Monodromy Zeta-functions of Deformations and Newton Diagrams

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ABSTRACT

For a one-parameter deformation of an analytic complex function germ of several variables, there is defined its monodromy zeta-function. We give a Varchenko type formula for this zeta-function if the deformation is non-degenerate with respect to its Newton diagram.

Key words: Deformations of singularities, monodromy, zeta-function, Newton diagram.

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1. Introduction

Let F be the germ of an analytic function on $(\mathbb{C}^{n+1}, 0)$, where $\mathbb{C}^{n+1} = \mathbb{C}_\sigma \times \mathbb{C}^n_{\mathbf{z}}$, σ is the coordinate on $\mathbb{C}$, and $\mathbf{z} = (z_1, z_2, \ldots, z_n)$ are the coordinates on $\mathbb{C}^n$. The germ F provides a deformation $f_\sigma = F(\sigma, \cdot)$ of the function germ $f = f_0$ on $(\mathbb{C}^n, 0)$. We give formulae for the monodromy zeta-functions of the deformations of the hypersurface germs $\{f = 0\} \cap (\mathbb{C}^*)^n$ and $\{f = 0\}$ at the origin in terms of the Newton diagram of F. A reason to study deformations of hypersurface germs and their monodromy zeta-functions was inspired by their connection with zeta-functions of deformations of polynomials: [3].

Let A be the complement to an arbitrary analytic hypersurface Y in $\mathbb{C}^n$: $A = \mathbb{C}^n \setminus Y$. Let $V = \{F = 0\} \cap (\mathbb{C}_\sigma \times A) \cap B_\varepsilon$, where $B_\varepsilon \subset \mathbb{C}^{n+1}$ is the closed ball of

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radius ε with the centre at the origin. Let $\mathbb{D}^*_\delta \subset \mathbb{C}_\sigma$ be the punctured disk of radius δ with the centre at the origin. For $0 < \delta \ll \varepsilon$ small enough the restriction to V of the projection $\mathbb{C}^{n+1} \to \mathbb{C}_\sigma$ onto the first factor provides a fibration over $\mathbb{D}^*_\delta$ ([7]). Denote by V_c the fibre over the point c. Consider the monodromy transformation $h_{F,A}\colon V_c \to V_c$ of the above fibration restricted to the loop $c \cdot \exp(2\pi i t)$, $t \in [0,1]$, $|c|$ is small enough.

The zeta-function of an arbitrary transformation $h\colon X \to X$ of a topological space X is the rational function $\zeta_h(t) = \prod_{i\geq 0}(\det(\mathrm{Id} - t h_*|_{H^c_i(X;\mathbb{C})}))^{(-1)^i}$, where $H^c_i(X;\mathbb{C})$ is the i-th homology group with closed support.

Definition 1.1. The zeta-function of the monodromy transformation $h_{F,A}$ will be called the monodromy zeta-function of the deformation f_σ on A: $\zeta_{f_\sigma|_A}(t) = \zeta_{h_{F,A}}(t)$.

For a power series $S = \sum c_{\mathbf{k}} \mathbf{y}^{\mathbf{k}}$, $\mathbf{y}^{\mathbf{k}} = y_1^{k_1} \cdots y_m^{k_m}$, one defines its Newton diagram as follows. Denote by $\mathbb{R}_+ \subset \mathbb{R}$ the set of non-negative real numbers. Denote by $\Gamma'(S)$ the convex hull of the union $\cup_{c_{\mathbf{k}}\neq 0}(\mathbf{k} + \mathbb{R}^m_+)$. The Newton diagram of the series S is the union of compact faces of $\Gamma'(S)$. For a germ G on $\mathbb{C}^m$ at the origin, its Newton diagram $\Gamma(G)$ is the Newton diagram of its Taylor series at the origin.

For a generic germ F on $(\mathbb{C}^{n+1}, 0)$ with fixed Newton diagram $\Gamma \in \mathbb{R}^{n+1}_+$ the zeta-functions $\zeta_{f_\sigma|_{(\mathbb{C}^*)^n}}(t)$, $\zeta_{f_\sigma|_{\mathbb{C}^n}}(t)$ are also fixed. We provide explicit formulas for these zeta-functions in terms of the Newton diagram Γ.

2. The main result (a Varchenko type formula)

Let F be a germ of a function on $(\mathbb{C}^{n+1}, 0)$. Let $\mathbf{k} = (k_0, k_1, \ldots, k_n)$ be the coordinates on $\mathbb{R}^{n+1}$ corresponding to the variables $\sigma, z_1, \ldots, z_n$ respectively. For $I \subset \{0, 1, \ldots, n\}$, denote by $\mathbb{R}^I$ and $\Gamma^I(F)$ the sets $\{\mathbf{k} \mid k_i = 0,\ i \notin I\} \subset \mathbb{R}^{n+1}$ and $\Gamma(F) \cap \mathbb{R}^I$ respectively.

An integer covector is called primitive if it is not a multiple of another integer covector. Let P^I be the set of primitive integer covectors in the dual space $(\mathbb{R}^I)^*$ such that all their components are strictly positive. For $\alpha \in P^I$, let $\Gamma^I_\alpha(F)$ be the subset of the diagram $\Gamma^I(F)$ where $\alpha|_{\Gamma^I(F)}$ reaches its minimal value: $\Gamma^I_\alpha(F) = \{\mathbf{x} \in \Gamma^I(F) \mid \alpha(\mathbf{x}) = \min(\alpha|_{\Gamma^I(F)})\}$ (for $\Gamma^I(F) = \emptyset$ we assume $\Gamma^I_\alpha(F) = \emptyset$). Consider the Taylor series of the germ F at the origin: $F = \sum F_{\mathbf{k}} \sigma^{k_0} z_1^{k_1} \ldots z_n^{k_n}$. Denote: $F_\alpha = \sum_{\mathbf{k}\in\Gamma^{\{0,1,\ldots,n\}}_\alpha} F_{\mathbf{k}} \sigma^{k_0} z_1^{k_1} \ldots z_n^{k_n}$.

Definition 2.1. A germ F of a function on $(\mathbb{C}^{n+1}, 0)$ is called non-degenerate with respect to its Newton diagram if for any $\alpha \in P^I$ the 1-form dF_α does not vanish on the germ $\{F_\alpha = 0\} \cap (\mathbb{C}^*)^{n+1}$ at the origin (see [9]).

For $I \in \{0, 1, \ldots, n\}$ such that $0 \in I$, we denote:

$$\zeta^I_F(t) = \prod_{\alpha\in P^I} \left(1 - t^{\alpha(\frac{\partial}{\partial k_0})}\right)^{(-1)^{l-1} l!\, V_l(\Gamma^I_\alpha(F))},$$

where $l = |I| - 1$, $\frac{\partial}{\partial k_0}$ is the vector in $\mathbb{R}^I$ with the single non-zero coordinate $k_0 = 1$, and $V_l(\cdot)$ denotes the l-dimensional integer volume, i.e., the volume in a rational l-dimensional affine hyperplane of $\mathbb{R}^I$ normalized in such a way that the volume of the minimal parallelepiped with integer vertices is equal to 1. We assume that $V_0(\mathrm{pt}) = 1$ and for $n \geq 0$ one has $V_n(\emptyset) = 0$.

Theorem 2.2. *Let F be non-degenerate with respect to its Newton diagram $\Gamma(F)$. Then one has*

$$\zeta_{f_\sigma|_{(\mathbb{C}^*)^n}}(t) = \zeta_F^{\{0,1\dots,n\}}(t), \tag{1}$$

$$\zeta_{f_\sigma|_{\mathbb{C}^n}}(t) = (1-t) \times \prod_{I:\, 0\in I\subset\{0,1,\dots,n\}} \zeta_F^I(t). \tag{2}$$

Remarks 2.3.

(i) The equation (1) implies the equation (2) because of the following multiplicative property of the zeta-function. Let $h\colon X \to X$ be a transformation of a CW-complex X. Let $Y \subset X$ be a subcomplex of X. Assume that $h(Y) \subset Y$, $h(X \setminus Y) \subset (X \setminus Y)$. Then $\zeta_{h|_X}(t) = \zeta_{h|_{X\setminus Y}}(t) \times \zeta_{h|_Y}(t)$.

One can see that $\zeta_{f_\sigma|_{\{0\}}}(t) = (1-t) \times \zeta_F^{\{0\}}(t)$. In fact, in the case $\Gamma^{\{0\}} = \emptyset$ one has $\zeta_{f_\sigma|_{\{0\}}}(t) = (1-t)$, $\zeta_F^{\{0\}}(t) = 1$. Otherwise $\zeta_{f_\sigma|_{\{0\}}}(t) = 1$, $\zeta_F^{\{0\}}(t) = (1-t)^{-1}$.

(ii) The zeta-function $\zeta_{f_\sigma|_{\mathbb{C}^n}}(t)$ coincides with the monodromy zeta-function of the germ of the function $\sigma\colon \{F = 0\} \to \mathbb{C}_\sigma$ at the origin. The main theorem of [8] provides a formula for the zeta-functions of germs of functions on complete intersections in non-degenerate cases. One can apply this formula to the germ σ and verify that the formula (2) agrees with the one of M. Oka. But (2) can not be deduced from the result of M. Oka because the function σ does not satisfy the condition of "convenience" ([8, page 17]).

Example 2.4.

(i) Let $F(\sigma, \mathbf{z}) = f(\mathbf{z}) - \sigma$. The monodromy zeta-function of the deformation f_σ coincides with the (ordinary) monodromy zeta-function $\zeta_f(t)$ of the germ f on $(\mathbb{C}^n, 0)$ (see, e.g., [9]). In this case the l-dimensional faces $\Gamma^I_\alpha(F)$ (where $l = |I| - 1 > 0$) are cones of integer height 1 over the corresponding $(l-1)$-dimensional faces $\Gamma^{I\setminus\{0\}}_{\alpha|_{\{k_0=0\}}}(f)$. One has:

$$V_l(\Gamma^I_\alpha(F)) = V_{l-1}\left(\Gamma^{I\setminus\{0\}}_{\alpha|_{\{k_0=0\}}}(f)\right) / l,$$

with $\alpha(\partial/\partial k_0) = \min(\alpha|_{\Gamma^{I\setminus\{0\}}(f)})$. This means that in this case the equation (2) coincides with the Varchenko formula ([9]).

(ii) For a deformation $F(\sigma, \mathbf{z})$ of the form $f_0(\mathbf{z}) - \sigma f_1(\mathbf{z})$, the fibre

$$(\{\sigma\} \times \{f_\sigma = 0\}) \cap B_\varepsilon$$

is the disjoint union of the sets

$$(\{\sigma\} \times \{f_0/f_1 = \sigma\}) \cap B_\varepsilon$$

and

$$(\{\sigma\} \times \{f_0 = f_1 = 0\}) \cap B_\varepsilon.$$

If $f_0(0) = f_1(0) = 0$, then $\zeta_{f_\sigma|_{\mathbb{C}^n}}(t) = (1-t) \times \zeta_{(f_0/f_1)|_{\mathbb{C}^n}}(t)$, otherwise $\zeta_{f_\sigma|_{\mathbb{C}^n}}(t) = \zeta_{(f_0/f_1)|_{\mathbb{C}^n}}(t)$ (the zeta-function of the meromorphic function f_0/f_1: [2]).
For $I \subset \{0,1,\dots,n\}$ such that $0 \in I$, and for a covector $\alpha \in P^I$, assume that the face $\Gamma^I_\alpha(F)$ has dimension l, where $l = |I|-1 > 1$. Then $\Gamma^I_\alpha(F)$ is the convex hull of the corresponding faces $\Delta^I_{\alpha,0} = \{0\} \times \Gamma^{I\setminus\{0\}}_{\alpha|_{\{k_0=0\}}}(f_0)$ and $\Delta^I_{\alpha,1} = \{1\} \times \Gamma^{I\setminus\{0\}}_{\alpha|_{\{k_0=0\}}}(f_1)$, which lie in the hyperplanes $\{k_0 = 0\}$ and $\{k_0 = 1\}$ respectively. It is not difficult to show (see, e.g., [4, Lemma 1]) that $V_l(\Gamma^I_\alpha(F)) = V^I_\alpha/l$, where

$$\begin{aligned} V^I_\alpha &= V_{l-1}(\Delta^I_{\alpha,0},\dots,\Delta^I_{\alpha,0}) + V_{l-1}(\Delta^I_{\alpha,0},\dots,\Delta^I_{\alpha,0},\Delta^I_{\alpha,1}) \\ &\quad + \dots + V_{l-1}(\Delta^I_{\alpha,0},\Delta^I_{\alpha,1},\dots,\Delta^I_{\alpha,1}) + V_{l-1}(\Delta^I_{\alpha,1},\dots,\Delta^I_{\alpha,1}). \end{aligned}$$

Here V_{l-1} denotes the $(l-1)$-dimensional Minkowski's mixed volume: see, e.g., [8]. Moreover, $\alpha(\partial/\partial k_0) = \min(\alpha|_{\Gamma^{I\setminus\{0\}}(f_0)}) - \min(\alpha|_{\Gamma^{I\setminus\{0\}}(f_1)})$, thus (2) coincides with the main result of [2].

3. A'Campo type formula

Proof of Theorem 2.2 uses an A'Campo type formula ([1]) written in terms of the integration with respect to the Euler characteristic ([3]).

For a constructible function Φ on a constructible set Z with values in a (multiplicative) Abelian group G, its integral $\int_Z \Phi^{d\chi}$ with respect to the Euler characteristic χ is defined as $\prod_{g\in G} g^{\chi(\Phi^{-1}(g))}$ (see [10]). Further we consider $G = \mathbb{C}(t)^*$ to be the multiplicative group of non-zero rational functions in the variable t.

Let F be a germ of an analytic function on $(\mathbb{C}^{n+1},0)$ defined on a neighbourhood U of the origin. Let Y be a hypersurface in $\mathbb{C}^n$. Denote $S = (\mathbb{C}_\sigma \times Y) \cup \{\sigma = 0\}$. Consider a resolution $\pi\colon (X,D) \to (U,0)$ of the germ of the hypersurface $\{F=0\} \cup S$ at the origin, where $D = \pi^{-1}(0)$ is the exceptional divisor.

Theorem 3.1. *Assume π to be an isomorphism outside of $\pi^{-1}(U \cap S)$. Then*

$$\zeta_{f_\sigma|_{\mathbb{C}^n\setminus Y}}(t) = \int_{D\cap W} \zeta_{\Sigma|_{W\setminus Z},x}(t)^{d\chi}, \tag{3}$$

where W is the proper preimage of $\{F = 0\}$ (i.e., the closure of $\pi^{-1}(V)$, $V = ((\{F=0\} \cap U) \setminus S)$), $\Sigma = \sigma \circ \pi$, $Z = \pi^{-1}(\mathbb{C}_\sigma \times Y)$ and $\zeta_{\Sigma|_{W\setminus Z},x}(t)$ is the monodromy zeta-function of the germ of the function Σ on the set $W \setminus Z$ at the point $x \in D \cap W$.

Proof. The map π provides an isomorphism $W \setminus (Z \cup \{\Sigma = 0\}) \to V$, which is also an isomorphism of fibrations provided by the maps Σ and σ over sufficiently small punctured neighbourhood of zero $\mathbb{D}^*_\delta \subset \mathbb{C}_\sigma$. Therefore the monodromy zeta-functions of this fibrations coincide, $\zeta_{f_\sigma|_{\mathbb{C}^n\setminus Y}}(t) = \zeta_{\Sigma|_{W\setminus Z}}(t)$ (the monodromy zeta-function of the "global" function Σ on $W \setminus Z$).

Applying the localization principle ([3]) to Σ we obtain:

$$\zeta_{f_\sigma|_{\mathbb{C}^n\setminus Y}}(t) = \int_{W\cap\{\Sigma=0\}} \zeta_{\Sigma|_{W\setminus Z},\, x}(t)^{d\chi}. \tag{4}$$

The integration is multiplicative with respect to subdivision of its domain. One has $W \cap \{\Sigma = 0\} = (D\cap W) \sqcup ((W \cap \{\Sigma = 0\}) \setminus D)$. Thus the right hand side of (4) is the product

$$\left[\int_{D\cap W} \zeta_{\Sigma|_{W\setminus Z},\, x}(t)^{d\chi}\right] \cdot \left[\int_{W\cap(\{\Sigma=0\}\setminus D)} \zeta_{\Sigma|_{W\setminus Z},\, x}(t)^{d\chi}\right].$$

The first factor coincide with the right hand side of (3); we prove that the second factor equals 1.

For a point $x \in D$, its neighbourhood $U(x) \subset X$ with a coordinate system $u_1, u_2, \dots, u_{n+1}$ is called *convenient* if each the of manifolds D, Z can be defined on $U(x)$ by an equation of type $\mathbf{u}^{\mathbf{k}} = 0$ and each of the functions Σ, $\tilde{F} = F \circ \pi$ has the form $a\,\mathbf{u}^{\mathbf{k}}$, where $a(0) \neq 0$. One can assume that X is covered by a finite number of convenient neighbourhoods.

For an arbitrary convenient neighbourhood U_0, choose an order of coordinates u_i on it such that $D = \{u_1 u_2 \cdots u_l = 0\}$.

Proposition 3.2. *The zeta-function $\zeta_{\Sigma|_{W\setminus Z},\, x}(t)$ at a point $x \in U_0 \setminus D$ is well-defined by the coordinates $u_{l+1}, u_{l+2}, \dots, u_{n+1}$ of x.*

Proof. The germ of the manifold Z at the point x is defined by an equation

$$u_{l+1}^{k_{1,l+1}} \cdots u_{n+1}^{k_{1,n+1}} = 0.$$

In a neighbourhood of x one has $\tilde{F} = a\, u_{l+1}^{k_{2,l+1}} \cdots u_{n+1}^{k_{2,n+1}}$, $\Sigma = b\, u_{l+1}^{k_{3,l+1}} \cdots u_{n+1}^{k_{3,n+1}}$, where $a(x) \neq 0$, $b(x) \neq 0$, $k_{1,j} \in \{0,1\}$; $k_{2,j}, k_{3,j} \geq 0$. The zeta-function $\zeta_{\Sigma|_{W\setminus Z},\, x}(t)$ is well-defined by the numbers $k_{i,j}$, $i = 1,2,3$, $j = l+1, \dots, n+1$, which do not depend on $u_1, \dots, u_l$. ☐

For a rational function $Q(t)$, we define a set

$$X_Q = \{x \in W \cap (\{\Sigma = 0\} \setminus D) \mid \zeta_{\Sigma|_{W\setminus Z},\, x}(t) = Q(t)\}.$$

It follows from the proposition above that for any convenient neighbourhood U_0 we have $\chi(U_0 \cap X_Q) = 0$. Thus for all $Q(t)$ we have $\chi(X_Q) = 0$ and

$$\int_{W\cap(\{\Sigma=0\}\setminus D)} \zeta_{\Sigma|_{W\setminus Z},\, x}(t)^{d\chi} = \prod_Q Q^{\chi(X_Q)} = 1.$$

☐

4. Proof of Theorem 2.2

Using the Newton diagram $\Gamma(F)$ of the germ F on $(\mathbb{C}^{n+1},0)$, one can construct a unimodular simplicial subdivision Λ of the set of covectors with non-negative coordinates $(\mathbb{R}^{n+1})^*_+$ (see, e.g., [9]). Consider the toroidal modification map

$$p : (X_\Lambda, D) \to (\mathbb{C}^{n+1}, 0),$$

corresponding to Λ. Let $U \subset \mathbb{C}^{n+1}$ be a small enough ball with the centre at the origin, $X = p^{-1}(U)$, $\pi = p|_X$. Let $Y = \{z_1 z_2 \cdots z_n = 0\} \subset \mathbb{C}^n_{\mathbf{z}}$. Then $S = (Y \times \mathbb{C}_\sigma) \cup \{\sigma = 0\}$ is the union of the coordinate hyperplanes of $\mathbb{C}^{n+1}$. Since F is non-degenerate with respect to its Newton diagram $\Gamma(F)$, π is a resolution of the germ $S \cup \{F = 0\}$ (see, e.g., [8]). Finally, π is an isomorphism outside of S, so the resolution (X, π) satisfies the assumptions of Theorem 3.1.

Compute the right hand side of (3). Let $x \in D \cap W$ be a point of the torus T_λ of dimension $n-l+1$, corresponding to an l-dimensional cone $\lambda \in \Lambda$. Let λ be generated by integer covectors $\alpha_1, \dots, \alpha_l$ and let λ lie on the border of a cone $\lambda' \in \Lambda$ generated by $\alpha_1, \dots, \alpha_l, \dots \alpha_{n+1}$. Let $(u_1, \dots, u_{n+1})$ be the coordinate system corresponding to the set $(\alpha_1, \dots, \alpha_{n+1})$. There exists a coordinate system $(u_1, \dots, u_l, w_{l+1}, \dots, w_{n+1})$ in a neighbourhood U' of the point x such that $w_i(x) = 0$, $i = l+1, \dots, n+1$ and $\tilde{F} = F \circ \pi = a\, u_1^{k_{1,1}} u_2^{k_{1,2}} \cdots u_l^{k_{1,l}} \cdot w_{n+1}^{k_{1,n+1}}$ (where $a(0) \neq 0$). The zero level set $\{\Sigma = 0\}$ is a normal crossing divisor contained in $\{u_1 u_2 \cdots u_l = 0\}$. Therefore $\Sigma = \sigma \circ \pi = u_1^{k_{2,1}} u_2^{k_{2,2}} \cdots u_l^{k_{2,l}}$. One has: $W \cap U' = \{w_{n+1} = 0\}$ and

$$(Z \cup \{\Sigma = 0\}) \cap U' = \{u_1 u_2 \cdots u_l = 0\}.$$

Thus $\zeta_{\Sigma|_{W \setminus Z}, x}(t) = \zeta_{g|_{\{u_i \neq 0,\, i \leq l\}}}(t)$, where g is the germ of the following function of n variables: $g(u_1, \dots, u_l, w_{l+1}, \dots, w_n) = u_1^{k_{2,1}} u_2^{k_{2,2}} \cdots u_l^{k_{2,l}}$.

Assume that one of the exponents $k_{2,1}, k_{2,2} \dots, k_{2,l}$ (say, $k_{2,1}$) is equal to zero. Then g does not depend on u_1. We may assume that the monodromy transformation of its Milnor fibre also does not depend on u_1. Denote $h = g|_{\{u_1 = 0\}}$. The monodromy transformations of the fibre of $g|_{\{u_2 u_3 \cdots u_l \neq 0\}}$ and one of $h|_{\{u_2 u_3 \cdots u_l \neq 0\}}$ are homotopy equivalent, so $\zeta_{g|_{\{u_2 u_3 \cdots u_l \neq 0\}}}(t) = \zeta_{h|_{\{u_2 u_3 \cdots u_l \neq 0\}}}(t)$. On the other hand the multiplicative property of the zeta-function implies that

$$\zeta_{g|_{\{u_i \neq 0,\, i \leq l\}}}(t) \times \zeta_{h|_{\{u_2 u_3 \cdots u_l \neq 0\}}}(t) = \zeta_{g|_{\{u_2 u_3 \cdots u_l \neq 0\}}}(t),$$

and thus $\zeta_{g|_{\{u_i \neq 0,\, i \leq l\}}}(t) = 1$.

Now assume that all the exponents $k_{2,1}, k_{2,2} \dots, k_{2,l}$ are positive. Then the non-zero fibre of the function g does not intersect $\{u_1 u_2 \dots u_l = 0\}$, so $\zeta_{g|_{\{u_i \neq 0,\, i \leq l\}}}(t) = \zeta_g(t)$. In the case $l > 1$ one has $\zeta_g(t) = 1$. In the case $l = 1$ one has: $g = u_1^{k_{2,1}}$, $\zeta_g(t) = 1 - t^{k_{2,1}}$.

We see that the integrand in (3) differs from 1 only at points x that lie in strata of dimension n. From here on $l = 1$. If all the components of $\alpha = \alpha_1$ are positive, then

$T_\lambda \subset D$. Otherwise, $T_\lambda \cap D = \emptyset$. From here on $\alpha \in P^{\{0,1,\dots,n\}}$ (see the definitions before Theorem 2.2).

Using the coordinates $(u_2, \dots, u_{n+1})$ on the torus $T_\lambda = \{u_1 = 0\}$ we obtain: $T_\lambda \cap W = \{Q_\alpha = 0\}$, where for the power series $F = \sum F_{\mathbf{k}} \sigma^{k_0} z_1^{k_1} \cdots z_n^{k_n}$ we denote $Q_\alpha = \sum_{\mathbf{k} \in \Gamma_\alpha^{\{0,\dots,n\}}(F)} F_{\mathbf{k}} u_2^{\alpha_2(\mathbf{k})} u_3^{\alpha_3(\mathbf{k})} \cdots u_{n+1}^{\alpha_{n+1}(\mathbf{k})}$. So $T_\lambda \cap W$ is the zero level set of the Laurent polynomial Q_α. Using results of [5,6] we obtain:

$$\chi(T_\lambda \cap W) = (-1)^{n-1} n!\, V_n(\Delta(Q_\alpha)),$$

where $\Delta(\cdot)$ denotes the Newton polyhedron. Since the polyhedra $\Delta(Q_\alpha)$ and $\Gamma_\alpha = \Gamma_\alpha^{\{0,1,\dots,n\}}(F)$ are isomorphic as subsets of integer lattices, their volumes are equal: $V_n(\Delta(Q_\alpha)) = V_n(\Gamma_\alpha)$. In a neighbourhood of a point $x \in T_\lambda \cap W$ one has $\Sigma = a\, u_1^{\alpha(\partial/\partial k_0)}$, where $a(x) \neq 0$. Therefore $\zeta_{\Sigma|_{W \setminus Z},\, x}(t) = 1 - t^{\alpha(\partial/\partial k_0)}$. Thus one has:

$$\int_{T_\lambda \cap W} \zeta_{\Sigma|_{W \setminus Z},\, x}(t)^{d\chi} = (1 - t^{\alpha(\frac{\partial}{\partial k_0})})^{\chi(T_\lambda \cap W)} = (1 - t^{\alpha(\frac{\partial}{\partial k_0})})^{(-1)^{n-1} n!\, V_n(\Gamma_\alpha)}. \quad (5)$$

Multiplying (5) for all strata $T_\lambda \subset D$ of dimension n we get (1).

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