1} \mapsto \ell^{N_j}_{p_2}\big) \le C\, N_j^{\beta - \frac{\xi}{p_1}}\, k^{\frac{1}{r}-\beta} e_k\big(\mathrm{id} : \ell^{N_j}_{p_1} \mapsto \ell^{N_j}_{p_2}\big). \tag{69}$$

Let us choose first r such that $\frac{1}{r} > \max(0, \frac{1}{p})$, and now $\beta > 0$ such that $\frac{1}{s} = \frac{1}{r} - \beta > \max(0, \frac{1}{p})$. Then (68) and (64) or (66), respectively, imply

$$L^{(e)}_{r,\infty}\big(D_\sigma : \ell^{N_j}_{p_1} \mapsto \ell^{N_j}_{p_2}\big) \le C N_j^{\beta - \frac{\xi}{p_1}} (N_j + 1)^{\frac{1}{r} - \beta - \frac{1}{p}} \le C 2^{jn(\frac{1}{r} - \frac{\xi}{p_1} - \frac{1}{p})}, \tag{70}$$

where the constant C depends on γ, but is independent of j.

Substep 2.2. Now, for given $M \in \mathbb{N}_0$, let

$$P \coloneqq \sum_{j=0}^{M} P_j \qquad \text{and} \qquad Q \coloneqq \sum_{j=M+1}^{\infty} P_j. \tag{71}$$

The expression $L^{(e)}_{r,\infty}(T)$ is a quasi-norm therefore there exists a number $0 < \varrho \le 1$ such that

$$L^{(e)}_{r,\infty}\Big(\sum_j T_j\Big)^\varrho \le \sum_j L^{(e)}_{r,\infty}(T_j)^\varrho, \tag{72}$$

see König [22, 1.c.5]. Hence, (59) and (65)–(72) yield

$$\begin{aligned}L^{(e)}_{r,\infty}(P : B_1 \mapsto B_2)^\varrho &\le \sum_{j=0}^{M} L^{(e)}_{r,\infty}(P_j : B_1 \mapsto B_2)^\varrho \le c_1 \sum_{j=0}^{M} 2^{jn\varrho(\frac{1}{r} - (\frac{\xi}{p_1} + \frac{1}{p}) - \frac{\theta}{n})}\\ &\le c_2\, 2^{nM\varrho(\frac{1}{r} - (\frac{\xi}{p_1} + \frac{1}{p}) - \frac{\theta}{n})}\end{aligned} \tag{73}$$

with a constant c_2 independent of M, if $\frac{1}{r} \geq \frac{\xi}{p_1} + \frac{1}{p} + \frac{\theta}{n}$. Hence

$$e_{2^{nM}}(P : B_1 \mapsto B_2) \leq c_3\, 2^{-nM(\frac{\theta}{n} + \frac{\xi}{p_1} + \frac{1}{p})}. \tag{74}$$

We proceed similarly to (73) and obtain

$$L^{(e)}_{r,\infty}(Q : B_1 \mapsto B_2)^{\varrho} \leq c_1 \sum_{j=M+1}^{\infty} 2^{jn\varrho(\frac{1}{r} - (\frac{\xi}{p_1} + \frac{1}{p}) - \frac{\theta}{n})} \leq c_2 2^{nM\varrho(\frac{1}{r} - (\frac{\xi}{p_1} + \frac{1}{p}) - \frac{\theta}{n})}$$

if

$$\max\Bigl(0, \frac{1}{p}, \frac{\xi}{p_1} + \frac{1}{p}\Bigr) < \frac{1}{r} < \frac{\xi}{p_1} + \frac{1}{p} + \frac{\theta}{n}.$$

This is always possible in view of our assumptions. Hence

$$e_{2^{nM}}(Q : B_1 \mapsto B_2) \leq c_3\, 2^{-nM(\frac{\theta}{n} + \frac{\xi}{p_1} + \frac{1}{p})}. \tag{75}$$

Summarizing we get from (74) and (75) (replacing nM by M),

$$\begin{aligned} e_{2^{M+1}}(\mathrm{id} : B_1 \mapsto B_2) &\leq e_{2^M}(P : B_1 \mapsto B_2) + e_{2^M}(Q : B_1 \mapsto B_2) \\ &\leq c\, 2^{-M(\frac{\theta}{n} + \frac{\xi}{p_1} + \frac{1}{p})}. \end{aligned}$$

Now by monotonicity of the entropy numbers the estimate from above follows.

Step 3. To estimate the entropy numbers from below we regard the diagonal operator D_σ between finite-dimensional sequence spaces $\ell_{p_1}^{N_j}$ and $\ell_{p_2}^{N_j}$. Let an operator $Q_j : \ell_{p_1}^{N_j}(w_\xi) \mapsto B_1$ be given by

$$(Q_j\lambda)_{u,l} := \begin{cases} \lambda_l & \text{if } u = j, \\ 0 & \text{otherwise}, \end{cases} \qquad u \in \mathbb{N}_0, \quad l \in \mathbb{N}_0.$$

It should be clear that $\|Q_j\| \leq 2^{j\theta}$. The identity operator $\mathrm{Id} : \ell_{p_1}^{N_j}(w_\xi) \mapsto \ell_{p_2}^{N_j}$ can be factorized by

$$\mathrm{Id} = P_j \circ \mathrm{id} \circ Q_j,$$

where the operator P_j is regarded as an operator acting between B_2 and $\ell_{p_2}^{N_j}$. Since $\|P_j\| = 1$, we get

$$\begin{aligned} e_k\bigl(\mathrm{Id} : \ell_{p_1}^{N_j}(w_\xi) \mapsto \ell_{p_2}^{N_j}\bigr) &\leq \|Q_j\|\|P_j\| e_k(\mathrm{id} : B_1 \mapsto B_2) \\ &\leq 2^{j\theta} e_k(\mathrm{id} : B_1 \mapsto B_2). \end{aligned} \tag{76}$$

First we consider the case $\xi > 0$. By the result of Gordon, König, and Schütt, see [17] or [22, p. 131], we have

$$\sup_{l\in\mathbb{N}} 2^{-k/(2l)} (\sigma_0 \cdot \sigma_1 \cdots \sigma_{l-1})^{1/l} \le e_{k+1}\big(D_\sigma : \ell_{p_1}^{N_j} \mapsto \ell_{p_1}^{N_j}\big) \le 6 \sup_{l\in\mathbb{N}} 2^{-k/(2l)} (\sigma_0\sigma_1 \cdots \sigma_{l-1})^{1/l}.$$

Let

$$C_{\xi,p_1} = \sup_{\lambda>0} e^{\xi/p_1} \frac{\lambda^{\xi/p_1}}{2^{\lambda/2}}.$$

Stirling's formula yields

$$\frac{e^{\frac{11\xi}{12p_1}}}{\sqrt{2}} \big(\sqrt{2\pi}\sqrt[6]{3}\big)^{-\xi/p_1} k^{-\xi/p_1} \le e_{k+1}\big(D_\sigma : \ell_{p_1}^{\gamma 2^{nj}} \mapsto \ell_{p_1}^{\gamma 2^{nj}}\big) \le 6C_{\xi,p_1} k^{-\xi/p_1} \tag{77}$$

and the constant C_{ξ,p_1} is independent of j and $\gamma = N_j 2^{-nj}$. Now using (57) and (77) we get

$$\begin{aligned} c\, k^{-\xi/p_1} &\le e_k\big(D_\sigma : \ell_{p_1}^{N_j} \mapsto \ell_{p_1}^{N_j}\big) \\ &\le e_k\big(D_\sigma : \ell_{p_1}^{N_j} \mapsto \ell_{p_2}^{N_j}\big)\big\|\mathrm{id} : \ell_{p_2}^{N_j} \mapsto \ell_{p_1}^{N_j}\big\| \\ &\le C 2^{jn/p} e_k\big(\mathrm{Id} : \ell_{p_1}^{N_j}(w_\xi) \mapsto \ell_{p_2}^{N_j}\big). \end{aligned} \tag{78}$$

Thus (76) and (78) imply for $k = 2^{nj}$,

$$c 2^{-jn(\frac{\theta}{n} + \frac{\xi}{p_1} + \frac{1}{p})} \le e_{2^{nj}}(\mathrm{id} : B_1 \mapsto B_2)\ .$$

When $\xi < 0$, note that for $\tilde{\sigma}_l = \sigma_l^{-1}$ we have $\mathrm{Id} = D_\sigma D_{\tilde{\sigma}}$. So by (57) and (77) we get

$$\begin{aligned} e_{2k}\big(\mathrm{Id} : \ell_{p_1}^{N_j} \mapsto \ell_{p_2}^{N_j}\big) &\le e_k\big(D_{\tilde{\sigma}} : \ell_{p_1}^{N_j} \mapsto \ell_{p_1}^{N_j}\big) e_k\big(D_\sigma : \ell_{p_1}^{N_j} \mapsto \ell_{p_2}^{N_j}\big) \\ &\le C\, k^{\xi/p_1}\, e_k\big(\mathrm{Id} : \ell_{p_1}^{N_j}(w_\xi) \mapsto \ell_{p_2}^{N_j}\big). \end{aligned}$$

We take $k = 2^{nj}$. Schütt's lower estimates of $e_k\big(\mathrm{Id} : \ell_{p_1}^{N_j} \mapsto \ell_{p_2}^{N_j}\big)$, $0 < p_1 \le p_2 \le \infty$, and (76) imply

$$\begin{aligned} c 2^{-nj\frac{1}{p}} &\le e_{2^{nj+1}}\big(\mathrm{Id} : \ell_{p_1}^{N_j} \mapsto \ell_{p_2}^{N_j}\big) \ \le\ c_1\ 2^{nj\frac{\xi}{p_1}} e_{2^{nj}}\big(\mathrm{Id} : \ell_{p_1}^{N_j}(w_\xi) \mapsto \ell_{p_2}^{N_j}\big) \\ &\le c_2\ 2^{nj(\frac{\xi}{p_1} + \frac{\theta}{p_1})}\ e_{2^{nj}}(\mathrm{id} : B_1 \mapsto B_2). \end{aligned}$$

If $0 < p_2 \le p_1 \le \infty$ one should use the estimate $e_k\big(\mathrm{Id} : \ell_{p_1}^{N_j} \mapsto \ell_{p_2}^{N_j}\big) \sim 2^{-\frac{k}{2N_j}} N_j^{-\frac{1}{p}}$ and (76). This finishes the proof. □

Before we present our result concerning the asymptotic behavior of entropy numbers for the compact embedding (51) with weights of type (50), we recall the following result of [23, Thm. 1; 24, Cors. 4.15, 4.16]. Let the weight $w^\beta(x)$, $\beta > 0$, be given by

$$w^\beta(x) = (1 + |x|^2)^{\beta/2}, \quad x \in \mathbb{R}^n. \tag{79}$$

Proposition 3.2. *Let the parameters satisfy* (26) *with* $\delta > 0$, *recall* (23). *Assume* $\beta > 0$. *Then the embedding*

$$\ell_{q_1}\big(2^{j\delta}\ell_{p_1}(w^\beta)\big) \longrightarrow \ell_{q_2}(\ell_{p_2})$$

is compact if, and only if,

$$\min\bigg(\delta, \frac{\beta}{p_1}\bigg) > \frac{n}{p^*}.$$

In that case one has for $\delta \neq \frac{\beta}{p_1}$,

$$e_k\Big(\ell_{q_1}\big(2^{j\delta}\ell_{p_1}(w^\beta)\big) \hookrightarrow \ell_{q_2}(\ell_{p_2})\Big) \sim k^{-\frac{\min(\delta,\beta/p_1)}{n} - \frac{1}{p_1} + \frac{1}{p_2}}, \quad k \in \mathbb{N}.$$

In case of $\delta = \frac{\beta}{p_1}$, *one has for* $\tau = \frac{s_1 - s_2}{n} + \frac{1}{q_2} - \frac{1}{q_1} > 0$ *that*

$$e_k\Big(\ell_{q_1}\big(2^{j\delta}\ell_{p_1}(w^\beta)\big) \hookrightarrow \ell_{q_2}(\ell_{p_2})\Big) \sim k^{-\frac{s_1-s_2}{n}}(1+\log k)^\tau, \quad k \in \mathbb{N},$$

whereas $\tau < 0$ *leads to*

$$e_k\Big(\ell_{q_1}\big(2^{j\delta}\ell_{p_1}(w^\beta)\big) \hookrightarrow \ell_{q_2}(\ell_{p_2})\Big) \sim k^{-\frac{s_1-s_2}{n}}, \quad k \in \mathbb{N}.$$

Remark 3.3. Note that the compactness assertion coincides with Proposition 2.6 (iii) with $\alpha = 0$. Embeddings of functions spaces with weights of type (79) have been studied by many authors for several years, in particular the limiting case $\delta = \frac{\beta}{p_1}$ attracted a lot of attention; we do not want to report on this history here. There are two-sided estimates for the case $\delta = \beta/p_1$, $\tau = 0$, in [23].

Recall that $A^s_{p,q}$ stands for either $B^s_{p,q}$ or $F^s_{p,q}$ if no distinction is needed.

Theorem 3.4. *Let the parameters satisfy* (26) *and let the weight* $w \in \mathcal{A}_\infty$ *be of type* (50) *with*

$$\frac{\beta}{p_1} > \frac{n}{p*}, \quad \alpha > -n, \quad \text{and} \quad \delta > \max\bigg(\frac{n}{p*}, \frac{\alpha}{p_1}\bigg). \tag{80}$$

(i) *If* $\frac{\beta}{p_1} < \delta$, *then*

$$e_k\big(A^{s_1}_{p_1,q_1}(\mathbb{R}^n, w) \hookrightarrow A^{s_2}_{p_2,q_2}(\mathbb{R}^n)\big) \sim k^{-(\frac{\beta}{np_1} + \frac{1}{p_1} - \frac{1}{p_2})}.$$

(ii) *If* $\frac{\beta}{p_1} > \delta$, *then*

$$e_k\big(A^{s_1}_{p_1,q_1}(\mathbb{R}^n, w) \hookrightarrow A^{s_2}_{p_2,q_2}(\mathbb{R}^n)\big) \sim k^{-\frac{s_1-s_2}{n}}.$$

(iii) *If* $\frac{\beta}{p_1} = \delta$ *and* $\tau = \frac{s_1-s_2}{n} + \frac{1}{q_2} - \frac{1}{q_1} > 0$*, then*

$$e_k\big(B^{s_1}_{p_1,q_1}(\mathbb{R}^n, w) \hookrightarrow B^{s_2}_{p_2,q_2}(\mathbb{R}^n)\big) \sim k^{-\frac{s_1-s_2}{n}}(1+\log k)^{\tau}.$$

(iv) *If* $\frac{\beta}{p_1} = \delta$ *and* $\tau < 0$*, then*

$$e_k\big(B^{s_1}_{p_1,q_1}(\mathbb{R}^n, w) \hookrightarrow B^{s_2}_{p_2,q_2}(\mathbb{R}^n)\big) \sim k^{-\frac{s_1-s_2}{n}}.$$

Remark 3.5. It is surprising that α, independently of its value (within the given bounds), does not influence the asymptotic behavior of entropy numbers.

Proof. First we consider the Besov spaces. It follows from Theorem 1.13 that

$$e_k\big(B^{s_1}_{p_1,q_1}(\mathbb{R}^n, w) \hookrightarrow B^{s_2}_{p_2,q_2}(\mathbb{R}^n)\big) \sim e_k\big(b^{\sigma_1}_{p_1,q_1}(w) \hookrightarrow b^{\sigma_2}_{p_2,q_2}\big),$$

where $\sigma_i = s_i + \frac{n}{2} - \frac{n}{p_i}$, see the proof of Proposition 2.1. So we can deal with sequence spaces. It should be clear that it is sufficient to regard the weights $w = w_{\alpha,\beta}$. Moreover,

$$\begin{aligned} e_k\big(b^{\sigma_1}_{p_1,q_1}(w_{\alpha,\beta}) \hookrightarrow b^{\sigma_2}_{p_2,q_2}\big) &\sim e_k\big(b^{\delta}_{p_1,q_1}(w_{\alpha,\beta}) \hookrightarrow b_{p_2,q_2}\big) \\ &\sim e_k\big(\ell_{q_1}(2^{j\delta}\ell_{p_1}(\widetilde{w}_{\alpha,\beta})) \hookrightarrow \ell_{q_2}(\ell_{p_2})\big), \end{aligned}$$

where $\delta = \sigma_1 - \sigma_2 = s_1 - \frac{n}{p_1} - s_2 + \frac{n}{p_2}$, see (23), and

$$\widetilde{w}_{\alpha,\beta}(j,l) = \begin{cases} 1 & \text{if } l = 0, \\ (2^{-j}l)^{\alpha/n} & \text{if } 2^{-j}l < 1, \\ (2^{-j}l)^{\beta/n} & \text{if } 2^{-j}l \geq 1. \end{cases}$$

We divide the identity operator

$$\mathrm{Id} : \ell_{q_1}\big(2^{j\delta}\ell_{p_1}(\widetilde{w}_{\alpha,\beta})\big) \longrightarrow \ell_{q_2}(\ell_{p_2})$$

into two parts

$$\mathrm{Id} = \mathrm{Id}_1 + \mathrm{Id}_2,$$

where

$$\mathrm{Id}_1 : \ell_{q_1}\big(2^{j\delta}\ell^{2^{jn}}_{p_1}(\widetilde{w}_{\alpha,\beta})\big) \longrightarrow \ell_{q_2}(\ell_{p_2})$$

and

$$\mathrm{Id}_2 : \ell_{q_1}\big(2^{j\delta}\widetilde{\ell}^{2^{jn}}_{p_1}(\widetilde{w}_{\alpha,\beta})\big) \longrightarrow \ell_{q_2}(\ell_{p_2}).$$

Now it follows from Lemma 3.1 with $\theta = \delta - \frac{\alpha}{p_1}$ and $\xi = \frac{\alpha}{n}$ that

$$e_k(\mathrm{Id}_1) \sim k^{-(\frac{\delta}{n}+\frac{1}{p_1}-\frac{1}{p_2})} = k^{-\frac{s_1-s_2}{n}}.$$

The estimate of $e_k(\mathrm{Id}_2)$ follows from the estimates for the weights $w^{\beta}(x) = (1+|x|^2)^{\beta/2}$, see Proposition 3.2. The corresponding estimates for Triebel-Lizorkin spaces follow by Proposition 1.8 and the properties of entropy numbers. □

We turn to the related double-weighted situation. Using the same method as above we get for entropy numbers:

Theorem 3.6. *Let the parameters satisfy* (28) *and let the weights* $w_{\alpha_1,\beta_1}, w_{\alpha_2,\beta_2} \in \mathcal{A}_\infty$ *be of type* (50) *with*

$$\frac{\beta_1}{p_1} - \frac{\beta_2}{p_2} > \frac{n}{p^*} \qquad \text{and} \qquad \delta > \max\Big(\frac{\alpha_1}{p_1} - \frac{\alpha_2}{p_2}, \frac{n}{p^*}\Big).$$

(i) *If* $\frac{\beta_1}{p_1} - \frac{\beta_2}{p_2} < \delta$, *then*

$$e_k\big(A^{s_1}_{p_1,q_1}(\mathbb{R}^n, w_{\alpha_1,\beta_1}) \hookrightarrow A^{s_2}_{p_2,q_2}(\mathbb{R}^n, w_{\alpha_2,\beta_2})\big) \sim k^{-\frac{1}{n}(\frac{n+\beta_1}{p_1} - \frac{n+\beta_2}{p_2})}.$$

(ii) *If* $\frac{\beta_1}{p_1} - \frac{\beta_2}{p_2} > \delta$, *then*

$$e_k\big(A^{s_1}_{p_1,q_1}(\mathbb{R}^n, w_{\alpha_1,\beta_1}) \hookrightarrow A^{s_2}_{p_2,q_2}(\mathbb{R}^n, w_{\alpha_2,\beta_2})\big) \sim k^{-\frac{s_1-s_2}{n}}.$$

(iii) *If* $\frac{\beta_1}{p_1} - \frac{\beta_2}{p_2} = \delta$ *and* $\tau = \frac{s_1-s_2}{n} + \frac{1}{q_2} - \frac{1}{q_1} > 0$, *then*

$$e_k\big(B^{s_1}_{p_1,q_1}(\mathbb{R}^n, w_{\alpha_1,\beta_1}) \hookrightarrow B^{s_2}_{p_2,q_2}(\mathbb{R}^n, w_{\alpha_2,\beta_2})\big) \sim k^{-\frac{s_1-s_2}{n}}(1+\log k)^\tau.$$

(iv) *If* $\frac{\beta_1}{p_1} - \frac{\beta_2}{p_2} = \delta$ *and* $\tau < 0$, *then*

$$e_k\big(B^{s_1}_{p_1,q_1}(\mathbb{R}^n, w_{\alpha_1,\beta_1}) \hookrightarrow B^{s_2}_{p_2,q_2}(\mathbb{R}^n, w_{\alpha_2,\beta_2})\big) \sim k^{-\frac{s_1-s_2}{n}}.$$

Remark 3.7. The proof of the above theorem is the same as the proof of Theorem 3.4. In particular we use Lemma 3.1 and Proposition 3.2. There is nothing substantially new in this approach, but it has some interesting consequences concerning embeddings of type (49). In particular, if we take $\alpha_2 = \beta_2 = 0$, then Theorem 3.6 coincides with Theorem 3.4. If $\alpha_1 = \alpha_2 = \alpha$ and $\beta_1 = \beta_2 = \beta$ and

$$\beta > 0, \qquad p_1 < p_2, \qquad \delta > \max(\alpha, 0)\Big(\frac{1}{p_1} - \frac{1}{p_2}\Big),$$

then

$$e_k\big(B^{s_1}_{p_1,q_1}(\mathbb{R}^n, w_{\alpha,\beta}) \hookrightarrow B^{s_2}_{p_2,q_2}(\mathbb{R}^n, w_{\alpha,\beta})\big) \sim \begin{cases} k^{-\frac{n+\beta}{n}\left(\frac{1}{p_1}-\frac{1}{p_2}\right)}, & \beta(\frac{1}{p_1} - \frac{1}{p_2}) < \delta, \\ k^{-\frac{s_1-s_2}{n}}, & \beta(\frac{1}{p_1} - \frac{1}{p_2}) > \delta, \\ k^{-\frac{s_1-s_2}{n}}(1+\log k)^\tau, & \beta(\frac{1}{p_1} - \frac{1}{p_2}) = \delta, \quad \tau > 0, \\ k^{-\frac{s_1-s_2}{n}}, & \beta(\frac{1}{p_1} - \frac{1}{p_2}) = \delta, \quad \tau < 0, \end{cases}$$

where $\tau = \frac{s_1-s_2}{n} + \frac{1}{q_2} - \frac{1}{q_1}$ as above.

4. Approximation numbers of compact embeddings

Finally, we study corresponding approximation numbers. For convenience we recall the well known estimates of approximation numbers of embeddings of finite-dimensional spaces, see [7, 12, 30].

Lemma 4.1. *Let* $N, k \in \mathbb{N}$.

(i) *If* $0 < p_1 \le p_2 \le 2$ *or* $2 \le p_1 \le p_2 \le \infty$, *then there is a positive constant* C *independent of* N *and* k *such that*

$$a_k\big(\mathrm{id}: \ell^N_{p_1} \mapsto \ell^N_{p_2}\big) \le C \begin{cases} 1 & \text{if} \quad k \le N, \\ 0 & \text{if} \quad k > N. \end{cases}$$

(ii) *If* $0 < p_1 < 2 < p_2 \le \infty$, $(p_1, p_2) \ne (1, \infty)$, *then there is a positive constant* C *independent of* N *and* k *such that*

$$a_k\big(\mathrm{id}: \ell^N_{p_1} \mapsto \ell^N_{p_2}\big) \le C \begin{cases} 1 & \text{if} \quad k \le N^{2/t}, \\ N^{1/t}k^{-1/2} & \text{if} \quad N^{2/t} < k \le N, \\ 0 & \text{if} \quad k > N, \end{cases}$$

where $\frac{1}{t} = \max\big(\frac{1}{p_2}, \frac{1}{p_1'}\big)$.

(iii) *Let* $0 < p_2 < p_1 \le \infty$. *Then*

$$a_k\big(\mathrm{id}: \ell^N_{p_1} \mapsto \ell^N_{p_2}\big) = \big(N - k + 1\big)^{1/p_2 - 1/p_1}, \qquad k = 1, \dots, N.$$

Moreover, if $k \le \frac{N}{4}$, *then we have an equivalence in* (i) *and* (ii).

Lemma 4.2.

(i) *Let* $0 < p_1 \le p_2 \le \infty$, $(p_1, p_2) \ne (1, \infty)$, $0 < q_1, q_2 \le \infty$, $\theta > 0$, *and* $\frac{\xi}{p_1} > -\frac{\theta}{n}$. *Then for all* $k \in \mathbb{N}$ *the following estimate*

$$a_k\Big(\mathrm{id}: \ell_{q_1}\big(2^{j\theta}\ell^{\gamma 2^{nj}}_{p_1}(w_\xi)\big) \to \ell_{q_2}\big(\ell^{\gamma 2^{nj}}_{p_2}\big)\Big) \sim k^{-\varkappa}$$

holds, where

$$\varkappa = \begin{cases} \frac{\theta}{n} + \frac{\xi}{p_1} & \text{for} \quad 0 < p_1 \le p_2 \le 2 \quad \text{or} \quad 2 \le p_1 < p_2 \le \infty, \\ \frac{\theta}{n} + \frac{\xi}{p_1} + \frac{1}{2} - \frac{1}{t} & \text{for} \quad 0 < p_1 < 2 < p_2 \le \infty \quad \text{and} \quad \frac{\theta}{n} + \frac{\xi}{p_1} > \frac{1}{t}, \\ \big(\frac{\theta}{n} + \frac{\xi}{p_1}\big)\frac{t}{2} & \text{for} \quad 0 < p_1 < 2 < p_2 \le \infty \quad \text{and} \quad \frac{\theta}{n} + \frac{\xi}{p_1} < \frac{1}{t}, \end{cases}$$

and $t = \min(p_2, p_1')$.

(ii) *Let $0 < p_2 < p_1 \le \infty$, $0 < q_1, q_2 \le \infty$, $\theta > 0$. We assume that $\frac{\xi}{p_1} > -\frac{\theta}{n} + \frac{1}{p^*}$. Then for all $k \in \mathbb{N}$ the following estimate*

$$a_k(\mathrm{id} : \ell_{q_1}(2^{j\theta}\ell_{p_1}^{\gamma 2^{nj}}(w_\xi)) \to \ell_{q_2}(\ell_{p_2}^{\gamma 2^{nj}})) \sim k^{-\left(\frac{\theta}{n}+\frac{\xi}{p_1}+\frac{1}{p_1}-\frac{1}{p_2}\right)}$$

holds.

Proof. Step 1. Preparations. We put $N_j = \gamma 2^{nj}$, $1/p = 1/p_1 - 1/p_2$, and use the notations (53), (54), and (55). Monotonicity arguments and elementary properties of the approximation numbers yield parallel to (56) and (57),

$$a_k\big(P_j : B_1 \mapsto B_2\big) \le 2^{-j\theta}\, a_k\big(\mathrm{Id} : \ell_{p_1}^{N_j}(w_\xi) \mapsto \ell_{p_2}^{N_j}\big), \tag{81}$$

and

$$a_k\big(\mathrm{Id} : \ell_{p_1}^{N_j}(w_\xi) \mapsto \ell_{p_2}^{N_j}\big) = a_k\big(D_\sigma : \ell_{p_1}^{N_j} \mapsto \ell_{p_2}^{N_j}\big), \tag{82}$$

where D_σ is again the diagonal operator defined by the sequence $\sigma_l = (1+l)^{-\frac{\xi}{p_1}}$. Using (81) and (82) we find

$$L_{r,\infty}^{(a)}(P_j : B_1 \mapsto B_2) \le c\, 2^{-j\theta}\, L_{r,\infty}^{(a)}(D_\sigma : \ell_{p_1}^{N_j} \mapsto \ell_{p_2}^{N_j}), \tag{83}$$

where $L_{r,\infty}^{(a)}(T)$ is the obvious counterpart of (58) for approximation numbers.

Step 2. We consider the estimate from above.
Substep 2.1. First we estimate $L_{r,\infty}^{(a)}(D_\sigma : \ell_{p_1}^{N_j} \mapsto \ell_{p_2}^{N_j})$. We prove that

$$L_{r,\infty}^{(a)}\big(D_\sigma : \ell_{p_1}^{N_j} \mapsto \ell_{p_2}^{N_j}\big) \le C\, 2^{jn(\frac{1}{r}-\frac{\xi}{p_1})}, \tag{84}$$

if $0 < p_1 \le p_2 \le 2$ or $2 \le p_1 \le p_2 \le \infty$. On the other hand, for $0 < p_1 < 2 < p_2 \le \infty$ and $(p_1, p_2) \ne (1, \infty)$ we show that

$$L_{r,\infty}^{(a)}\big(D_\sigma : \ell_{p_1}^{N_j} \mapsto \ell_{p_2}^{N_j}\big) \le C 2^{jn(\frac{1}{r}-\frac{1}{2}+\frac{1}{t}-\frac{\xi}{p_1})} \quad \text{if } \frac{1}{r} > \max\Big(\frac{1}{2}, \frac{\xi}{p_1} + \frac{1}{2} - \frac{1}{t}\Big), \tag{85}$$

$$L_{r,\infty}^{(a)}\big(D_\sigma : \ell_{p_1}^{N_j} \mapsto \ell_{p_2}^{N_j}\big) \le C\, 2^{jn(\frac{2}{rt}-\frac{\xi}{p_1})} \quad \text{if } \max\Big(0, \frac{t\xi}{2p_1}\Big) < \frac{1}{r} < \frac{1}{2}. \tag{86}$$

The last inequality holds if $\frac{\xi}{p_1} < \frac{1}{t}$.

Let us choose $K \in \mathbb{N}$ such that $2^{n(K-1)} \le N_j < 2^{nK}$ and let $\Pi_i : \ell_{p_1}^{N_j} \mapsto \ell_{p_1}^{N_j}$ be a projection defined in the following way

$$(\Pi_i \lambda)_l := \begin{cases} \lambda_l & \text{if } \ 2^{n(i-1)} \le l < \min(2^{ni}, N_j), \\ 0 & \text{otherwise,} \end{cases}$$

for $\lambda = \{\lambda_l\}$, and $i = 1, 2, \ldots, K$. Then

$$D_\sigma = \sum_{i=1}^{K} D_\sigma \circ \Pi_i.$$

Multiplicativity of approximation numbers yields

$$a_k\big(D_\sigma \circ \Pi_i : \ell_{p_1}^{N_j} \mapsto \ell_{p_2}^{N_j}\big) \le C\, 2^{-n(i-1)\frac{\xi}{p_1}}\, a_k\big(\mathrm{id} : \ell_{p_1}^{2^{n(i-1)}} \mapsto \ell_{p_2}^{2^{n(i-1)}}\big), \quad i \in \mathbb{N}.$$

Let r be some real positive number. Now part (i) of Lemma 4.1 implies

$$L_{r,\infty}^{(a)}\big(D_\sigma \circ \Pi_i : \ell_{p_1}^{N_j} \mapsto \ell_{p_2}^{N_j}\big) \le C\, 2^{n(i-1)(\frac{1}{r}-\frac{\xi}{p_1})} \tag{87}$$

if $0 < p_1 \le p_2 \le 2$ or $2 \le p_1 \le p_2 \le \infty$. In a similar way part (ii) of Lemma 4.1 implies

$$L_{r,\infty}^{(a)}\big(D_\sigma \circ \Pi_i : \ell_{p_1}^{N_j} \mapsto \ell_{p_2}^{N_j}\big) \le C\, 2^{n(i-1)(\frac{1}{t}+\frac{1}{r}-\frac{1}{2}-\frac{\xi}{p_1})} \qquad \text{if } \frac{1}{2} < \frac{1}{r}, \tag{88}$$

$$L_{r,\infty}^{(a)}\big(D_\sigma \circ \Pi_i : \ell_{p_1}^{N_j} \mapsto \ell_{p_2}^{N_j}\big) \le C\, 2^{n(i-1)(\frac{2}{rt}-\frac{\xi}{p_1})} \qquad \text{if } 0 < \frac{1}{r} < \frac{1}{2}, \tag{89}$$

for $0 < p_1 < 2 < p_2 \le \infty$.

If $0 < p_1 \le p_2 \le 2$ or $2 \le p_1 \le p_2 \le \infty$ we choose $\frac{1}{r} > \max\big(\frac{\xi}{p_1}, 0\big)$. Then the (adapted) argument of (72) and (87) gives

$$\begin{aligned} L_{r,\infty}^{(a)}\big(D_\sigma : \ell_{p_1}^{N_j} \mapsto \ell_{p_2}^{N_j}\big)^\varrho &\le \sum_{i=1}^{K} L_{r,\infty}^{(a)}(D_\sigma \circ \Pi_i : \ell_{p_1}^{N_j} \mapsto \ell_{p_2}^{N_j})^\varrho \\ &\le c_1 \sum_{i=0}^{K-1} 2^{-\varrho n i\left(\frac{\xi}{p_1}-\frac{1}{r}\right)} \\ &\le c_2\, 2^{\varrho n K\left(\frac{1}{r}-\frac{\xi}{p_1}\right)} \end{aligned}$$

for some ϱ, $0 < \varrho \le 1$. This implies (84) in view of our choice of K.

Now let $0 < p_1 < 2 < p_2 \le \infty$. We choose r such that $\frac{1}{r} > \frac{1}{2}$ and that $\frac{1}{r} - \frac{1}{2} + \frac{1}{t} - \frac{\xi}{p_1} > 0$. The formula (88) yields

$$\begin{aligned} L_{r,\infty}^{(a)}\big(D_\sigma : \ell_{p_1}^{N_j} \mapsto \ell_{p_2}^{N_j}\big)^\varrho &\le \sum_{i=1}^{K} L_{r,\infty}^{(a)}\big(D_\sigma \circ \Pi_i : \ell_{p_1}^{N_j} \mapsto \ell_{p_2}^{N_j}\big)^\varrho \\ &\le c_1 \sum_{i=0}^{K-1} 2^{-\varrho n i\left(\frac{\xi}{p_1}-\frac{1}{t}-\frac{1}{r}+\frac{1}{2}\right)} \\ &\le c_2\, 2^{\varrho n K\left(\frac{1}{r}-\frac{1}{2}+\frac{1}{t}-\frac{\xi}{p_1}\right)}. \end{aligned}$$

The constant c_2 is independent of K. This implies (85). Similarly, if $\max\big(0, \frac{t\xi}{2p_1}\big) < \frac{1}{r} < \frac{1}{2}$, then (89) yields

$$\begin{aligned} L^{(a)}_{r,\infty}\big(D_\sigma : \ell^{N_j}_{p_1} \mapsto \ell^{N_j}_{p_2}\big)^\varrho &\le \sum_{i=1}^{K} L^{(a)}_{r,\infty}\big(D_\sigma \circ \Pi_i : \ell^{N_j}_{p_1} \mapsto \ell^{N_j}_{p_2}\big)^\varrho \\ &\le c_1 \sum_{i=0}^{K-1} 2^{-\varrho n i\left(\frac{\xi}{p_1} - \frac{2}{rt}\right)} \\ &\le c_2\, 2^{\varrho n K\left(\frac{2}{rt} - \frac{\xi}{p_1}\right)} \end{aligned}$$

and the constant c_2 is independent of K. This implies (86).

Substep 2.2. Now let $0 < p_2 < p_1 \le \infty$. We prove that

$$L^{(a)}_{r,\infty}\big(D_\sigma : \ell^{N_j}_{p_1} \mapsto \ell^{N_j}_{p_2}\big) \le C\, 2^{jn(\frac{1}{r} + \frac{1}{p} - \frac{\xi}{p_1})}. \tag{90}$$

We can deal with that case similar to Substep 2.1 using now part (iii) of Lemma 4.1. We obtain

$$L^{(a)}_{r,\infty}\big(D_\sigma \circ \Pi_i : \ell^{N_j}_{p_1} \mapsto \ell^{N_j}_{p_2}\big) \le C\, 2^{n(i-1)(\frac{1}{r} + \frac{1}{p} - \frac{\xi}{p_1})}$$

if $0 < r < \infty$. This leads to

$$\begin{aligned} L^{(a)}_{r,\infty}\big(D_\sigma : \ell^{N_j}_{p_1} \mapsto \ell^{N_j}_{p_2}\big)^\varrho &\le \sum_{i=1}^{K} L^{(a)}_{r,\infty}\big(D_\sigma \circ \Pi_i : \ell^{N_j}_{p_1} \mapsto \ell^{N_j}_{p_2}\big)^\varrho \\ &\le c_1 \sum_{i=0}^{K-1} 2^{-\varrho n i\left(\frac{\xi}{p_1} - \frac{1}{p} - \frac{1}{r}\right)} \\ &\le c_2\, 2^{\varrho n K\left(\frac{1}{r} + \frac{1}{p} - \frac{\xi}{p_1}\right)}, \end{aligned}$$

if $\frac{1}{r} > \max\Big(0, \frac{\xi}{p_1} - \frac{1}{p}\Big)$. This implies (90).

Substep 2.3. First we regard the case $0 < p_1 \le p_2 \le 2$ or $2 \le p_1 \le p_2 \le \infty$. For given $M \in \mathbb{N}_0$, let P and Q be given by (71). Then (83), (84), and (71) yield

$$\begin{aligned} L^{(a)}_{r,\infty}(P : B_1 \mapsto B_2)^\varrho &\le \sum_{j=0}^{M} L^{(a)}_{r,\infty}(P_j : B_1 \mapsto B_2)^\varrho \le c_1 \sum_{j=0}^{M} 2^{-jn\varrho(\frac{\theta}{n} - \frac{1}{r} + \frac{\xi}{p_1})} \\ &\le c_2\, 2^{-Mn\varrho(\frac{\theta}{n} - \frac{1}{r} + \frac{\xi}{p_1})}, \end{aligned} \tag{91}$$

with a constant c_2 independent of M, if $\frac{1}{r} > \max\big(0, \frac{\xi}{p_1} + \frac{\theta}{n}\big)$. Hence for every $\theta > 0$ and any ξ we have

$$a_{2^{nM}}(P : B_1 \mapsto B_2) \le c_3\, 2^{-nM(\frac{\theta}{n} + \frac{\xi}{p_1})}.$$

Since $\frac{\xi}{p_1} > -\frac{\theta}{n}$ we can find a number $r > 0$ such that $\max\big(0, \frac{\xi}{p_1}\big) < \frac{1}{r} < \frac{\theta}{n} + \frac{\xi}{p_1}$. In a similar way to (91) we obtain

$$L^{(a)}_{r,\infty}(Q : B_1 \mapsto B_2)^\varrho \le c_1 \sum_{j=M+1}^{\infty} 2^{-jn\varrho(\frac{\theta}{n}-\frac{1}{r}+\frac{\xi}{p_1})} \le c_2 2^{-Mn\varrho(\frac{\theta}{n}-\frac{1}{r}+\frac{\xi}{p_1})}.$$

Consequently,

$$a_{2^{nM}}(Q : B_1 \mapsto B_2) \le c_3\, 2^{-nM(\frac{\theta}{n}+\frac{\xi}{p_1})}.$$

This implies the estimate from above for $0 < p_1 \le p_2 \le 2$ or $2 \le p_1 \le p_2 \le \infty$.

Substep 2.4. Let $0 < p_1 < 2 < p_2 \le \infty$, $(p_1, p_2) \ne (1, \infty)$. Using (85) instead of (84) we get

$$L^{(a)}_{r,\infty}(P : B_1 \mapsto B_2)^\varrho \le \sum_{j=0}^{M} L^{(a)}_{r,\infty}(P_j : B_1 \mapsto B_2)^\varrho \le c_3 2^{-Mn\varrho(\frac{\theta}{n}-\frac{1}{r}+\frac{\xi}{p_1}+\frac{1}{2}-\frac{1}{t})},$$

if $\frac{1}{r} > \max\big(\frac{\theta}{n} + \frac{\xi}{p_1} + \frac{1}{2} - \frac{1}{t}, \frac{1}{2}\big)$, and consequently

$$a_{2^{nM}}(P : B_1 \mapsto B_2) \le c_3\, 2^{-nM(\frac{\theta}{n}+\frac{\xi}{p_1}+\frac{1}{2}-\frac{1}{t})} \tag{92}$$

for $0 < p_1 < 2 < p_2 \le \infty$, $(p_1, p_2) \ne (1, \infty)$. In a parallel way we conclude

$$L^{(a)}_{r,\infty}(Q : B_1 \mapsto B_2)^\varrho \le c_4\, 2^{-nM\varrho(\frac{\theta}{n}-\frac{1}{r}+\frac{\xi}{p_1}+\frac{1}{2}-\frac{1}{t})} \tag{93}$$

if $\max\big(\frac{1}{2}, \frac{\xi}{p_1} + \frac{1}{2} - \frac{1}{t}\big) < \frac{1}{r} < \frac{\theta}{n} + \frac{\xi}{p_1} + \frac{1}{2} - \frac{1}{t}$. Now, (92) and (93) imply the estimates from above for $\frac{\theta}{n} + \frac{\xi}{p_1} > \frac{1}{t}$. On the other hand,

$$L^{(a)}_{r,\infty}(Q : B_1 \mapsto B_2)^\varrho \le c_4\, 2^{-nM\varrho(\frac{\theta}{n}+\frac{\xi}{p_1}-\frac{2}{tr})} \tag{94}$$

if $\max\big(0, \frac{t\xi}{2p_1}\big) < \frac{1}{r} < \min\big(\frac{1}{2}, \frac{t}{2}\big(\frac{\theta}{n} + \frac{\xi}{p_1}\big)\big)$ by (86).

For $\frac{\xi}{p_1} > -\frac{\theta}{n}$ satisfying $\frac{\theta}{n} + \frac{\xi}{p_1} < \frac{1}{t}$ we take $k = \big[2^{nM\frac{2}{t}}\big]$. Then $\frac{\xi}{p_1} < \frac{1}{t}$, and (94) leads to

$$a_k(Q : B_1 \mapsto B_2) \le c\, k^{-\frac{1}{r}} k^{-\frac{t}{2}(\frac{\theta}{n}+\frac{\xi}{p_1}-\frac{2}{tr})} \le c\, k^{-\frac{t}{2}(\frac{\theta}{n}+\frac{\xi}{p_1})}. \tag{95}$$

By standard arguments the above estimates hold for any positive integer k. Moreover, (92) implies

$$a_k(P : B_1 \mapsto B_2) \le C\, k^{-(\frac{\theta}{n}+\frac{\xi}{p_1}+\frac{1}{2}-\frac{1}{t})} \le C\, k^{-\frac{t}{2}(\frac{\theta}{n}+\frac{\xi}{p_1})} \tag{96}$$

since $\frac{\theta}{n}+\frac{\xi}{p_1} < \frac{1}{t}$. Now (95) and (96) give us the estimates from above in the remaining case.

Substep 2.5. Let $0 < p_2 < p_1 \le \infty$. Using (83) and (90) we get

$$L^{(a)}_{r,\infty}(P : B_1 \mapsto B_2)^{\varrho} \le \sum_{j=0}^{M} L^{(a)}_{r,\infty}(P_j : B_1 \mapsto B_2)^{\varrho} \le c_3 2^{-Mn\varrho(\frac{\theta}{n}-\frac{1}{r}-\frac{1}{p}+\frac{\xi}{p_1})},$$

if $\frac{1}{r} > \max\big(\frac{\theta}{n} + \frac{\xi}{p_1} - \frac{1}{p}, 0\big)$, and consequently

$$a_{2^{nM}}(P : B_1 \mapsto B_2) \le c_3\, 2^{-nM(\frac{\theta}{n}+\frac{\xi}{p_1}-\frac{1}{p})}.$$

In a similar way one obtains

$$L^{(a)}_{r,\infty}(Q)^{\varrho} \le c_4 2^{-nM\varrho(\frac{\theta}{n}-\frac{1}{r}+\frac{\xi}{p_1}-\frac{1}{p})} \quad \text{if} \quad \max\Big(0, \frac{\xi}{p_1} - \frac{1}{p}\Big) < \frac{1}{r} < \frac{\theta}{n} + \frac{\xi}{p_1} - \frac{1}{p}.$$

Consequently

$$a_k(Q : B_1 \mapsto B_2) \le c\, k^{-(\frac{\theta}{n}+\frac{\xi}{p_1}-\frac{1}{p})}.$$

This finishes the proof of the estimates from above.

Step 3. We estimate the approximation numbers from below. We can regard the following commutative diagram

$$\begin{array}{ccc} \ell^{N_j}_{p_1} & \xrightarrow{\ R\ } & \ell_{q_1}\big(2^{j\theta}\ell^{N_j}_{p_1}(w_\xi)\big) \\ \Big\downarrow{\scriptstyle \mathrm{Id}} & & \Big\downarrow{\scriptstyle \mathrm{id}} \\ \ell^{N_j}_{p_2} & \xleftarrow{\ P\ } & \ell_{q_2}\big(\ell^{N_j}_{p_2}\big) \end{array}$$

where the operator $R : \ell^{N_j}_{p_1} \mapsto \ell_{q_1}\big(2^{j\theta}\ell^{N_j}_{p_1}(w_\xi)\big)$ is given by

$$(R\lambda)_{u,i} := \begin{cases} \lambda_i & \text{if } u = j, \\ 0 & \text{otherwise}, \end{cases} \qquad \text{with} \quad u \in \mathbb{N}_0, \quad 0 \le i \le N_j,$$

and P is a projection. Since $\frac{\xi}{p_1} > -\frac{\theta}{n}$, we have $\|R\| \le c\, 2^{jn(\frac{\theta}{n}+\frac{\xi}{p_1})}$. Moreover, it should be clear that $\|P\| = 1$. So, by Lemma 4.1:

(i) If $0 < p_1 \le p_2 \le 2$ or $2 \le p_1 \le p_2 \le \infty$, then taking $k = \gamma 2^{nj-2}$ we get

$$1 \le C\, a_k\big(\mathrm{Id} : \ell^{N_j}_{p_1} \mapsto \ell^{N_j}_{p_2}\big) \le C 2^{nj(\frac{\theta}{n}+\frac{\xi}{p_1})}\, a_k\Big(\mathrm{id} : \ell_{q_1}\big(2^{j\theta}\ell^{N_j}_{p_1}(w_\xi)\big) \mapsto \ell_{q_2}\big(\ell^{N_j}_{p_2}\big)\Big).$$

(ii) If $0 < p_1 < 2 < p_2 \le \infty$ and $\frac{\theta}{n} + \frac{\xi}{p_1} > \frac{1}{t}$, then taking once more $k = \gamma 2^{nj-2}$ we have

$$\begin{aligned} 2^{-j(\frac{1}{2}-\frac{1}{t})} &\le C a_k\big(\mathrm{Id} : \ell^{N_j}_{p_1} \mapsto \ell^{N_j}_{p_2}\big) \\ &\le C 2^{jn(\frac{\theta}{n}+\frac{\xi}{p_1})}\, a_k\Big(\mathrm{id} : \ell_{q_1}\big(2^{j\theta}\ell^{N_j}_{p_1}(w_\xi)\big) \mapsto \ell_{q_2}\big(\ell^{N_j}_{p_2}\big)\Big). \end{aligned}$$

(iii) If $0 < p_1 < 2 < p_2 \le \infty$ and $\frac{\theta}{n} + \frac{\xi}{p_1} < \frac{1}{t}$, then we take $k = \left[2^{nj\frac{2}{t}}\right]$. So,

$$1 \le C\, a_k\big(\mathrm{Id} : \ell_{p_1}^{N_j} \mapsto \ell_{p_2}^{N_j}\big) \le C 2^{jn(\frac{\theta}{n}+\frac{\xi}{p_1})} a_k\Big(\mathrm{id} : \ell_{q_1}\big(2^{j\theta}\ell_{p_1}^{N_j}(w_\xi)\big) \mapsto \ell_{q_2}\big(\ell_{p_2}^{N_j}\big)\Big).$$

(iv) If $0 < p_2 < p_1 \le \infty$, then we take $k = \frac{N_j}{2}$ and get

$$N_j^{\frac{1}{p}} \le C a_k\big(\mathrm{Id} : \ell_{p_1}^{N_j} \mapsto \ell_{p_2}^{N_j}\big) \le C 2^{jn(\frac{\theta}{n}+\frac{\xi}{p_1})} a_k\Big(\mathrm{id} : \ell_{q_1}\big(2^{j\theta}\ell_{p_1}^{N_j}(w_\xi)\big) \mapsto \ell_{q_2}\big(\ell_{p_2}^{N_j}\big)\Big).$$

This finishes the proof. □

Theorem 4.3. *Let the parameters satisfy* (26) *and let the weight* $w \in \mathcal{A}_\infty$ *be of type* (50) *with* (80). *We assume that* $\frac{\beta}{p_1} \ne \delta$. *Then*

$$a_k\big(A^{s_1}_{p_1,q_1}(\mathbb{R}^n, w_{\alpha,\beta}) \hookrightarrow A^{s_2}_{p_2,q_2}(\mathbb{R}^n)\big) \sim k^{-\varkappa},$$

where

(i) $\varkappa = \frac{\min(\delta,\beta/p_1)}{n}$ *if* $0 < p_1 \le p_2 \le 2$ *or* $2 \le p_1 \le p_2 \le \infty$,

(ii) $\varkappa = \frac{\min(\delta,\beta/p_1)}{n} + \frac{1}{p_2} - \frac{1}{p_1}$ *if* $p_1\left(\frac{\beta}{n}+1\right)^{-1} < p_2 < p_1 < \infty$,

(iii) $\varkappa = \frac{\min(\delta,\beta/p_1)}{n} + \frac{1}{2} - \frac{1}{\min(p_1',p_2)}$ *if* $0 < p_1 < 2 < p_2 \le \infty$, $(p_1,p_2) \ne (1,\infty)$ *and* $\min\big(\frac{\beta}{p_1},\delta\big) > \frac{n}{\min(p_1',p_2)}$,

(iv) $\varkappa = \frac{\min(\delta,\beta/p_1)}{n} \cdot \frac{\min(p_1',p_2)}{2}$ *if* $0 < p_1 < 2 < p_2 \le \infty$, $(p_1,p_2) \ne (1,\infty)$ *and* $\min\big(\frac{\beta}{p_1},\delta\big) < \frac{n}{\min(p_1',p_2)}$.

Remark 4.4. The proof is similar to the proof of Theorem 3.4. The approximation numbers in double weighted situation can be treated in the same way as in Theorem 3.6 related to entropy numbers.

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