2-Microlocal Besov and Triebel-Lizorkin Spaces of Variable Integrability

Henning KEMPKA

Mathematical Institut
Friedrich Schiller University Jena
D-07737 Jena — Germany
khenning@minet.uni-jena.de

Received: October 14, 2008
Accepted: November 13, 2008

ABSTRACT

We introduce 2-microlocal Besov and Triebel-Lizorkin spaces with variable integrability and give a characterization by local means. These spaces cover spaces of variable exponent, spaces of variable smoothness and weighted spaces that have been studied in recent years.

Key words: Besov spaces, Triebel-Lizorkin spaces, 2-microlocal spaces, variable smoothness, variable integrability, local means.

2000 Mathematics Subject Classification: 42B35,46E30

Introduction

In this paper we combine two concepts generalizing Besov and Triebel-Lizorkin spaces. The first is the concept of 2-microlocal spaces, which initially appeared in the book of Peetre [26]. Furthermore, 2-microlocal spaces have been studied by Bony [4] in connection with pseudodifferential operators and were widely elaborated by Jaffard and Meyer [14]. They are an useful tool to measure local regularity of functions [19]. The approach is Fourier analytical. A distribution $f \in \mathcal{S}'(\mathbb{R}^n)$ belongs to these spaces

This paper has been supported by the DFG grant SCHM 969/7-1.

if for an admissible sequence of weights $\boldsymbol{w} = \{w_j\}_{j=0}^{\infty}$ the related norm is finite

$$\|f \mid B_{pq}^{s,\mathrm{mloc}}(\mathbb{R}^n, \boldsymbol{w})\| = \Big(\sum_{j=0}^{\infty} 2^{jsq} \|(\varphi_j \hat{f})^{\vee} w_j \mid L_p(\mathbb{R}^n)\|^q\Big)^{1/q}$$

$$\|f \mid F_{pq}^{s,\mathrm{mloc}}(\mathbb{R}^n, \boldsymbol{w})\| = \Big\|\Big(\sum_{j=0}^{\infty} 2^{jsq} |(\varphi_j \hat{f})^{\vee}(x) w_j(x)|^q\Big)^{1/q} \Big| L_p(\mathbb{R}^n)\Big\|,$$

where $\{\varphi_j\}$ is a smooth resolution of unity, $s \in \mathbb{R}$, and $0 < p, q \leq \infty$ ($p \neq \infty$ in the F-case). If the weights satisfy $w_j(x) = 1$, then we obtain the usual Besov and Triebel-Lizorkin spaces, studied in detail by Triebel in [32–34].

At the beginning, the weight functions have been the 2-microlocal weights $w_j(x) = (1 + 2^j|x - x_0|)^{s'}$ with $x_0 \in \mathbb{R}^n$ and $s' \in \mathbb{R}$. Besov spaces with these weights were examined by Jaffard [13], Meyer [21], and Xu [36]. Using more general weights Moritoh and Yamada [22] studied wavelet characterizations of 2-microlocal Besov spaces. The above definition with weights satisfying Definition 1.1 was given in [15] by Kempka and characterizations by local means, atoms, and wavelets have been established [15,17]. Leaving the world of tempered distributions and admitting ultra distributions Besov studied $B_{pq}^{s,\mathrm{mloc}}(\mathbb{R}^n, \boldsymbol{w})$ and $F_{pq}^{s,\mathrm{mloc}}(\mathbb{R}^n, \boldsymbol{w})$ with an even more general definition of the weights and proved a characterization by differences in [3].

The spaces $F_{pq}^{s,\mathrm{mloc}}(\mathbb{R}^n, \boldsymbol{w})$ with the 2-microlocal weights were treated by Andersson in [2].

The second concept on which we rely are spaces with variable integrability. It can be traced back to Orlicz [25] and an overview is given by Kováčik and Rákosník in [18].

Let $\Omega \subset \mathbb{R}^n$ be an open set and $p\colon \Omega \to (0, \infty]$ a measurable function bounded away from zero. Then we denote $\Omega_\infty = \{\, x \in \Omega : p(x) = \infty \,\}$ and $\Omega_0 = \Omega \setminus \Omega_\infty$. For measurable $f\colon \Omega \to \mathbb{C}$ we define the modular $\varrho_{L_{p(\cdot)}}$ by

$$\varrho_{L_{p(\cdot)}}(f) = \int_{\Omega_0} |f(x)|^{p(x)}\,dx + \operatorname*{ess\text{-}sup}_{x \in \Omega_\infty} |f(x)|.$$

Then $L_{p(\cdot)}(\Omega)$ is the collection of all f such that $\varrho_{L_{p(\cdot)}}(f/\lambda) < \infty$ for some $\lambda > 0$. The spaces $L_{p(\cdot)}(\Omega)$ are (quasi)-Banach spaces if they are equipped with the Luxemburg norm

$$\|f \mid L_{p(\cdot)}(\Omega)\| = \inf\{\, \lambda > 0 : \varrho_{L_{p(\cdot)}}(f/\lambda) < 1 \,\}.$$

Variable exponent Lebesgue spaces share many properties with the usual $L_p(\Omega)$ spaces with fixed exponent (see [18]). One crucial property which is missing is the translation invariance. The variable exponent spaces have interesting applications in fluid dynamics, PDE's, and image processing. In that connection, Sobolev spaces with variable exponent have been introduced and studied in detail in [1,12].

From the point of view of Harmonic Analysis, the breakthrough for variable exponent spaces was achieved by Diening, when he showed in [7] that the Hardy-Littlewood maximal operator is bounded on $L_{p(\cdot)}(\Omega)$ for p satisfying some regularity condition inside a large ball B_R and being constant outside. Inspired by this work, Cruz-Uribe, Fiorenza, and Neugebauer in [6] and Nekvinda in [24] changed the constant condition for p with less restrictive conditions.

Our goal is to combine both approaches to define 2-microlocal Besov and Triebel-Lizorkin spaces with variable integrability and to give a characterization by local means for them. Diening, Hästö, and Roudenko recently presented in [9] Triebel-Lizorkin spaces with variable $s(\cdot)$, $p(\cdot)$, and $q(\cdot)$. These spaces are contained in our approach.

The paper is organized as follows. In section 1 we introduce the basic notation in the theory of 2-microlocal spaces and spaces of variable integrability. In section 2 we define the spaces $B^{\boldsymbol{w}}_{p(\cdot),q}(\mathbb{R}^n)$ and $F^{\boldsymbol{w}}_{p(\cdot),q}(\mathbb{R}^n)$, which combine both approaches. Further, we present the local means characterization of these spaces in the abstract form of Rychkov [28] and in the sense of Triebel [33, 2.4.6, 2.5.3]. The proofs are carried out in section 3, based on the techniques in [15, 28, 37]. The last section is devoted to the study of the spaces $F^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$. They are a generalization of $F^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$, studied in [9]. We modify the proofs in the previous section and obtain the local means characterization for $F^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$.

1. Preliminaries and definitions

As usual $\mathbb{R}^n$ denotes the $n-$dimensional Euclidean space, $\mathbb{N}$ is the collection of all natural numbers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. $\mathbb{Z}$ and $\mathbb{C}$ stand for the sets of integers and complex numbers, respectively.

The Schwartz space $\mathcal{S}(\mathbb{R}^n)$ is the space of all complex valued rapidly decreasing infinitely differentiable functions on $\mathbb{R}^n$ and by $\mathcal{S}'(\mathbb{R}^n)$ we denote its dual space. We denote by $\mathcal{F}$ and $\mathcal{F}^{-1}$ the Fourier transform and its inverse on $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}'(\mathbb{R}^n)$ respectively and we use the symbols $\hat{f}$ and $\check{f}$ for $\mathcal{F}f$ and $\mathcal{F}^{-1}f$.

The constant c stands for unimportant positive constants. So the value of the constant c may change from one occurrence to another. By $a_k \sim b_k$ we mean that there are two constants $c_1, c_2 > 0$ such that $c_1 a_k \le b_k \le c_2 a_k$ for all admissible k.

1.1. 2-microlocal spaces

As in [15] we introduce a sequence of admissible weight functions $\boldsymbol{w} = \{w_j\}$.

Definition 1.1 (Admissible weight sequence)**.** Let $\alpha \ge 0$ and let $\alpha_1, \alpha_2 \in \mathbb{R}$, $\alpha_1 \le \alpha_2$. A sequence of non-negative measurable functions $\boldsymbol{w} = \{w_j\}_{j=0}^{\infty}$ belongs to the class $\mathcal{W}^{\alpha}_{\alpha_1,\alpha_2}$ if, and only if,

(i) There exists a constant $\mathrm{C} > 0$ such that

$$0 < w_j(x) \le \mathrm{C} w_j(y)(1 + 2^j|x - y|)^{\alpha} \quad \text{for all } j \in \mathbb{N}_0 \text{ and all } x, y \in \mathbb{R}^n.$$

(ii) For all $j \in \mathbb{N}_0$ and all $x \in \mathbb{R}^n$ we have

$$2^{\alpha_1} w_j(x) \le w_{j+1}(x) \le 2^{\alpha_2} w_j(x).$$

Such a system $\{w_j\}_{j=0}^{\infty} \in \mathcal{W}^{\alpha}_{\alpha_1,\alpha_2}$ is called *admissible weight sequence.*

This definition coincides with Definition 3.1 in [3] for the case of the usual distributions ($w(x) = \log(1 + |x|)^d$ in the notation of Besov).

Remark 1.2. If we use $\boldsymbol{w} \in \mathcal{W}^{\alpha}_{\alpha_1,\alpha_2}$ without any restrictions, then $\alpha \ge 0$ and $\alpha_1, \alpha_2 \in \mathbb{R}$ are arbitrary but fixed numbers.

Examples of admissible weight sequences are the 2-microlocal weights $w_j(x) = (1+2^j \operatorname{dist}(x,U))^{s'}$, where $U \subset \mathbb{R}^n$ and $s' \in \mathbb{R}$ or $w_j(x) = (1+2^j \log(1+\operatorname{dist}(x,U)))^{s'}$. Further examples are given in [15].

Furthermore, we need a smooth resolution of unity.

Definition 1.3 (Resolution of unity)**.** A system $\boldsymbol{\varphi} = \{\varphi_j\}_{j=0}^{\infty} \subset \mathcal{S}(\mathbb{R}^n)$ belongs to the class $\Phi(\mathbb{R}^n)$ if, and only if,

(i) $\operatorname{supp} \varphi_0 \subseteq \{\, x \in \mathbb{R}^n : |x| \le 2 \,\}$ and $\operatorname{supp} \varphi_j \subseteq \{\, x \in \mathbb{R}^n : 2^{j-1} \le |x| \le 2^{j+1} \,\}$;

(ii) for each $\beta \in \mathbb{N}_0^n$ there exist constants $c_\beta > 0$ such that

$$2^{j|\beta|} \sup_{x \in \mathbb{R}^n} |D^\beta \varphi_j(x)| \le c_\beta \quad \text{holds for all } j \in \mathbb{N}_0;$$

(iii) for all $x \in \mathbb{R}^n$ we have

$$\sum_{j=0}^{\infty} \varphi_j(x) = 1.$$

Remark 1.4. Such a resolution of unity can easily be constructed. Consider the following example. Let $\varphi_0 \in \mathcal{S}(\mathbb{R}^n)$ with $\varphi_0(x) = 1$ for $|x| \le 1$ and $\operatorname{supp} \varphi_0 \subseteq \{\, x \in \mathbb{R}^n : |x| \le 2 \,\}$. For $j \ge 1$ we define

$$\varphi_j(x) = \varphi_0(2^{-j}x) - \varphi_0(2^{-j+1}x).$$

It is obvious that $\boldsymbol{\varphi} = \{\varphi_j\}_{j \in \mathbb{N}_0} \in \Phi(\mathbb{R}^n)$.

Now, we are able to give the definitions of the 2-microlocal Besov and Triebel-Lizorkin spaces.

Definition 1.5. Let $\boldsymbol{w} = \{w_j\}_{j \in \mathbb{N}_0} \in \mathcal{W}^{\alpha}_{\alpha_1,\alpha_2}$ and let $\{\varphi_j\}_{j \in \mathbb{N}_0} \in \Phi(\mathbb{R}^n)$ be a resolution of unity. Further, let $0 < q \le \infty$ and $s \in \mathbb{R}$.

(i) For $0 < p \leq \infty$ we define

$$B^{s,\mathrm{mloc}}_{pq}(\mathbb{R}^n, \boldsymbol{w}) = \left\{ f \in \mathcal{S}'(\mathbb{R}^n) : \|f \mid B^{s,\mathrm{mloc}}_{pq}(\mathbb{R}^n, \boldsymbol{w})\|_\varphi < \infty \right\}$$

where

$$\|f \mid B^{s,\mathrm{mloc}}_{pq}(\mathbb{R}^n, \boldsymbol{w})\|_\varphi = \left(\sum_{j=0}^{\infty} 2^{jsq} \|w_j (\varphi_j \hat{f})\check{} \mid L_p(\mathbb{R}^n)\|^q \right)^{1/q}.$$

(ii) For $0 < p < \infty$ we define

$$F^{s,\mathrm{mloc}}_{pq}(\mathbb{R}^n, \boldsymbol{w}) = \left\{ f \in \mathcal{S}'(\mathbb{R}^n) : \|f \mid F^{s,\mathrm{mloc}}_{pq}(\mathbb{R}^n, \boldsymbol{w})\|_\varphi < \infty \right\}$$

where

$$\|f \mid F^{s,\mathrm{mloc}}_{pq}(\mathbb{R}^n, \boldsymbol{w})\|_\varphi = \left\| \left(\sum_{j=0}^{\infty} 2^{jsq} |(\varphi_j \hat{f})\check{}(x) w_j(x)|^q \right)^{1/q} \middle| L_p(\mathbb{R}^n) \right\|.$$

We use the usual modifications if p or q are infinity.

If $w_j(x) = 1$, then we obtain the usual Besov an Triebel-Lizorkin spaces.

By a Fourier multiplier theorem for weighted Lebesgue spaces of entire analytic functions [29, Theorem 1.7.5] it is easy to show that the definition of the spaces $B^{s,\mathrm{mloc}}_{pq}(\mathbb{R}^n, \boldsymbol{w})$ is independent of the chosen resolution of unity [15, Theorem 2.12]. The same holds for $F^{s,\mathrm{mloc}}_{pq}(\mathbb{R}^n, \boldsymbol{w})$ if we generalize Theorem 1.9.1 in [29] and replace the weight function ϱ there by an admissible weight sequence $\{\varrho_j\}$ satisfying Definition 1.1. We will not go into detail, because our results prove that the spaces $F^{s,\mathrm{mloc}}_{pq}(\mathbb{R}^n, \boldsymbol{w})$ are independent of the resolution of unity if the resolution is constructed as in Remark 1.4. Therefore, we can suppress φ in the notation of the norm.

1.2. Variable exponent spaces

We take over the notation of spaces of variable integrability from [9]. As usual we denote by Ω an open set from $\mathbb{R}^n$. A measurable function $p\colon \Omega \to (0, \infty]$ is called a variable exponent function if it is bounded away from zero. For a set $A \subset \Omega$ we denote $p^+_A = \text{ess-sup}_{x \in A}\, p(x)$ and $p^-_A = \text{ess-inf}_{x \in A}\, p(x)$; we use the abbreviations $p^+ = p^+_\Omega$ and $p^- = p^-_\Omega$. The variable exponent Lebesgue space $L_{p(\cdot)}(\Omega)$ consists of all measurable functions f such that for some $\lambda > 0$ the modular $\varrho_{L_{p(\cdot)}}(f/\lambda)$ is finite, where

$$\varrho_{L_{p(\cdot)}}(f) = \int_{\Omega_0} |f(x)|^{p(x)}\, dx + \underset{x \in \Omega_\infty}{\text{ess-sup}} |f(x)|.$$

The Luxemburg norm of a function $f \in L_{p(\cdot)}(\Omega)$ is given by

$$\|f \mid L_{p(\cdot)}(\Omega)\| = \inf\{\, \lambda > 0 : \varrho_{L_{p(\cdot)}}(f/\lambda) \leq 1 \,\}.$$

If w is a positive measurable weight function, we define the weighted variable exponent Lebesgue space by $\|f \mid L_{p(\cdot)}(\Omega, w)\| = \|wf \mid L_{p(\cdot)}(\Omega)\|$. We denote the class of all measurable $p\colon \mathbb{R}^n \to (0,\infty]$ such that $p^- > 0$ by $\mathcal{P}(\mathbb{R}^n)$.

The Hardy-Littlewood maximal operator $\mathcal{M}$ for $f \in L_1^{\mathrm{loc}}(\mathbb{R}^n)$ is defined as

$$(\mathcal{M}f)(x) = \sup_B \frac{1}{|B|} \int_B |f(y)|\, dy,$$

where the supremum is taken over all balls B with $x \in B$ and $|B|$ is the Lebesgue measure of B. By $\mathcal{B}(\mathbb{R}^n)$ we denote the class of all $p \in \mathcal{P}(\mathbb{R}^n)$ such that the Hardy-Littlewood maximal operator is bounded on $L_{p(\cdot)}(\mathbb{R}^n)$. In order to cover concrete classes of $\mathcal{B}(\mathbb{R}^n)$ we introduce the following notation.

Definition 1.6. Let $g \in C(\mathbb{R}^n)$. We say that g is *locally* log-*Hölder continuous*, abbreviated $g \in C^{\log}_{\mathrm{loc}}(\mathbb{R}^n)$, if there exists $c_{\log} > 0$ such that

$$|g(x) - g(y)| \le \frac{c_{\log}}{\log(e + 1/|x-y|)}$$

holds for all $x, y \in \mathbb{R}^n$.

We say that g is *globally* log-*Hölder continuous*, abbreviated $g \in C^{\log}(\mathbb{R}^n)$, if g is locally log-Hölder continuous and there exists $g_\infty \in \mathbb{R}$ such that

$$|g(x) - g_\infty| \le \frac{c_{\log}}{\log(e + |x|)}$$

holds for all $x \in \mathbb{R}^n$.

From [8, Theorem 3.6], it is known that $\mathcal{M}$ is bounded on $L_{p(\cdot)}(\mathbb{R}^n)$ if $p \in \mathcal{P}(\mathbb{R}^n)$ belongs to the class $C^{\log}(\mathbb{R}^n)$ and $1 < p^- \le p^+ \le \infty$ (see also [9, 24]).

2. The spaces $B^{\boldsymbol{w}}_{p(\cdot),q}(\mathbb{R}^n)$ and $F^{\boldsymbol{w}}_{p(\cdot),q}(\mathbb{R}^n)$

2.1. Definition of the spaces

Now, we combine both concepts to receive the definition of the 2-microlocal Besov and Triebel-Lizorkin spaces with variable integrability. We use all notation introduced before. We put the factor 2^{js} inside of the weight sequence. This changes α_1 and α_2 but the notation becomes clearer.

Definition 2.1. Let $\boldsymbol{w} \in \mathcal{W}^{\alpha}_{\alpha_1,\alpha_2}$, $\{\varphi_j\}_{j\in\mathbb{N}_0}$ a resolution of unity from Remark 1.4, $p \in \mathcal{P}(\mathbb{R}^n)$ with $0 < p^- \le p^+ \le \infty$, and $0 < q \le \infty$.

(i) The space $B^{\boldsymbol{w}}_{p(\cdot),q}(\mathbb{R}^n)$ is defined as

$$B^{\boldsymbol{w}}_{p(\cdot),q}(\mathbb{R}^n) = \left\{ f \in \mathcal{S}'(\mathbb{R}^n) : \|f \mid B^{\boldsymbol{w}}_{p(\cdot),q}(\mathbb{R}^n)\|_\varphi < \infty \right\}$$

where

$$\|f \mid B^{\boldsymbol{w}}_{p(\cdot),q}(\mathbb{R}^n)\|_\varphi = \Big(\sum_{j=0}^{\infty}\|w_j(\varphi_j\hat{f})^{\check{}} \mid L_{p(\cdot)}(\mathbb{R}^n)\|^q\Big)^{1/q}.$$

(ii) If $p^+ < \infty$ then $F^{\boldsymbol{w}}_{p(\cdot),q}(\mathbb{R}^n)$ is defined by

$$F^{\boldsymbol{w}}_{p(\cdot),q}(\mathbb{R}^n) = \big\{f \in \mathcal{S}'(\mathbb{R}^n) : \|f \mid F^{\boldsymbol{w}}_{p(\cdot),q}(\mathbb{R}^n)\|_\varphi < \infty\big\}$$

where

$$\|f \mid F^{\boldsymbol{w}}_{p(\cdot),q}(\mathbb{R}^n)\|_\varphi = \left\|\Big(\sum_{j=0}^{\infty}\|(\varphi_j\hat{f})^{\check{}}(x)w_j(x)\|^q\Big)^{1/q} \,\middle|\, L_{p(\cdot)}(\mathbb{R}^n)\right\|.$$

This definition clearly generalizes both concepts. For p constant and $w_j(x) = 2^{js}(1+2^j|x-x_0|)^{s'}$ we obtain the 2-microlocal spaces, considered at the beginning. Moreover, the usual Besov and Triebel-Lizorkin spaces are contained for p constant and $w_j(x) = 2^{js}$. If we fix the weight sequence by $w_j(x) = 2^{js}$, then we derive the spaces of variable integrability considered by Xu [37]. If we set the weight sequence to $w_j(x) = 2^{js(x)}$ then we obtain the Triebel-Lizorkin spaces in [9] for constant q.

2.2. Local means characterization

Our main result is the local means characterization of these spaces. To that end, we define the Peetre maximal operator. It was introduced by Jaak Peetre in [27]. The operator assigns to each system $\{\psi_j\}_{j\in\mathbb{N}_0} \subset \mathcal{S}(\mathbb{R}^n)$, to each distribution $f \in \mathcal{S}'(\mathbb{R}^n)$, and to each number $a > 0$ the following quantities:

$$\sup_{y\in\mathbb{R}^n}\frac{|(\psi_k\hat{f})^{\check{}}(y)|}{1+|2^k(y-x)|^a}, \qquad x \in \mathbb{R}^n, \quad k \in \mathbb{N}_0. \tag{1}$$

Since $\psi_k \in \mathcal{S}(\mathbb{R}^n)$ for all $k \in \mathbb{N}_0$ the operator is well-defined because $(\psi_k\hat{f})^{\check{}} = c(\check{\psi}_k * f)$ is well-defined for every distribution $f \in \mathcal{S}'(\mathbb{R}^n)$.

Given a system $\{\psi_k\}_{k\in\mathbb{N}_0} \subset \mathcal{S}(\mathbb{R}^n)$ we set $\Psi_k = \hat{\psi}_k \in \mathcal{S}(\mathbb{R}^n)$ and reformulate the Peetre maximal operator (1) for every $f \in \mathcal{S}'(\mathbb{R}^n)$ and $a > 0$ as

$$(\Psi_k^* f)_a(x) = \sup_{y\in\mathbb{R}^n}\frac{|(\Psi_k * f)(y)|}{1+|2^k(y-x)|^a}, \quad x \in \mathbb{R}^n \text{ and } k \in \mathbb{N}_0.$$

We start with two given functions $\psi_0, \psi_1 \in \mathcal{S}(\mathbb{R}^n)$. We define

$$\psi_j(x) = \psi_1(2^{-j+1}x), \quad \text{for } x \in \mathbb{R}^n \text{ and } j \in \mathbb{N}. \tag{2}$$

Furthermore, for all $j \in \mathbb{N}_0$ we write $\Psi_j = \hat{\psi}_j$.

Now, we state the main theorem.

Theorem 2.2. *Let $\boldsymbol{w} = \{w_k\}_{k\in\mathbb{N}_0} \in \mathcal{W}^{\alpha}_{\alpha_1,\alpha_2}$, $0 < q \le \infty$, $p \in \mathcal{P}(\mathbb{R}^n)$ and let $a \in \mathbb{R}$, $R \in \mathbb{N}_0$ with $R > \alpha_2$. Further, let ψ_0, ψ_1 belong to $\mathcal{S}(\mathbb{R}^n)$ with*

$$D^{\beta}\psi_1(0) = 0, \quad \text{for } 0 \le |\beta| < R, \tag{3}$$

and

$$|\psi_0(x)| > 0 \quad \text{on} \quad \{\, x \in \mathbb{R}^n : |x| < \varepsilon \,\}, \tag{4}$$

$$|\psi_1(x)| > 0 \quad \text{on} \quad \{\, x \in \mathbb{R}^n : \varepsilon/2 < |x| < 2\varepsilon \,\} \tag{5}$$

for some $\varepsilon > 0$.

(i) *If there exists $0 < p_0 < p^-$ with $p(\cdot)/p_0 \in \mathcal{B}(\mathbb{R}^n)$, then for $a > \frac{n}{p_0} + \alpha$*

$$\|f \mid B^{\boldsymbol{w}}_{p(\cdot),q}(\mathbb{R}^n)\| \sim \|(\Psi_k * f)w_k \mid \ell_q(L_{p(\cdot)})\| \sim \|(\Psi_k^* f)_a w_k \mid \ell_q(L_{p(\cdot)})\|$$

holds for all $f \in \mathcal{S}'(\mathbb{R}^n)$.

(ii) *If $p^+ < \infty$ and if there exists $p_0 < \min(p^-, q)$ with $p(\cdot)/p_0 \in \mathcal{B}(\mathbb{R}^n)$, then for $a > \frac{n}{p_0} + \alpha$*

$$\|f \mid F^{\boldsymbol{w}}_{p(\cdot),q}(\mathbb{R}^n)\| \sim \|(\Psi_k * f)w_k \mid L_{p(\cdot)}(\ell_q)\| \sim \|(\Psi_k^* f)_a w_k \mid L_{p(\cdot)}(\ell_q)\|$$

holds for all $f \in \mathcal{S}'(\mathbb{R}^n)$.

The proof relies on [28] and will be shifted to the next section. Moreover, Theorem 2.2 shows that the definition of the 2-microlocal spaces of variable integrability is independent of the resolution of unity from Remark 1.4.

Remarks 2.3.

(i) If $R = 0$, then there are no moment conditions (3) on ψ_1.

(ii) If $p \in C^{\log}(\mathbb{R}^n)$, then for every $p_0 < p^-$ we have $p(\cdot)/p_0 \in \mathcal{B}(\mathbb{R}^n)$, see [8, Theorem 3.6].

Next we reformulate Theorem 2.2 in the sense of [33, subsections 2.4.6, 2.5.3].

Let $B = \{\, x \in \mathbb{R}^n : |x| < 1 \,\}$ be the unit ball and $k \in \mathcal{S}(\mathbb{R}^n)$ a function with support in B. For a distribution $f \in \mathcal{S}'(\mathbb{R}^n)$ the corresponding local means are defined for $x \in \mathbb{R}^n$ and $t > 0$ by (at least formally)

$$k(t,f)(x) = \int_{\mathbb{R}^n} k(y) f(x + ty)\, dy = t^{-n} \int_{\mathbb{R}^n} k\Big(\frac{y - x}{t}\Big) f(y)\, dy.$$

Let $k_0, k^0 \in \mathcal{S}(\mathbb{R}^n)$ be two functions satisfying

$$\operatorname{supp} k_0 \subseteq B, \qquad \operatorname{supp} k^0 \subseteq B,$$

and

$$\hat{k_0}(0) \neq 0, \qquad \hat{k^0}(0) \neq 0.$$

For $N \in \mathbb{N}_0$ we define the iterated Laplacian

$$k(y) := \Delta^N k^0(y) = \left(\sum_{j=1}^{n} \frac{\partial^2}{\partial y_j^2}\right)^N k^0(y), \qquad y \in \mathbb{R}^n.$$

It follows easily that

$$\check{k}(x) = |x|^{2N} \check{k^0}(x)$$

and that implies

$$D^\beta \check{k}(0) = 0 \qquad \text{for} \qquad 0 \leq |\beta| < 2N. \tag{6}$$

Using this notation we come to the usual local means characterization.

Theorem 2.4. *Let $\boldsymbol{w} = \{w_j\}_{j\in\mathbb{N}_0} \in \mathcal{W}^\alpha_{\alpha_1,\alpha_2}$, $0 < q \leq \infty$, and $p \in \mathcal{P}(\mathbb{R}^n)$. Furthermore, let $N \in \mathbb{N}_0$ with $2N > \alpha_2$ and let $k_0, k^0 \in \mathcal{S}(\mathbb{R}^n)$ and the function k be defined as above.*

(i) *If there exists a $p_0 \leq p^-$ with $p(\cdot)/p_0 \in \mathcal{B}(\mathbb{R}^n)$, then*

$$\|k_0(1,f)w_0 \mid L_{p(\cdot)}(\mathbb{R}^n)\| + \left(\sum_{j=1}^{\infty} \|k(2^{-j},f)w_j \mid L_{p(\cdot)}(\mathbb{R}^n)\|^q\right)^{1/q}$$

is an equivalent norm on $B^{\boldsymbol{w}}_{p(\cdot),q}(\mathbb{R}^n)$ for all $f \in \mathcal{S}'(\mathbb{R}^n)$.

(ii) *If $p^+ < \infty$ and if there exists a $p_0 \leq \min(p^-, q)$ with $p(\cdot)/p_0 \in \mathcal{B}(\mathbb{R}^n)$, then*

$$\|k_0(1,f)w_0 \mid L_{p(\cdot)}(\mathbb{R}^n)\| + \left\|\left(\sum_{j=1}^{\infty} |k(2^{-j},f)(\cdot)w_j(\cdot)|^q\right)^{1/q} \,\middle|\, L_{p(\cdot)}(\mathbb{R}^n)\right\|$$

is an equivalent norm on $F^{\boldsymbol{w}}_{p(\cdot),q}(\mathbb{R}^n)$ for all $f \in \mathcal{S}'(\mathbb{R}^n)$.

Proof. We put

$$\psi_0 = \check{k}_0, \qquad \psi_1 = \check{k}(\cdot/2).$$

Then the Tauberian conditions (4) and (5) are satisfied and due to (6) also the moment conditions (3) are fulfilled. If we define ψ_j for $j \in \mathbb{N}_0$ as in (2), then we get

$$(\psi_j \hat{f})\check{}(x) = c(\check{\psi}_j * f)(x) = c \int_{\mathbb{R}^n} (\mathcal{F}\psi_j)(y) f(x+y)\, dy. \tag{7}$$

For $j = 0$ we get $\mathcal{F}\psi_0 = k_0$ and for $j \geq 1$ we obtain

$$(\mathcal{F}\psi_j)(y) = (\mathcal{F}\psi_1(2^{-j+1}\cdot))(y) = 2^{(j-1)n}(\mathcal{F}\psi_1)(2^{j-1}y) = 2^{jn}k(2^j y).$$

This and the equation (7) lead to

$$(\psi_j \hat{f})\check{}(x) = c2^{jn}\int_{\mathbb{R}^n} k(2^j y)f(x+y)\,dy = ck(2^{-j}, f)(x), \quad j \in \mathbb{N}_0, \quad x \in \mathbb{R}^n.$$

Together with Theorem 2.2 the proof is complete. □

2.3. Connection to known spaces

For p *constant* and $w_j(x) = 2^{js}$ we clearly get back to the usual Besov and Triebel-Lizorkin spaces $B^s_{pq}(\mathbb{R}^n)$ and $F^s_{pq}(\mathbb{R}^n)$. Consequently, we get the entire scale of function spaces which are included in $B^s_{pq}(\mathbb{R}^n)$ and $F^s_{pq}(\mathbb{R}^n)$, as the Lebesgue, Sobolev, Hardy, Hölder-Zygmund, ... spaces (see [33] for details).

If p is constant, then the spaces from Definition 2.1 coincide with the spaces in [3] presented by Besov. On the one hand, our approach is more general because negative smoothness is allowed ($\alpha_1, \alpha_2 \in \mathbb{R}$ in contrast to $0 \leq \alpha_1 \leq \alpha_2$ in [3]); on the other hand, Besov's approach is more general because he used the machinery of ultra-distributions to allow exponential growth for the weights.

Regarding p constant the entire literature on 2-microlocal spaces $B^{s,s'}_{pq}(\mathbb{R}^n, U)$ with

$$w_j(x) = 2^{js}(1 + 2^j \operatorname{dist}(x, U))^{s'}$$

mentioned before ([2, 4, 13–15, 17, 19, 20, 22]) is included in this approach.

Moreover, also the spaces of generalized smoothness are contained in this approach ([11, 23]) by taking

$$w_j(x) = 2^{js}\Psi(2^{-j}), \quad \text{or, more general,} \quad w_j(x) = \sigma_j.$$

Here, $\{\sigma_j\}$ is an admissible sequence that means there exist $d_0, d_1 > 0$ with $d_0\sigma_j \leq \sigma_{j+1} \leq d_1\sigma_j$. One immediately recognizes, that this is the second condition in Definition 1.1 for constant weight functions.

Schneider in [30] has defined Besov spaces $B^{\mathbb{S},s_0}_p(\mathbb{R}^n)$ where $\mathbb{S}(\cdot)$ has to be a lower semi-continuous function which describes the local regularity and $s_0 \leq \mathbb{S}(\cdot)$ is the global minimum smoothness. Schneider gave a wavelet characterization of these spaces. This approach is not contained in our definition of $B^{\boldsymbol{w}}_{p(\cdot),q}(\mathbb{R}^n)$, but it is closely connected for p constant, see [30] and [16].

If $w_j(x) = 2^{js}w_0(x)$ for all $j \in \mathbb{N}$, then we obtain the weighted Besov and Triebel-Lizorkin spaces for p constant, see [10, chapter 4].

For *variable* $p \in \mathcal{P}(\mathbb{R}^n)$ and $w_j(x) = 2^{js}$ the spaces considered by Xu in [37] are contained in the above scale.

Moreover, also the classical spaces of variable integrability are contained in $F^{\boldsymbol{w}}_{p(\cdot),q}(\mathbb{R}^n)$ for $w_j(x) = 2^{js}$. For example we have $F^s_{p(\cdot),2}(\mathbb{R}^n) = \mathcal{L}^{s,p(\cdot)}(\mathbb{R}^n)$, where $\mathcal{L}^{s,p(\cdot)}(\mathbb{R}^n)$ are the Bessel potential spaces (fractional Sobolev spaces) of variable integrability which were introduced in [1] and in [12]. The integrability p has to belong to $C^{\log}(\mathbb{R}^n)$ with $1 < p^- \le p^+ < \infty$ and $s \ge 0$ (Theorem 4.5 in [9]). As a special case we get under this conditions that $F^k_{p(\cdot),2}(\mathbb{R}^n) = W^k_{p(\cdot)}(\mathbb{R}^n)$, where $k \in \mathbb{N}_0$.

If one chooses in [9] the parameter q as a constant function, then the spaces from Definition 2.1 include the Triebel-Lizorkin spaces of [9]. Our conditions on the weight sequence would also allow $w_j(x) = 2^{js(x)}$ where the function s can be negative, whereas in [9] $s(x) \ge 0$. The next lemma shows that the conditions on $s\colon \mathbb{R}^n \to \mathbb{R}$ in [9] define an admissible weight sequence.

Lemma 2.5. *Let $s\colon \mathbb{R}^n \to \mathbb{R}$ be a bounded and measurable function which satisfies the local log-Hölder condition, i.e.,*

$$|s(x) - s(y)| \le \frac{c_s}{\log\bigl(\mathrm{e} + \frac{1}{|x-y|}\bigr)}, \tag{8}$$

for some constant $c_s > 0$ and all $x, y \in \mathbb{R}^n$.

Then $w_j(x) = 2^{js(x)}$ is an admissible weight sequence for sufficiently large $\alpha > 0$ and $\alpha_1, \alpha_2 \in \mathbb{R}$ with respect to Definition 1.1.

Proof. The second condition in Definition 1.1 is easily proved with the help of the boundedness of s and we derive $\alpha_1 = -\|s\|_\infty$ and $\alpha_2 = \|s\|_\infty$. Now, we prove the first condition on the weights. We can define

$$D(x, y) = \frac{w_j(x)}{w_j(y)} = 2^{j(s(x)-s(y))}$$

and the first condition in Definition 1.1 is equivalent to

$$D(x, y) \le \mathrm{C}(1 + 2^j|x - y|)^\alpha,$$

for $\mathrm{C} > 0$ and $\alpha \ge 0$ independently chosen of $j \in \mathbb{N}_0$ and $x, y \in \mathbb{R}^n$. From now on we choose $j \ge 2$, the remaining cases $j \le 1$ can easily be incorporated in the constant C.

- *First case, $|x - y| \ge 1/2$.* We have the following estimate:

$$D(x, y) = 2^{j(s(x)-s(y))} \le \frac{2^{2j\|s\|_\infty+1}}{2} \le |x - y| 2^{j(2\|s\|_\infty+1)} \le (1 + 2^j|x - y|)^\alpha,$$

for $\alpha \ge 2\|s\|_\infty + 1$.

- *Second case, $2^{-k-1} \le |x - y| \le 2^{-k}$ for $k \ge 2$.* We use (8) and obtain

$$D(x, y) \le 2^{j|s(x)-s(y)|} \le 2^{j\frac{c_s}{\log\left(\mathrm{e}+\frac{1}{|x-y|}\right)}}.$$

Furthermore,

$$\log\Big(\mathrm{e}+\frac{1}{|x-y|}\Big) \geq \log(\mathrm{e}+2^k) \geq c \cdot k,$$

which gives

$$D(x,y) \leq 2^{j\frac{\tilde{c}_s}{k}}, \quad \text{where } \tilde{c}_s = c_s \operatorname{ld} \mathrm{e}.$$

For $k \geq j-1$ we get

$$D(x,y) \leq 2^{2\tilde{c}_s} \leq \mathrm{C}(1+2^j|x-y|)^\alpha, \quad \text{for } \alpha \geq 1 \text{ and } \mathrm{C} = 2^{2\tilde{c}_s}.$$

If $k \leq j-2$ we wish to have the following estimates:

$$D(x,y) \leq 2^{j\frac{\tilde{c}_s}{k}} \leq 2^{(j-k-1)\alpha} \leq (1+2^{j-k-1})^\alpha \leq (1+2^j|x-y|)^\alpha. \tag{9}$$

To show the second inequality in (9) we need an $\alpha \geq 1$ with

$$\frac{j\tilde{c}_s}{k} \leq \alpha(j-k-1) \quad \text{that means} \quad \alpha \geq \tilde{c}_s \frac{j}{k(j-1-k)}.$$

But this is satisfied with $\alpha = 4\tilde{c}_s$ which is independent of $2 \leq k \leq j-2$. This finishes the proof, and we have $w_j(x) = 2^{js(x)} \in \mathcal{W}^\alpha_{\alpha_1,\alpha_2}$ if

$$\begin{aligned} -\alpha_1 \geq \|s\|_\infty &\leq \alpha_2, \\ \mathrm{C} &\geq \max(2^{2c_s \operatorname{ld} \mathrm{e}}, 2, 2^{4\|s\|_\infty}), \end{aligned}$$

and

$$\alpha \geq \max(4c_s \operatorname{ld} \mathrm{e}, 2\|s\|_\infty + 1). \qquad \square$$

3. Proof of Theorem 2.2

In the following subsections we prove two theorems, which together give the proof of Theorem 2.2.

3.1. Helpful lemmas

Before proving the local means characterization we recall some technical lemmas without proof, which appeared in the papers of Rychkov [28] and Xu [37]. The first lemma describes the use of the so called moment conditions.

Lemma 3.1 ([28, Lemma 1]). *Let $g, h \in \mathcal{S}(\mathbb{R}^n)$ and let $M \in \mathbb{N}_0$. Suppose that*

$$(D^\beta \hat{g})(0) = 0 \quad \textit{for} \quad 0 \leq |\beta| < M. \tag{10}$$

Then for each $N \in \mathbb{N}_0$ there is a constant C_N such that

$$\sup_{z\in\mathbb{R}^n} |(g_t * h)(z)|(1+|z|^N) \leq C_N t^M, \quad \textit{for} \quad 0 < t < 1,$$

where $g_t(x) = t^{-n} g(x/t)$.

Remark 3.2. If $M = 0$, then condition (10) is empty.

The next lemma is a discrete convolution inequality which we will need later on.

Lemma 3.3 ([37, Lemma 3]). *Let $0 < q \leq \infty$, $\delta > 0$, and $p \in \mathcal{P}(\mathbb{R}^n)$ with $0 < p^- \leq p^+ \leq \infty$. Let $\{g_k\}_{k\in\mathbb{N}_0}$ be a sequence of non-negative measurable functions on $\mathbb{R}^n$ and denote*

$$G_\nu(x) = \sum_{k=0}^{\infty} 2^{-|\nu-k|\delta} g_k(x), \qquad x \in \mathbb{R}^n, \quad \nu \in \mathbb{N}_0.$$

Then there exist constants $C_1, C_2 \geq 0$ such that

$$\|G_k \mid \ell_q(L_{p(\cdot)})\| \leq C_1 \|g_k \mid \ell_q(L_{p(\cdot)})\|$$

and

$$\|G_k \mid L_{p(\cdot)}(\ell_q)\| \leq C_2 \|g_k \mid L_{p(\cdot)}(\ell_q)\|.$$

Lemma 3.4 ([5, Corollary 2.1]). *If $p \in \mathcal{B}(\mathbb{R}^n)$, then, for all $1 < q \leq \infty$,*

$$\|\mathcal{M} f_j \mid L_{p(\cdot)}(\ell_q)\| \leq c \|f_j \mid L_{p(\cdot)}(\ell_q)\|,$$

where $\mathcal{M}$ is the Hardy-Littlewood maximal operator.

Lemma 3.5 ([28, Lemma 3]). *Let $0 < r \leq 1$ and let $\{\gamma_\nu\}_{\nu\in\mathbb{N}_0}$, $\{\beta_\nu\}_{\nu\in\mathbb{N}_0}$ be two sequences taking values in $(0,\infty)$. Assume that for some $N^0 \in \mathbb{N}_0$,*

$$\gamma_\nu = O(2^{\nu N^0}), \quad \text{for } \nu \to \infty. \tag{11}$$

Furthermore, we assume that for any $N \in \mathbb{N}$

$$\gamma_\nu \leq C_N \sum_{k=0}^{\infty} 2^{-kN} \beta_{k+\nu} \gamma_{k+\nu}^{1-r}, \quad \nu \in \mathbb{N}_0 \quad C_N < \infty$$

holds, then for any $N \in \mathbb{N}$

$$\gamma_\nu^r \leq C_N \sum_{k=0}^{\infty} 2^{-kN} \beta_{k+\nu}, \quad \nu \in \mathbb{N}_0$$

holds with the same constants C_N.

3.2. Comparison of different Peetre maximal operators

In this subsection we present an inequality between different Peetre maximal operators.

We start with two given functions $\psi_0, \psi_1 \in \mathcal{S}(\mathbb{R}^n)$. We define

$$\psi_j(x) = \psi_1(2^{-j+1}x), \quad \text{for } x \in \mathbb{R}^n \text{ and } j \in \mathbb{N}.$$

Furthermore, for all $j \in \mathbb{N}_0$ we write $\Psi_j = \hat{\psi}_j$ and in an analogous manner we define Φ_j from two starting functions $\phi_0, \phi_1 \in \mathcal{S}(\mathbb{R}^n)$.

Using this notation we are ready to formulate the theorem.

Theorem 3.6. *Let $\boldsymbol{w} = \{w_j\}_{j\in\mathbb{N}_0} \in \mathcal{W}^{\alpha}_{\alpha_1,\alpha_2}$, $0 < q \le \infty$, $p \in \mathcal{P}(\mathbb{R}^n)$ with $0 < p^- \le p^+ \le \infty$, and $a \in \mathbb{R}$ with $a > 0$. Moreover, let $R \in \mathbb{N}_0$ with $R > \alpha_2$,*

$$D^\beta \psi_1(0) = 0, \quad 0 \le |\beta| < R, \tag{12}$$

and for some $\varepsilon > 0$

$$|\phi_0(x)| > 0 \quad \textit{on} \quad \{\, x \in \mathbb{R}^n : |x| < \varepsilon \,\} \tag{13}$$

$$|\phi_1(x)| > 0 \quad \textit{on} \quad \{\, x \in \mathbb{R}^n : \varepsilon/2 < |x| < 2\varepsilon \,\} \tag{14}$$

then

$$\|(\Psi_k^* f)_a w_k \mid \ell_q(L_{p(\cdot)})\| \le c \|(\Phi_k^* f)_a w_k \mid \ell_q(L_{p(\cdot)})\|$$

and

$$\|(\Psi_k^* f)_a w_k \mid L_{p(\cdot)}(\ell_q)\| \le c \|(\Phi_k^* f)_a w_k \mid L_{p(\cdot)}(\ell_q)\|$$

holds for every $f \in \mathcal{S}'(\mathbb{R}^n)$.

Proof. We have the fixed resolution of unity from Remark 1.4 and define the functions $\{\lambda_j\}_{j\in\mathbb{N}_0}$ by

$$\lambda_j(x) = \frac{\varphi_j\left(\frac{2x}{\varepsilon}\right)}{\phi_j(x)}.$$

It follows from the *Tauberian conditions* (13) and (14) that they satisfy

$$\sum_{j=0}^{\infty} \lambda_j(x)\phi_j(x) = 1 \qquad\qquad x \in \mathbb{R}^n \tag{15}$$

$$\lambda_j(x) = \lambda_1(2^{-j+1}x), \quad x \in \mathbb{R}^n, \quad j \in \mathbb{N} \tag{16}$$

$$\operatorname{supp} \lambda_0 \subset \{\, x \in \mathbb{R}^n : |x| \le \varepsilon \,\} \quad \text{and} \quad \operatorname{supp} \lambda_1 \subset \{\, x \in \mathbb{R}^n : \varepsilon/2 \le |x| \le 2\varepsilon \,\}. \tag{17}$$

Furthermore, we denote $\Lambda_k = \hat{\lambda}_k$ for $k \in \mathbb{N}_0$ and obtain together with (15) the following identities (convergence in $\mathcal{S}'(\mathbb{R}^n)$):

$$f = \sum_{k=0}^{\infty} \Lambda_k * \Phi_k * f, \qquad \Psi_\nu * f = \sum_{k=0}^{\infty} \Psi_\nu * \Lambda_k * \Phi_k * f. \tag{18}$$

We have

$$
\begin{aligned}
|(\Psi_\nu * \Lambda_k * \Phi_k * f)(y)| &\le \int_{\mathbb{R}^n} |(\Psi_\nu * \Lambda_k)(z)||(\Phi_k * f)(y-z)|\,dz \\
&\le (\Phi_k^* f)_a(y) \int_{\mathbb{R}^n} |(\Psi_\nu * \Lambda_k)(z)|(1+|2^k z|^a)\,dz \qquad (19)\\
&=: (\Phi_k^* f)_a(y) I_{\nu,k},
\end{aligned}
$$

where

$$
I_{\nu,k} := \int_{\mathbb{R}^n} |(\Psi_\nu * \Lambda_k)(z)|(1+|2^k z|^a)\,dz.
$$

According to Lemma 3.1 we get

$$
I_{\nu,k} \le c \begin{cases} 2^{(k-\nu)R}, & k \le \nu, \\ 2^{(\nu-k)(a+1+|\alpha_1|)}, & \nu \le k. \end{cases} \tag{20}
$$

Namely, we have for $1 \le k < \nu$ with the change of variables $2^k z \mapsto z$

$$
\begin{aligned}
I_{\nu,k} &= 2^{-n} \int_{\mathbb{R}^n} |(\Psi_{\nu-k} * \Lambda_1(\cdot/2))(z)|(1+|z|^a)\,dz \\
&\le c \sup_{z\in\mathbb{R}^n} |(\Psi_{\nu-k} * \Lambda_1(\cdot/2))(z)|(1+|z|)^{a+n+1} \le c\, 2^{(k-\nu)R}.
\end{aligned}
$$

Similarly, we get for $1 \le \nu < k$ with the substitution $2^\nu z \mapsto z$

$$
\begin{aligned}
I_{\nu,k} &= 2^{-n} \int_{\mathbb{R}^n} |(\Psi_1(\cdot/2) * \Lambda_{k-\nu})(z)|(1+|2^{k-\nu} z|^a)\,dz \\
&\le c\, 2^{(\nu-k)(M-a)}.
\end{aligned}
$$

M can be taken arbitrarily large because Λ_1 has infinite vanishing moments. Taking $M = 2a + |\alpha_1| + 1$ we derive (20) for the cases $k, \nu \ge 1$ with $k \ne \nu$. The missing cases can be treated separately in an analogous manner. The needed moment conditions are always satisfied by (12) and (17). The case $k = \nu = 0$ is covered by the constant c in (20).

Furthermore, we have

$$
\begin{aligned}
(\Phi_k^* f)_a(y) &\le (\Phi_k^* f)_a(x)(1+|2^k(x-y)|^a) \\
&\le (\Phi_k^* f)_a(x)(1+|2^\nu(x-y)|^a) \max(1, 2^{(k-\nu)a}).
\end{aligned}
$$

We put this into (19) and get

$$
\sup_{y\in\mathbb{R}^n} \frac{|(\Psi_\nu * \Lambda_k * \Phi_k * f)(y)|}{1+|2^\nu(x-y)|^a} \le c(\Phi_k^* f)_a(x) \begin{cases} 2^{(k-\nu)R}, & k \le \nu \\ 2^{(\nu-k)(1+|\alpha_1|)}, & k \ge \nu. \end{cases}
$$

Multiplying both sides with $w_\nu(x)$ and using

$$w_\nu(x) \leq w_k(x) \begin{cases} 2^{(k-\nu)(-\alpha_2)}, & k \leq \nu, \\ 2^{(\nu-k)\alpha_1}, & k \geq \nu, \end{cases}$$

leads us to

$$\sup_{y\in\mathbb{R}^n} \frac{|(\Psi_\nu * \Lambda_k * \Phi_k * f)(y)|}{1+|2^\nu(x-y)|^a} w_\nu(x) \leq c(\Phi_k^* f)_a(x) w_k(x) \begin{cases} 2^{(k-\nu)(R-\alpha_2)}, & k \leq \nu, \\ 2^{(\nu-k)}, & k \geq \nu. \end{cases}$$

This inequality together with (18) gives for $\delta := \min(1, R-\alpha_2) > 0$

$$(\Psi_\nu^* f)_a(x) w_\nu(x) \leq c \sum_{k=0}^{\infty} 2^{-|k-\nu|\delta} (\Phi_k^* f)_a(x) w_k(x), \quad x \in \mathbb{R}^n. \tag{21}$$

Then, Lemma 3.3 yields immediately the desired result. □

Remark 3.7. The conditions (12) are usually called *moment conditions* while (13) and (14) are the so called *Tauberian conditions*.

If $R = 0$ in Theorem 3.6, then there are no moment conditions on ψ_1.

3.3. Boundedness of the Peetre maximal operator

We will present a theorem which describes the boundedness of the Peetre maximal operator. We use the same notation introduced at the beginning of the last subsection. Especially, we have the functions $\psi_k \in \mathcal{S}(\mathbb{R}^n)$ and $\Psi_k = \hat{\psi}_k \in \mathcal{S}(\mathbb{R}^n)$ for all $k \in \mathbb{N}_0$.

Theorem 3.8. *Let* $\{w_k\}_{k\in\mathbb{N}_0} \in \mathcal{W}^\alpha_{\alpha_1,\alpha_2}$*,* $a \in \mathbb{R}$*, and* $0 < q \leq \infty$*,* $p \in \mathcal{P}(\mathbb{R}^n)$*. For some* $\varepsilon > 0$ *we assume* $\psi_0, \psi_1 \in \mathcal{S}(\mathbb{R}^n)$ *with*

$$|\psi_0| > 0 \quad \text{on} \quad \{x \in \mathbb{R}^n : |x| < \varepsilon\},$$
$$|\psi_1| > 0 \quad \text{on} \quad \{x \in \mathbb{R}^n : \varepsilon/2 < |x| < 2\varepsilon\}.$$

(i) *If there exists* $0 < p_0 < p^-$ *with* $p(\cdot)/p_0 \in \mathcal{B}(\mathbb{R}^n)$*, then for* $a > \frac{n}{p_0} + \alpha$

$$\|(\Psi_k^* f)_a w_k \mid \ell_q(L_{p(\cdot)})\| \leq c\|(\Psi_k * f) w_k \mid \ell_q(L_{p(\cdot)})\| \tag{22}$$

holds for all $f \in \mathcal{S}'(\mathbb{R}^n)$*.*

(ii) *If* $p^+ < \infty$ *and if there exists* $0 < p_0 < \min(p^-, q)$ *with* $p(\cdot)/p_0 \in \mathcal{B}(\mathbb{R}^n)$*, then for* $a > \frac{n}{p_0} + \alpha$

$$\|(\Psi_k^* f)_a w_k \mid L_{p(\cdot)}(\ell_q)\| \leq c\|(\Psi_k * f) w_k \mid L_{p(\cdot)}(\ell_q)\| \tag{23}$$

holds for all $f \in \mathcal{S}'(\mathbb{R}^n)$*.*

Proof. As in the last proof we find the functions $\{\lambda_j\}_{j\in\mathbb{N}_0}$ with the properties (16), (17), and

$$\sum_{k=0}^{\infty} \lambda_k(2^{-\nu}x)\psi_k(2^{-\nu}x) = 1 \quad \text{for all } \nu \in \mathbb{N}_0.$$

Instead of (18) we get the identity

$$\Psi_\nu * f = \sum_{k=0}^{\infty} \Lambda_{k,\nu} * \Psi_{k,\nu} * \Psi_\nu * f, \tag{24}$$

where

$$\Lambda_{k,\nu}(\xi) = [\lambda_k(2^{-\nu}\cdot)]\check{}\,(\xi) = 2^{\nu n}\Lambda_k(2^\nu\xi) \qquad \text{for all } \nu, k \in \mathbb{N}_0.$$

The $\Psi_{k,\nu}$ are defined similarly. For $k \geq 1$ and $\nu \in \mathbb{N}_0$ we have $\Psi_{k,\nu} = \Psi_{k+\nu}$ and with the notation

$$\sigma_{k,\nu}(x) = \begin{cases} \psi_0(2^{-\nu}x), & \text{if } k = 0, \\ \psi_\nu(x), & \text{otherwise} \end{cases}$$

we get $\psi_k(2^{-\nu}x)\psi_\nu(x) = \sigma_{k,\nu}(x)\psi_{k+\nu}(x)$. Hence, we can rewrite (24) as

$$\Psi_\nu * f = \sum_{k=0}^{\infty} \Lambda_{k,\nu} * \hat{\sigma}_{k,\nu} * \Psi_{k+\nu} * f. \tag{25}$$

For $k \geq 1$ we get from Lemma 3.1

$$|(\Lambda_{k,\nu} * \hat{\sigma}_{k,\nu})(z)| = 2^{\nu n}|(\Lambda_k * \Psi)(2^\nu z)| \leq C_M 2^{\nu n} \frac{2^{-kM}}{(1 + |2^\nu z|^a)} \tag{26}$$

for all $k, \nu \in \mathbb{N}_0$ and arbitrary large $M \in \mathbb{N}$. For $k = 0$ we get the estimate (26) by using Lemma 3.1 with $M = 0$. This together with (25) gives us

$$|(\Psi_\nu * f)(y)| \leq C_M 2^{\nu n} \sum_{k=0}^{\infty} \int_{\mathbb{R}^n} \frac{2^{-kM}}{(1 + |2^\nu(y-z)|^a)} |(\Psi_{k+\nu} * f)(z)|\, dz. \tag{27}$$

For fixed $r \in (0, 1]$ we divide both sides of (27) by $(1 + |2^\nu(x-y)|^a)$ and we take the supremum with respect to $y \in \mathbb{R}^n$. Using the inequalities

$$(1 + |2^\nu(y-z)|^a)(1 + |2^\nu(x-y)|^a) \geq c(1 + |2^\nu(x-z)|^a),$$
$$|(\Psi_{k+\nu} * f)(z)| \leq |(\Psi_{k+\nu} * f)(z)|^r (\Psi^*_{k+\nu} f)_a(x)^{1-r}(1 + |2^{k+\nu}(x-z)|^a)^{1-r},$$

and

$$\frac{(1 + |2^{k+\nu}(x-z)|^a)^{1-r}}{(1 + |2^\nu(x-z)|^a)} \leq \frac{2^{ka}}{(1 + |2^{k+\nu}(x-z)|^a)^r},$$

we get

$$(\Psi_\nu^* f)_a(x) \\ \leq C_M \sum_{k=0}^{\infty} 2^{-k(M+n-a)} (\Psi_{k+\nu}^* f)_a(x)^{1-r} \int_{\mathbb{R}^n} \frac{2^{(k+\nu)n} |(\Psi_{k+\nu} * f)(z)|^r}{(1+|2^{k+\nu}(x-z)|^a)^r}\, dz.$$

Now, we apply Lemma 3.5 with

$$\gamma_\nu = (\Psi_\nu^* f)_a(x), \quad \beta_\nu = \int_{\mathbb{R}^n} \frac{2^{\nu n}|(\Psi_\nu * f)(z)|^r}{(1+|2^\nu(x-z)|^a)^r}\, dz, \qquad \nu \in \mathbb{N}_0,$$

$N = M+n-a$, $C_N = C_M+n-a$, and N^0 in (11) equals the order of the distribution $f \in \mathcal{S}'(\mathbb{R}^n)$.

By Lemma 3.5 we obtain for every $N \in \mathbb{N}$, $x \in \mathbb{R}^n$,and $\nu \in \mathbb{N}_0$

$$(\Psi_\nu^* f)_a(x)^r \leq C_N \sum_{k=0}^{\infty} 2^{-kNr} \int_{\mathbb{R}^n} \frac{2^{(k+\nu)n} |(\Psi_{k+\nu} * f)(z)|^r}{(1+|2^{k+\nu}(x-z)|^a)^r}\, dz, \tag{28}$$

provided that $(\Psi_\nu^* f)_a(x) < \infty$. Since $f \in \mathcal{S}'(\mathbb{R}^n)$, we see that $(\Psi_\nu^* f)_a(x) < \infty$ for all $x \in \mathbb{R}^n$ and all $\nu \in \mathbb{N}_0$ at least if $a > N^0$, where N^0 is the order of the distribution. Thus we have (28) with C_N independent of $f \in \mathcal{S}'(\mathbb{R}^n)$ for $a \geq N^0$ and therefore with $C_N = C_{N,f}$ for all $a > 0$. (The right side of (28) decreases as a increases.) One can easily check that (28) with $C_N = C_{N,f}$ implies that if for some $a > 0$ the right side of (28) is finite, then $(\Psi_\nu^* f)_a(x) < \infty$. Now, repeating the above argument resurrects the independence of C_N. If the right side of (28) is infinite, there is nothing to prove.

We point out that (28) holds also for $r > 1$, where the proof is much simpler. We only have to take (27) with $a+n$ instead of a, divide both sides by $(1+|2^\nu(x-y)|^a)$ and apply Hölder's inequality with respect to k and then z.

Multiplying (28) by $w_\nu(x)^r$ we derive with the properties of our weight sequence

$$(\Psi_\nu^* f)_a(x)^r w_\nu(x)^r \\ \leq C'_N \sum_{k=0}^{\infty} 2^{-k(N+\alpha_1)r} \int_{\mathbb{R}^n} \frac{2^{(k+\nu)n} |(\Psi_{k+\nu} * f)(z)|^r w_{k+\nu}(z)^r}{(1+|2^{k+\nu}(x-z)|^{a-\alpha})^r}\, dz, \tag{29}$$

for all $x \in \mathbb{R}^n$, $\nu \in \mathbb{N}_0$ and all $N \in \mathbb{N}$.

Now, we choose an $r > 0$ with $\frac{n}{a-\alpha} < r$. Then the function

$$\frac{1}{(1+|z|)^{r(a-\alpha)}}$$

is in $L_1(\mathbb{R}^n)$, and by the majorant property of the Hardy-Littlewood maximal operator (see [31, chapter 2]) it follows

$$(\Psi_\nu^* f)_a(x)^r w_\nu(x)^r \leq C'_N \sum_{k=0}^{\infty} 2^{-k(N+\alpha_1)r} M(|\Psi_{k+\nu} * f|^r w_{k+\nu}^r)(x). \tag{30}$$

We fix $N > 0$ such that $N + \alpha_1 > 0$ and denote

$$g_k(x) = M(|\Psi_k * f|^r w_k^r)(x).$$

From (30) we derive

$$G_\nu(x) = (\Psi_\nu^* f)_a(x)^r w_\nu(x)^r \le C \sum_{k\ge\nu}^{\infty} 2^{-k(N+\alpha_1)r} g_k(x).$$

So, for $0 < \delta < N + \alpha_1$, we apply the $\ell_{q/r}(L_{p(\cdot)/r})$ and the $L_{p(\cdot)/r}(\ell_{q/r})$ norm and we derive by Lemma 3.3

$$\|(\Psi_k^* f)_a(x)^r w_k(x)^r \mid \ell_{q/r}(L_{p(\cdot)/r})\| \le c\|M(|\Psi_k * f|^r w_k^r)(x) \mid \ell_{q/r}(L_{p(\cdot)/r})\| \tag{31}$$

and

$$\|(\Psi_k^* f)_a(x)^r w_k(x)^r \mid L_{p(\cdot)/r}(\ell_{q/r})\| \le c\|M(|\Psi_k * f|^r w_k^r)(x) \mid L_{p(\cdot)/r}(\ell_{q/r})\|. \tag{32}$$

To get rid of the maximal operator in (31), we rewrite the $\ell_{q/r}$ norm. Then we choose $\frac{n}{a-\alpha} < r < p_0$ and we get from Theorem 1.2 in [5] that $p(\cdot)/r \in \mathcal{B}(\mathbb{R}^n)$. This gives us (22).

For (32) we choose $\frac{n}{a-\alpha} < r < p_0$ and we obtain by Lemma 3.4 that (23) holds and the proof is complete. □

4. Complement: The case of $F^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$

The spaces $F^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ are a further generalization of the Triebel-Lizorkin spaces. Also the third index q may now dependent on $x \in \mathbb{R}^n$. These spaces were introduced by Diening, Hästö, and Roudenko in [9] and in this work they proved a characterization of these spaces by decomposition in molecules and atoms. Furthermore, they showed that the definition of the spaces $F^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ is independent of the chosen resolution of unity, if $s(\cdot), p(\cdot), q(\cdot)$ satisfy the *standing assumptions* (see [9]).

To study these spaces turned out to be useful in the connection with trace theorems. Due to

$$\operatorname{tr} F^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n) = F^{s(\cdot)-1/p(\cdot)}_{p(\cdot),p(\cdot)}(\mathbb{R}^{n-1})$$

(see Theorem 3.13 in [9]) we see the necessity of taking s and q variable if p is not constant. Quite recently, Vybíral in [35] has proved embeddings of Sobolev and Jawerth type for $F^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$.

We want to concentrate on this scale and generalize the smoothness function $s(\cdot)$ by the sequence of admissible weights which yields more general spaces, cf. Lemma 2.5.

Definition 4.1. Let $\boldsymbol{w} \in \mathcal{W}^{\alpha}_{\alpha_1,\alpha_2}$, $\{\varphi_j\}_{j\in\mathbb{N}_0}$ a resolution of unity from Remark 1.4. Further, let $p, q \in \mathcal{P}(\mathbb{R}^n)$ with $0 < p^- \leq p^+ < \infty$ and $0 < q^- \leq q^+ \leq \infty$. The space $F^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ is defined by

$$F^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\mathbb{R}^n) = \big\{ f \in S' : \|f \mid F^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)\|_{\varphi} < \infty \big\},$$

where

$$\|f \mid F^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)\|_{\varphi} = \Bigg\| \bigg(\sum_{j=0}^{\infty} |(\varphi_j \hat{f})\check{\ }(x) w_j(x)|^{q(x)} \bigg)^{1/q(x)} \ \Bigg| \ L_{p(\cdot)}(\mathbb{R}^n) \Bigg\|.$$

Fortunately, the arguments in the above proofs were mostly pointwise and at the end we used Lemma 3.3 or Lemma 3.4. That means, the local means characterization of the spaces $F^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ can be obtained by the same proofs as above; we only have to find the corresponding counterparts to the mentioned lemmas.

Furthermore, we obtain that the definition of this spaces is independent of the resolution of unity φ; so we can suppress φ in the notation of the norm.

First, let us give a modified version of Lemma 3.3.

Lemma 4.2. *Let $p, q \in \mathcal{P}(\mathbb{R}^n)$ with $0 < q^- \leq q^+ \leq \infty$ and $0 < p^- \leq p^+ \leq \infty$. For any sequence $\{g_j\}_{j\in\mathbb{N}_0}$ of nonnegative measurable functions on $\mathbb{R}^n$ denote for some $\delta > 0$*

$$G_j(x) = \sum_{k=0}^{\infty} 2^{-|k-j|\delta} g_k(x) \quad \text{for all } x \in \mathbb{R}^n.$$

Then with constant $c = c(p, q, \delta)$ we have

$$\|\{G_j\}_{j\in\mathbb{N}_0} \mid L_{p(\cdot)}(\ell_{q(\cdot)})\| \leq c\, \|\{g_j\}_{j\in\mathbb{N}_0} \mid L_{p(\cdot)}(\ell_{q(\cdot)})\|. \tag{33}$$

Proof. For fixed $x \in \mathbb{R}^n$ we show that there exists a constant $c = c(q, \delta)$ such that

$$\|G_j \mid \ell_{q(\cdot)}\| \leq c \|g_j \mid \ell_{q(\cdot)}\|, \tag{34}$$

holds, where c is independent of $x \in \mathbb{R}^n$.

- *First step: $q(x) \leq 1$.* We use the embedding $\ell_q(x) \hookrightarrow \ell_1$ (remember that $x \in \mathbb{R}^n$ is fixed), where the embedding constant is 1, and we derive

$$\begin{aligned}\sum_{j\in\mathbb{N}_0} G_j^{q(x)}(x) &\leq \sum_{j\in\mathbb{N}_0} \sum_{k\in\mathbb{N}_0} 2^{-|j-k|\delta q(x)} g_k^{q(x)}(x) \\ &\leq \sum_{k\in\mathbb{N}_0} g_k^{q(x)}(x) \sum_{j\in\mathbb{Z}} 2^{-|j|\delta q(x)}.\end{aligned}$$

We estimate the constant in (34),

$$c(q, \delta, x) = \Big(\sum_{j\in\mathbb{Z}} 2^{-|j|\delta q(x)} \Big)^{1/q(x)} \leq \Big(\sum_{j\in\mathbb{Z}} 2^{-|j|\delta q^-} \Big)^{1/q^-} := c(q, \delta),$$

which clearly is independent of $x \in \mathbb{R}^n$.

- *Second step:* $1 \le q(x) < \infty$. We define

$$\gamma_k = 2^{-|k|\delta} \quad \text{for all } k \in \mathbb{Z},$$
$$\beta_k = g_k(x) \quad \text{for } k \in \mathbb{N}_0 \quad \text{and} \quad \beta_k = 0 \quad \text{for } k < 0.$$

Then we have $G_k(x) = (\gamma * \beta)(k)$ and by Young's inequality

$$\|G_k \mid \ell_{q(x)}\| \le \|\gamma_k \mid \ell_1\| \, \|\beta_k \mid \ell_{q(x)}\|.$$

Hence, we derive (34) with $c(\delta) = \|2^{-|k|\delta} \mid \ell_1\|$. The case $q(x) = \infty$ is an easy modification of the above arguments.

Considering all cases, we obtain (34) by defining

$$c(q,\delta) = \max\Big\{ \sum_{j\in\mathbb{Z}} 2^{-|j|\delta}, \Big(\sum_{j\in\mathbb{Z}} 2^{-|j|\delta q^-} \Big)^{1/q^-} \Big\}$$

and (33) follows from (34) by using the monotony of the modular $\varrho_{L_{p(\cdot)}}$, see (2.7) in [18]. □

To get a modified version of Lemma 3.4 for $q(\cdot)$ not constant is a bit more difficult, it is even impossible. In [9, section 5] it is shown that the maximal operator can not be bounded on $L_{p(\cdot)}(\ell_{q(\cdot)})$ when $q(\cdot)$ is not constant. Fortunately, they found a replacement which is useful for us. We introduce the function $\eta_{\nu,m}(x) = 2^{\nu n}(1 + |2^\nu x|)^{-m}$ which we need for the formulation.

Lemma 4.3 (Theorem 3.2 in [9]). *Let $p, q \in C^{\log}(\mathbb{R}^n)$ with $1 < p^- \le p^+ < \infty$ and $1 < q^- \le q^+ < \infty$. Then the inequality*

$$\big\| \|\eta_{\nu,m} * f_\nu \mid \ell_{q(\cdot)}\| \bigm| L_{p(\cdot)}(\mathbb{R}^n) \big\| \le c \, \|f_\nu \mid L_{p(\cdot)}(\ell_{q(\cdot)})\|$$

holds for every sequence $\{f_\nu\}$ of $L_1^{\text{loc}}(\mathbb{R}^n)$ functions and constant $m > n$.

Now, it is easy to derive the following theorems.

Theorem 4.4. *Let $\boldsymbol{w} = \{w_j\}_{j\in\mathbb{N}_0} \in \mathcal{W}^{\alpha}_{\alpha_1,\alpha_2}$, $p, q \in \mathcal{P}(\mathbb{R}^n)$ with $0 < q^- \le q^+ \le \infty$ and $0 < p^- \le p^+ \le \infty$. Moreover, let $a \in \mathbb{R}$ with $a > 0$ and $R \in \mathbb{N}_0$ with $R > \alpha_2$,*

$$D^\beta \psi_1(0) = 0, \quad 0 \le |\beta| < R,$$

and for some $\varepsilon > 0$

$$|\phi_0(x)| > 0 \quad \text{on} \quad \{ x \in \mathbb{R}^n : |x| < \varepsilon \},$$
$$|\phi_1(x)| > 0 \quad \text{on} \quad \{ x \in \mathbb{R}^n : \varepsilon/2 < |x| < 2\varepsilon \}.$$

Then

$$\|(\Psi_k^* f)_a w_k \mid L_{p(\cdot)}(\ell_{q(\cdot)})\| \le c \, \|(\Phi_k^* f)_a w_k \mid L_{p(\cdot)}(\ell_{q(\cdot)})\| \tag{35}$$

holds for every $f \in \mathcal{S}'(\mathbb{R}^n)$.

Proof. We take the same proof as the one of Theorem 3.6. We get again the estimate (21). Now, we use Lemma 4.2 on (21) and obtain (35). □

We present now the counterpart of Theorem 3.8.

Theorem 4.5. *Let $\{w_k\}_{k\in\mathbb{N}_0} \in \mathcal{W}^{\alpha}_{\alpha_1,\alpha_2}$, $a \in \mathbb{R}$, and $p, q \in \mathcal{P}(\mathbb{R}^n)$ with $0 < q^- \le q^+ < \infty$ and $0 < p^- \le p^+ < \infty$. For some $\varepsilon > 0$ we assume $\psi_0, \psi_1 \in \mathcal{S}(\mathbb{R}^n)$ with*

$$|\psi_0| > 0 \quad on \quad \{\, x \in \mathbb{R}^n : |x| < \varepsilon \,\},$$
$$|\psi_1| > 0 \quad on \quad \{\, x \in \mathbb{R}^n : \varepsilon/2 < |x| < 2\varepsilon \,\}.$$

If there exists $0 < p_0 < \min(p^-, q^-)$ with $\frac{p(\cdot)}{p_0}, \frac{q(\cdot)}{q_0} \in C^{\log}(\mathbb{R}^n)$, then for $a > \frac{n}{p_0} + \alpha$

$$\|(\Psi_k^* f)_a w_k \mid L_{p(\cdot)}(\ell_{q(\cdot)})\| \le c\|(\Psi_k * f)w_k \mid L_{p(\cdot)}(\ell_{q(\cdot)})\| \tag{36}$$

holds for all $f \in \mathcal{S}'(\mathbb{R}^n)$.

Proof. Again we look on the proof of Theorem 3.8 and we get by pointwise estimates inequality (29). Using for $r < p_0$ and $N > -\alpha_1$ the monotony of the $L_{p(\cdot)/r}(\ell_{q(\cdot)/r})$ norm and Lemma 4.2 on this inequality we obtain

$$\begin{aligned}\|(\Psi_\nu^* f)_a^r w_\nu^r \mid L_{p(\cdot)/r}(\ell_{q(\cdot)/r})\| \\ \le c\Big\|\|[|(\Psi_k * f)|^r w_k^r] * \eta_{k,(a-\alpha)r} \mid \ell_{q(\cdot)/r}\| \;\Big|\; L_{p(\cdot)/r}\Big\|.\end{aligned} \tag{37}$$

It is easy to see that for $r < p_0$ we have $p(\cdot)/r, q(\cdot)/r \in C^{\log}(\mathbb{R}^n)$ with $1 < p^-/r \le p^+/r < \infty$ and $1 < q^-/r \le q^+/r < \infty$. Therefore, all conditions in Lemma 4.3 are satisfied and we get using $\frac{n}{a-\alpha} < r < p_0$ with Lemma 4.3 applied on (37)

$$\|(\Psi_\nu^* f)_a^r w_\nu^r \mid L_{p(\cdot)/r}(\ell_{q(\cdot)/r})\| \le c\||(\Psi_k * f)|^r w_k^r \mid L_{p(\cdot)/r}(\ell_{q(\cdot)})\|,$$

which gives (36). □

Remark 4.6. It is desirable to replace $q^+ < \infty$ in Theorem 4.5 by $q^+ \le \infty$ to get the connection to the usual space $F^s_{p,q}(\mathbb{R}^n)$ of constant parameters. Unfortunately, the restriction $q^+ < \infty$ is necessary due to the use of Lemma 4.3.

As easy corollaries of the above theorems we get the local means characterization of the spaces $F^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$.

Corollary 4.7. *Let $\boldsymbol{w} = \{w_k\}_{k\in\mathbb{N}_0} \in \mathcal{W}^{\alpha}_{\alpha_1,\alpha_2}$, $p, q \in \mathcal{P}(\mathbb{R}^n)$ with $0 < p^- \le p^+ < \infty$ and $0 < q^- \le q^+ < \infty$, and let $a \in \mathbb{R}$, $R \in \mathbb{N}_0$ with $R > \alpha_2$. Further, let ψ_0, ψ_1 belong to $\mathcal{S}(\mathbb{R}^n)$ with*

$$D^\beta \psi_1(0) = 0, \quad for\ 0 \le |\beta| < R.$$

and

$$|\psi_0(x)| > 0 \quad on \quad \{\, x \in \mathbb{R}^n : |x| < \varepsilon \,\},$$
$$|\psi_1(x)| > 0 \quad on \quad \{x \in \mathbb{R}^n : \varepsilon/2 < |x| < 2\varepsilon \,\}$$

for some $\varepsilon > 0$. If there exists $p_0 < \min(p^-, q^-)$ with $p(\cdot)/p_0, q(\cdot)/p_0 \in C^{\log}(\mathbb{R}^n)$, then for $a > \frac{n}{p_0} + \alpha$

$$\|f \mid F^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)\| \sim \|(\Psi_k * f)w_k \mid L_{p(\cdot)}(\ell_{q(\cdot)})\| \sim \|(\Psi^*_k f)_a w_k \mid L_{p(\cdot)}(\ell_{q(\cdot)})\|$$

holds for all $f \in \mathcal{S}'(\mathbb{R}^n)$.

Keeping the same notation as introduced in section 2.2 we can formulate

Corollary 4.8. *Let $\boldsymbol{w} = \{w_j\}_{j\in\mathbb{N}_0} \in \mathcal{W}^{\alpha}_{\alpha_1,\alpha_2}$, and $p, q \in \mathcal{P}(\mathbb{R}^n)$ with $0 < p^- \leq p^+ < \infty$ and $0 < q^- \leq q^+ < \infty$. Furthermore, let $N \in \mathbb{N}_0$ with $2N > \alpha_2$ and let $k_0, k^0 \in \mathcal{S}(\mathbb{R}^n)$ and the function k be defined in section 2.2. If there exists a $p_0 \leq \min(p^-, q^-)$ with $p(\cdot)/p_0, q(\cdot)/p_0 \in C^{\log}(\mathbb{R}^n)$, then*

$$\|k_0(1,f)w_0 \mid L_{p(\cdot)}(\mathbb{R}^n)\| + \left\| \left(\sum_{j=1}^{\infty} |k(2^{-j}, f)(\cdot) w_j(\cdot)|^{q(\cdot)} \right)^{1/q(\cdot)} \,\middle|\, L_{p(\cdot)}(\mathbb{R}^n) \right\|$$

is an equivalent norm on $F^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ for all $f \in \mathcal{S}'(\mathbb{R}^n)$.

References

[1] A. Almeida and S. Samko, *Characterization of Riesz and Bessel potentials on variable Lebesgue spaces*, J. Funct. Spaces Appl. **4** (2006), no. 2, 113–144.

[2] P. Andersson, *Two-microlocal spaces, local norms and weighted spaces* (1997), 35–58. paper in Ph.D. Thesis.

[3] O. V. Besov, *Equivalent normings of spaces of functions of variable smoothness*, Tr. Mat. Inst. Steklova **243** (2003), 87–95 (Russian); English transl., Proc. Steklov Inst. Math. **4 (243)** (2003), 80–88.

[4] J.-M. Bony, *Second microlocalization and propagation of singularities for semilinear hyperbolic equations*, Hyperbolic equations and related topics (Katata/Kyoto, 1984), Academic Press, Boston, MA, 1986, pp. 11–49.

[5] D. Cruz-Uribe, A. Fiorenza, J. M. Martell, and C. Pérez, *The boundedness of classical operators on variable L^p spaces*, Ann. Acad. Sci. Fenn. Math. **31** (2006), no. 1, 239–264.

[6] D. Cruz-Uribe, A. Fiorenza, and C. J. Neugebauer, *The maximal function on variable L^p spaces*, Ann. Acad. Sci. Fenn. Math. **28** (2003), no. 1, 223–238.

[7] L. Diening, *Maximal function on generalized Lebesgue spaces $L^{p(\cdot)}$*, Math. Inequal. Appl. **7** (2004), no. 2, 245–253.

[8] L. Diening, P. Harjulehto, P. Hästö, Y. Mizuta, and T. Shimomura, *Maximal functions in variable exponent spaces: Limiting cases of the exponent*, preprint.

[9] L. Diening, P. Hästö, and S. Roudenko, *Function spaces of variable smoothness and integrability*, J. Funct. Anal., to appear.

[10] D. E. Edmunds and H. Triebel, *Function spaces, entropy numbers, differential operators*, Cambridge Tracts in Mathematics, vol. 120, Cambridge University Press, Cambridge, 1996.

[11] W. Farkas and H.-G. Leopold, *Characterisations of function spaces of generalised smoothness*, Ann. Mat. Pura Appl. (4) **185** (2006), no. 1, 1–62.

[12] P. Gurka, P. Harjulehto, and A. Nekvinda, *Bessel potential spaces with variable exponent*, Math. Inequal. Appl. **10** (2007), no. 3, 661–676.

[13] S. Jaffard, *Pointwise smoothness, two-microlocalization and wavelet coefficients*, Publ. Mat. **35** (1991), no. 1, 155–168. Conference on Mathematical Analysis (El Escorial, 1989).

[14] S. Jaffard and Y. Meyer, *Wavelet methods for pointwise regularity and local oscillations of functions*, Mem. Amer. Math. Soc. **123** (1996), no. 587, x+110.

[15] H. Kempka, *Local characterization of generalized 2-microlocal spaces*, Jenaer Schriften zur Math. & Inf., vol. 20/06, 2006.

[16] ———, *Generalized 2-microlocal Besov spaces*, dissertation, Jena, 2008.

[17] ———, *Atomic, molecular and wavelet decomposition of generalized 2-microlocal Besov spaces*, J. Funct. Spaces Appl., to appear.

[18] O. Kováčik and J. Rákosník, *On spaces $L^{p(x)}$ and $W^{k,p(x)}$*, Czechoslovak Math. J. **41 (116)** (1991), no. 4, 592–618.

[19] J. Lévy Véhel and S. Seuret, *The 2-microlocal formalism*, Fractal geometry and applications: A jubilee of Benoît Mandelbrot, Part 2, Proc. Sympos. Pure Math., vol. 72, Amer. Math. Soc., Providence, RI, 2004, pp. 153–215.

[20] Y. Meyer, *Wavelets, vibrations and scalings*, CRM Monograph Series, vol. 9, American Mathematical Society, Providence, RI, 1998.

[21] Y. Meyer and H. Xu, *Wavelet analysis and chirps*, Appl. Comput. Harmon. Anal. **4** (1997), no. 4, 366–379.

[22] S. Moritoh and T. Yamada, *Two-microlocal Besov spaces and wavelets*, Rev. Mat. Iberoamericana **20** (2004), no. 1, 277–283.

[23] S. Moura, *Function spaces of generalised smoothness*, Dissertationes Math. **398** (2001), 88.

[24] A. Nekvinda, *Hardy-Littlewood maximal operator on $L^{p(x)}(\mathbb{R})$*, Math. Inequal. Appl. **7** (2004), no. 2, 255–265.

[25] W. Orlicz, *Über konjugierte Exponentenfolgen*, Studia Math. **3** (1931), 200–212.

[26] J. Peetre, *New thoughts on Besov spaces*, Duke University Mathematics Series, vol. 1, Mathematics Department, Duke University, Durham, N.C., 1976.

[27] ———, *On spaces of Triebel-Lizorkin type*, Ark. Mat. **13** (1975), 123–130.

[28] V. S. Rychkov, *On a theorem of Bui, Paluszyński, and Taibleson*, Tr. Mat. Inst. Steklova **227** (1999), 286–298 (Russian); English transl., Proc. Steklov Inst. Math. **4 (227)** (1999), 280–292.

[29] H.-J. Schmeisser and H. Triebel, *Topics in Fourier analysis and function spaces*, Mathematik und ihre Anwendungen in Physik und Technik, vol. 42, Akademische Verlagsgesellschaft Geest & Portig K.-G., Leipzig, 1987.

[30] J. Schneider, *Function spaces of varying smoothness*, I, Math. Nachr. **280** (2007), no. 16, 1801–1826.

[31] E. M. Stein and G. Weiss, *Introduction to Fourier analysis on Euclidean spaces*, Princeton Mathematical Series, vol. 32, Princeton University Press, Princeton, N.J., 1971.

[32] H. Triebel, *Theory of function spaces*, Mathematik und ihre Anwendungen in Physik und Technik, vol. 38, Akademische Verlagsgesellschaft Geest & Portig K.-G., Leipzig, 1983.

[33] ———, *Theory of function spaces*, II, Monographs in Mathematics, vol. 84, Birkhäuser Verlag, Basel, 1992.

[34] ———, *Theory of function spaces*, III, Monographs in Mathematics, vol. 100, Birkhäuser Verlag, Basel, 2006.

[35] J. Vybíral, *Sobolev and Jawerth embeddings for spaces with variable smoothness and integrability*, preprint.

[36] H. Xu, *Généralisation de la théorie des chirps à divers cadres fonctionnels et application à leur analyse par ondelettes*, Université Paris IX Dauphine, 1996.

[37] J. Xu, *Variable Besov and Triebel-Levizorkin spaces*, Ann. Acad. Sci. Fenn. Math. **33** (2008), no. 2, 511–522.