Divergent Cesàro Means of Jacobi-Sobolev Expansions

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ABSTRACT

Let μ be the Jacobi measure supported on the interval $[-1,1]$. Let introduce the Sobolev-type inner product

$$\langle f,g\rangle = \int_{-1}^{1} f(x)g(x)\,d\mu(x) + Mf(1)g(1) + Nf'(1)g'(1),$$

where $M,N \geq 0$. In this paper we prove that, for certain indices δ, there are functions whose Cesàro means of order δ in the Fourier expansion in terms of the orthonormal polynomials associated with the above Sobolev inner product are divergent almost everywhere on $[-1,1]$.

Key words: Jacobi-Sobolev type polynomials, Fourier expansion, Cesàro mean.

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Introduction

Let $d\mu(x) = (1-x)^{\alpha}(1+x)^{\beta}\,dx$, $\alpha > -1$, $\beta > -1$, be the Jacobi measure supported on the interval $[-1,1]$. Let f and g functions in $L^2(\mu)$ such that there exists the first derivative in 1. We can introduce the discrete Sobolev-type inner product

$$\langle f,g\rangle = \int_{-1}^{1} f(x)g(x)\,d\mu(x) + Mf(1)g(1) + Nf'(1)g'(1) \qquad (1)$$

where $M \geq 0$, $N \geq 0$. We denote by $\{q_n^{(\alpha,\beta)}\}_{n\geq 0}$ the sequence of orthonormal polynomials with respect to the inner product (1) (see [1]). These polynomials are known in the literature as Jacobi-Sobolev type polynomials. For $M = N = 0$, the classical Jacobi orthonormal polynomials appear. We will denote them $\{p_n^{(\alpha,\beta)}\}_{n\geq 0}$.

For every function f such that $\langle f, q_n^{(\alpha,\beta)}\rangle$ exists for $n = 0, 1, \ldots$, the Fourier expansion in Jacobi-Sobolev type polynomials is

$$\sum_{n=0}^{\infty} c_n(f) q_n^{(\alpha,\beta)}(x), \tag{2}$$

where

$$c_n(f) = \langle f, q_n^{(\alpha,\beta)}\rangle.$$

The Cesàro means of order δ of the Fourier expansion (2) are defined by (see [9, p. 76–77])

$$\sigma_N^\delta f(x) = \sum_{n=0}^{N} \frac{A_{N-n}^\delta}{A_N^\delta} c_n(f) q_n^{(\alpha,\beta)}(x),$$

where $A_k^\delta = \binom{k+\delta}{k}$.

In this contribution we will prove that there are functions such that their Cesàro means of order δ diverge almost everywhere on $[-1, 1]$. A similar result, when $M = N = 0$, has been obtained in [6].

Notice that, for an appropriate function f, the study of the convergence of Fourier series in terms of the polynomials associated to the Sobolev inner product

$$\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)\, d\mu(x) + Mf(c)g(c) + Nf'(c)g'(c)$$

when $c \in [-1, 1]$ has been presented [7] and when $c \in (1, \infty)$ in ([3, 4]) some analog results have been deduced.

Throughout this paper positive constants are denoted by $c, c_1, \ldots$. and they may vary at every occurrence. The notation $u_n \sim v_n$ means $c_1 \leq u_n/v_n \leq c_2$ for sufficiently large n, and by $u_n \cong v_n$ we mean that the sequence u_n/v_n converges to 1.

1. Jacobi-Sobolev type polynomials

Some basic properties of the polynomials $q_n^{(\alpha,\beta)}$ (see [1]) that we will need in the sequel, are given in below:

$$q_n^{(\alpha,\beta)}(x) = A_n p_n^{(\alpha,\beta)}(x) + B_n(x-1)p_{n-1}^{(\alpha+2,\beta)}(x) + C_n(x-1)^2 p_{n-2}^{(\alpha+4,\beta)}(x) \tag{3}$$

where

(i) if $M > 0$ and $N > 0$ then

$$A_n \cong -cn^{-2\alpha-2}, \qquad B_n \cong cn^{-2\alpha-2}, \qquad C_n \cong 1,$$

(ii) if $M = 0$ and $N > 0$ then

$$A_n \cong \frac{-1}{\alpha+2}, \qquad B_n \cong 1, \qquad C_n \cong \frac{1}{\alpha+2},$$

(iii) if $M > 0$ and $N = 0$ then

$$A_n \cong cn^{-2\alpha-2}, \qquad B_n \cong 1, \qquad C_n \cong 0.$$

$$|q_n^{(\alpha,\beta)}(1)| \sim \begin{cases} n^{-\alpha-3/2} & \text{if } M > 0,\ N \geq 0, \\ n^{\alpha+1/2} & \text{if } M = 0,\ N \geq 0. \end{cases} \tag{4}$$

$$(q_n^{(\alpha,\beta)})'(1) \sim n^{-\alpha-7/2} \quad \text{if} \quad M \geq 0, \ \ N > 0. \tag{5}$$

$$\max_{x\in[-1,1]} |q_n^{(\alpha,\beta)}(x)| \sim n^{\beta+1/2} \quad \text{if} \quad -1/2 \leq \alpha \leq \beta. \tag{6}$$

$$|q_n^{(\alpha,\beta)}(\cos\theta)| = \begin{cases} O(\theta^{-\alpha-1/2}(\pi-\theta)^{-\beta-1/2}) & \text{if } c/n \leq \theta \leq \pi - c/n, \\ O(n^{\alpha+1/2}) & \text{if } 0 \leq \theta \leq c/n, \\ O(n^{\beta+1/2}) & \text{if } \pi - c/n \leq \theta \leq \pi, \end{cases} \tag{7}$$

for $\alpha \geq -1/2,\ \beta \geq -1/2$, and $n \geq 1$.

The asymptotic behavior of $q_n^{(\alpha,\beta)}$, when $x \in [-1+\epsilon, 1-\epsilon]$ and $\epsilon > 0$, is given by

$$q_n^{(\alpha,\beta)}(x) = s_n^{\alpha,\beta}(1-x)^{-\alpha/2-1/4}(1+x)^{-\beta/2-1/4}\cos(k\theta+\gamma) + O(n^{-1}), \tag{8}$$

where $x = \cos\theta$, $k = n + \frac{\alpha+\beta+1}{2}$, $\gamma = -(\alpha+1)\frac{\pi}{2}$, and $\lim_{n\to\infty} s_n^{\alpha,\beta} = (\frac{2}{\pi})^{1/2}$.

The Mehler-Heine formula for Jacobi orthonormal polynomials is (see [8, Theorem 8.1.1 and (4.3.4)]

$$\lim_{n\to\infty} (-1)^n n^{-\beta-1/2} p_n^{(\alpha,\beta)}\Big(\cos\Big(\pi - \frac{z}{n}\Big)\Big) = 2^{-\frac{\alpha+\beta}{2}}(z/2)^{-\beta}J_\beta(z), \tag{9}$$

where α, β are real numbers and $J_\beta(z)$ is the Bessel function. This formula holds uniformly for $|z| \leq R$, for R a given positive real number.

From (9)

$$\lim_{n\to\infty} (-1)^n n^{-\beta-1/2} p_n^{(\alpha,\beta)}\Big(\cos\Big(\pi - \frac{z}{n+j}\Big)\Big) = 2^{-\frac{\alpha+\beta}{2}}(z/2)^{-\beta}J_\beta(z) \tag{10}$$

holds uniformly for $|z| \leq R$, R a fixed positive real number, and uniformly on $j \in N \cup \{0\}$.

Lemma 1.1. *Let $\alpha, \beta > -1$ and $M, N \geq 0$. There exists a positive constant c such that*

$$\lim_{n\to\infty} (-1)^n n^{-\beta-1/2} q_n^{(\alpha,\beta)}\Big(\cos\Big(\pi - \frac{z}{n}\Big)\Big) = c(z/2)^{-\beta} J_\beta(z),$$

uniformly for $|z| \leq R$, $R > 0$ fixed.

Proof. Here we will only analyze the case when $M = 0$ and $N > 0$. The proof of the other cases can be done in a similar way. From (3) we have

$$\begin{aligned}
(-1)^n n^{-\beta-1/2} q_n^{(\alpha,\beta)}\Big(\cos\Big(\pi - \frac{z}{n+j}\Big)\Big) &= A_n (-1)^n n^{-\beta-1/2} p_n^{(\alpha,\beta)}\Big(\cos\Big(\pi - \frac{z}{n+j}\Big)\Big) \\
&- B_n\Big(\cos\Big(\pi - \frac{z}{n+j}\Big) - 1\Big)(-1)^{n-1} n^{-\beta-1/2} p_{n-1}^{(\alpha+2,\beta)}\Big(\cos\Big(\pi - \frac{z}{n+j}\Big)\Big) \\
&+ C_n\Big(\cos\Big(\pi - \frac{z}{n+j}\Big) - 1\Big)^2 (-1)^{n-2} n^{-\beta-1/2} p_{n-2}^{(\alpha+4,\beta)}\Big(\cos\Big(\pi - \frac{z}{n+j}\Big)\Big)
\end{aligned}$$

where $j \in N \cup \{0\}$.

Finally, if $n \to \infty$ and using (3) and (10) we get

$$\begin{aligned}
&\lim_{n\to\infty} (-1)^n n^{-\beta-1/2} q_n^{(\alpha,\beta)}\Big(\cos\Big(\pi - \frac{z}{n+j}\Big)\Big) \\
&\qquad = \Big(-\frac{1}{\alpha+2} 2^{-\frac{\alpha+\beta}{2}} + 2\cdot 2^{-\frac{\alpha+\beta+2}{2}} + \frac{1}{\alpha+2} 4\cdot 2^{-\frac{\alpha+\beta+4}{2}}\Big)\Big(\frac{z}{2}\Big)^{-\beta} J_\beta(z) \\
&\qquad = 2^{-\frac{\alpha+\beta}{2}}\Big(\frac{z}{2}\Big)^{-\beta} J_\beta(z).
\end{aligned}$$

□

For every function f such that $\langle f, q_n^{(\alpha,\beta)}\rangle$ exists for $n = 0, 1, \ldots$, the Fourier-Sobolev coefficients of the series (2) can be written as

$$c_n(f) = \langle f, q_n^{(\alpha,\beta)}\rangle = c_n'(f) + M f(1) q_n^{(\alpha,\beta)}(1) + N f'(1)(q_n^{(\alpha,\beta)})'(1), \tag{11}$$

where

$$c_n'(f) = \int_{-1}^{1} f(x) q_n^{(\alpha,\beta)}(x)(1-x)^\alpha (1+x)^\beta\, dx.$$

Next, we will estimate the following integral involving Jacobi-Sobolev type polynomials

$$\int_{-1}^{1} |q_n^{(\alpha,\beta)}(x)|^q (1-x)^\alpha (1+x)^\beta\, dx$$

where $1 \leq q < \infty$. For $M = N = 0$ the calculation of this integral appears in [8, p. 391, Exercise 91] (see also [5, (2.2)]).

First we compute an upper bound for this integral:

Theorem 1.2. *Let $M \geq 0$ and $N \geq 0$. For $\alpha \geq -1/2$*

$$\int_0^1 (1-x)^\alpha |q_n^{(\alpha,\beta)}(x)|^q \, dx = \begin{cases} O(1) & \text{if } 2\alpha > q\alpha - 2 + q/2, \\ O(\log n) & \text{if } 2\alpha = q\alpha - 2 + q/2, \\ O(n^{q\alpha + q/2 - 2\alpha - 2}) & \text{if } 2\alpha < q\alpha - 2 + q/2. \end{cases}$$

For $\beta \geq -1/2$

$$\int_{-1}^0 (1+x)^\beta |q_n^{(\alpha,\beta)}(x)|^q \, dx = \begin{cases} O(1) & \text{if } 2\beta > q\beta - 2 + q/2, \\ O(\log n) & \text{if } 2\beta = q\beta - 2 + q/2, \\ O(n^{q\beta + q/2 - 2\beta - 2}) & \text{if } 2\beta < q\beta - 2 + q/2. \end{cases}$$

Proof. From (7), for $q\alpha + q/2 - 2\alpha - 2 \neq 0$, we have

$$\begin{aligned} \int_0^1 (1-x)^\alpha |q_n^{(\alpha,\beta)}(x)|^q \, dx &= O(1) \int_0^{\pi/2} \theta^{2\alpha+1} |q_n^{(\alpha,\beta)}(\cos\theta)|^q \, d\theta \\ &= O(1) \int_0^{n^{-1}} \theta^{2\alpha+1} n^{q\alpha+q/2} \, d\theta \\ &\quad + O(1) \int_{n^{-1}}^{\pi/2} \theta^{2\alpha+1} \theta^{-q\alpha-q/2} \, d\theta \\ &= O(n^{q\alpha+q/2-2\alpha-2}) + O(1), \end{aligned}$$

and for $q\alpha + q/2 - 2\alpha - 2 = 0$ we have

$$\int_0^1 (1-x)^\alpha |q_n^{(\alpha,\beta)}(x)|^q \, dx = O(\log n).$$

For the proof of the second part we can proceed in a similar way. □

Now, a technique similar to the used in [8, Theorem 7.34] yields:

Theorem 1.3. *Let $M \geq 0$ and $N \geq 0$. For $\beta > -1/2$*

$$\int_{-1}^0 (1+x)^\beta |q_n^{(\alpha,\beta)}(x)|^q \, dx \sim n^{q\beta+q/2-2\beta-2}$$

where $\frac{4(\beta+1)}{2\beta+1} < q < \infty$.

Proof. For the proof of this theorem it is enough to find a lower bound for the integral.

Let $\beta \geq -1/2$, $M \geq 0$ and $N \geq 0$. According to Lemma 1.1, we have

$$\begin{aligned}\int_{\pi/2}^{\pi} (\pi-\theta)^{2\beta+1} |q_n^{(\alpha,\beta)}(\cos\theta)|^q \, d\theta &> \int_{\pi-1/n}^{\pi} (\pi-\theta)^{2\beta+1} |q_n^{(\alpha,\beta)}(\cos\theta)|^q \, d\theta \\ &= \int_0^1 (z/n)^{2\beta+1} |q_n^{(\alpha,\beta)}(\cos(\pi - z/n)|^q \; n^{-1} \, dz \\ &\cong c\int_0^1 (z/n)^{2\beta+1} n^{q\beta+q/2} |(z/2)^{-\beta} J_\beta(z)|^q \; n^{-1} \, dz \\ &\sim n^{q\beta+q/2-2\beta-2}. \qquad \square\end{aligned}$$

2. Divergent Cesàro means of Jacobi-Sobolev expansions

If the expansion (2) is Cesàro summable of order δ on a set, say E, of positive measure in $[-1,1]$, then from [9, Theorem 3.1.22] (see also [6, Lemma 1.1]) we get

$$\left|c_n(f) q_n^{(\alpha,\beta)}(x)\right| = O(n^\delta), \qquad x \in E.$$

From the Egorov's theorem there exists a subset $E_1 \subset E$ of positive measure such that

$$\left|c_n(f) q_n^{(\alpha,\beta)}(x)\right| = O(n^\delta)$$

uniformly for $x \in E_1$. Hence, from (8), we have

$$|n^{-\delta} c_n(f)(\cos(k\theta+\gamma) + O(n^{-1}))| \leq c$$

uniformly for $x = \cos\theta \in E_1$. Using the Cantor-Lebesgue Theorem, (see [6, subsection 1.5] as well as [9, p. 316]), we get

$$\left|\frac{c_n(f)}{n^\delta}\right| \leq c, \qquad \forall n \geq 1. \tag{12}$$

Now we will prove our main result:

Theorem 2.1. *Let α, β, p, and δ be given numbers such that*

$$\begin{aligned} &\beta > -1/2, && -\frac{1}{2} \leq \alpha \leq \beta, \\ &1 \leq p < \frac{4(\beta+1)}{2\beta+3}, && 0 \leq \delta < \frac{2\beta+2}{p} - \frac{2\beta+3}{2}. \end{aligned}$$

There exists $f \in L^p(\mu)$, supported on $[-1,0]$, whose Cesàro means $\sigma_N^\delta f(x)$ are divergent almost everywhere on $[-1,1]$.

Proof. Assume that

$$1 \le p < \frac{4(\beta+1)}{2\beta+3}, \quad \delta < \frac{2\beta+2}{p} - \frac{2\beta+3}{2}.$$

For q conjugate to p, from the last inequalities, we get

$$\frac{4(\beta+1)}{2\beta+1} < q \le \infty, \quad \delta < \beta + \frac{1}{2} - \frac{2\beta}{q} - \frac{2}{q}.$$

For the linear functional $c'_n(f) = \int_{-1}^{1} f(x) q_n^{(\alpha,\beta)}(x)\, d\mu(x)$, from the uniform boundedness principle, (6) and Theorem 1.3, it follows that there is $f \in L^p(\mu)$, supported on $[-1,0]$, such that

$$\frac{c'_n(f)}{n^\delta} \to \infty, \quad \text{when} \quad n \to \infty.$$

Hence, from (4), (5), and (11), we obtain

$$\frac{c_n(f)}{n^\delta} \to \infty, \quad \text{when} \quad n \to \infty.$$

Since this result is contrary with (12) $\sigma_N^\delta f(x)$ is divergent almost everywhere. □

Remark 2.2. Using formulae in [2], which relate the Riesz and Cesàro means of order $\delta \ge 0$, we conclude that Theorem 2.1 also holds for Riesz means.

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