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On the Existence of Weak Solutions for a Semilinear Singular Hyperbolic System

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ABSTRACT. In this paper we prove the existence of a weak solution for the semilinear singular real hyperbolic system

$$\begin{cases} \dfrac{\partial u}{\partial t}+\dfrac{\partial u}{\partial r}+\dfrac{u-v}{r}+k(u^2+v^2)u=0 \\ & ,r\in \mathfrak{R}_+, t\in \mathfrak{R}, \\ \dfrac{\partial v}{\partial t}-\dfrac{\partial v}{\partial r}+\dfrac{u-v}{r}+k(u^2+v^2)v=0 \end{cases}$$

where $k(r)$ is a smooth, bounded and positive function of the type $r^n, n\geq 3$, in a neighbourhood of zero. The initial data (u_0, v_0) belong to $(H^2(\mathfrak{R}_+))^2$ and verify

$$u_0(0)=v_0(0), \frac{\partial u_0}{\partial r}(0)=-\frac{\partial v_0}{\partial r}(0), (ru_0, rv_0)\in (L^2(\mathfrak{R}_+))^2.$$

1. INTRODUCTION

Let us consider the semilinear singular hyperbolic system

$$\begin{cases} \dfrac{\partial u}{\partial t}+\dfrac{\partial u}{\partial r}+\dfrac{u-v}{r}+k(u^2+v^2)u=0 \\ \\ \dfrac{\partial v}{\partial t}-\dfrac{\partial v}{\partial r}+\dfrac{u-v}{r}+k(u^2+v^2)v=0 \end{cases} \tag{1.1}$$

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in the domain $D=\{(r,t)|r\in\mathfrak{R}_+,\ t\in\mathfrak{R}\}$, where $(u,v):D^2\to\mathfrak{R}^2$, $k:\bar{\mathfrak{R}}_+\to\bar{\mathfrak{R}}_+$, $k\in W^{1,\infty}(\mathfrak{R}_+)$ and $k(r)\leq Mr^3$, $k'(r)\leq Mr^2$, $r\in[0,r_1]$ for a certain $M>0$ and $r_1>0$.

The linear part of system (1.1) is associated (for complex u and v) to a simplified model for the linear Dirac system (cf. [2] and [3]).

In order to study the Cauchy problem for the system (1.1) we regularise this system as follows:

$$\begin{cases} \dfrac{\partial u}{\partial t}+\dfrac{\partial u}{\partial r}+\dfrac{1}{r+\delta}(u-v)+k(u^2+v^2)u=0 \\ \\ \dfrac{\partial v}{\partial t}-\dfrac{\partial v}{\partial r}+\dfrac{1}{r+\delta}(u-v)+k(u^2+v^2)v=0 \end{cases} \tag{1.2}$$

where $0<\delta<1$. Given the initial data $\binom{u_0}{v_0}$, with a suitable smoothness, we first study the Cauchy problem for (1.2) under the boundary condition

$$u(0,t)-v(0,t)=0,\ t\in\mathfrak{R}. \tag{1.3}$$

Then we obtain some estimates on the solution, independent of δ, and we pass to the limit, when $\delta\to 0$, in order to obtain a weak solution for the Cauchy problem for (1.1). More precisely, we prove the following theorem (where $H^1_{0,\ell}(]0,R[)=\{u\in H^1(]0,R[)|u(0)=0\}$):

Theorem 1.1: Let $u_0,\ v_0\in H^2(\mathfrak{R}_+)$ be such that

$$u_0(0)=v_0(0),\frac{\partial u_0}{\partial r}(0)=-\frac{\partial v_0}{\partial r}(0) \text{ and } ru_0,\ rv_0\in L^2(\mathfrak{R}_+).$$

Then, there exists (u,v) such that $ru, rv\in L^\infty(\mathfrak{R},\ L^2(\mathfrak{R}_+))$, $r^2u,\ r^2v\in C([-T,T];L^2(]0,R[))\cap L^2(-T,T;\quad H^1_{0,\ell}\quad(]0.R[))$, $\dfrac{\partial u}{\partial v},\dfrac{\partial u}{\partial t}\in L^2(-T,T;$ $L^2(]0,R[))$, for each $R>0$ and $T>0$, $u(r,0)=u_0(r)$, $v(r,0)=v_0(r)$, $r\in\mathfrak{R}_+$, and (u,v) verifies (1.1) in $(\mathcal{D}'(\mathfrak{R}_+\times\mathfrak{R}))^2$.

Our previous papers [2] and [3] are concerned with nonlocal nonlinear complex perturbations (nonlinear Dirac system) of the principal part of the system (1.1). In this paper we deal with a local nonlinear perturbation which is, as far as we know, the only one that can be analysed by this method.

2. ESTIMATES FOR THE REGULARISED PROBLEM

As in [2] and [3], we consider the skew-adjoint operator in $(L^2(\mathfrak{R}_+))^2$ defined by

$$D(A)=\left\{\binom{u}{v}\in(H^1(\mathfrak{R}_+))^2 \,|\, u-v\in H^1_0(\mathfrak{R}_+)\right\},$$

$$A\binom{u}{v}=\begin{pmatrix}-\dfrac{\partial u}{\partial r}\\[2ex] \dfrac{\partial v}{\partial r}\end{pmatrix}.$$

We put $S(t)=e^{At}$, $t\in\mathfrak{R}$, and $S(t)$ is defined by

$$S(t)\binom{u}{v}=\binom{u(r-t)}{v(r+t)} \text{ with}\begin{cases} u(-r)=v(r)\\ v(-r)=u(r)\end{cases}, r>0$$

We have, in $D(A)$ (with $L^p=(L^p(\mathfrak{R}_+))^2$, $H^m=(H^m(\mathfrak{R}_+))^2$),

$$\|S(t)\binom{u}{v}\|_{L^\infty}=\|\binom{u}{v}\|_{L^\infty},\ \|S(t)\binom{u}{v}\|_{H^1}=\|\binom{u}{v}\|_{H^1},\ t\in\mathfrak{R}.$$

Now, let be $F: D(A)\to D(A)$, $D(A)$ with the H^1 norm, defined by

$$F\binom{u}{v}=-\frac{1}{r+\delta}\begin{bmatrix}1 & -1\\ 1 & -1\end{bmatrix}\binom{u}{v}-k(u^2+v^2)\binom{u}{v}.$$

This map is locally Lipschitz continuous. Hence, for $\binom{u_0}{v_0}\in D(A)$ there exists $T>0$ and an unique $\binom{u}{v}\in C([-T,T];D(A))\cap C^1([-T,T];L^2)$ such that

$$\binom{u}{v}=S(t)\binom{u_0}{v_0}+\int_0^t S(t-\tau)\,F\left[\binom{u}{v}(\tau)\right]d\tau \tag{2.1}$$

We have $\|F\binom{u}{v}\|_{H^1}\leq c(\delta)(1+\|\binom{u}{v}\|^2_{L^\infty})\|\binom{u}{v}\|_{H^1}$, and, for $t\in[-T,T]$,

$$\begin{cases}\dfrac{\partial}{\partial t}u^2+\dfrac{\partial}{\partial r}u^2+\dfrac{2}{r+\delta}(u-v)\,u+2k(u^2+v^2)\,u^2=0\\[2ex] \dfrac{\partial}{\partial t}v^2-\dfrac{\partial}{\partial r}v^2+\dfrac{2}{r+\delta}(u-v)\,v+2k(u^2+v^2)\,v^2=0\end{cases}$$

Hence, since $\binom{u^2}{v^2}\in C([-T,T];D(A))\cap C^1([-T,T];L^2)$ and $k\geq 0$,

$$\binom{u^2}{v^2}(t)\leq S(t)\binom{u^2_0}{v^2_0}-\int_0^t S(t-\tau)\Big[\frac{2}{r+\delta}\binom{(u-v)\,u}{(u-v)\,v}(\tau)\Big]\,d\tau,\ t\in[0,T],$$

$$\Big\|\binom{u^2}{v^2}(t)\Big\|_{L^\infty}\leq c\Big\|\binom{u^2_0}{v^2_0}\Big\|_{L^\infty}+\frac{c}{\delta}\int_0^t\Big\|\binom{u^2}{v^2}(\tau)\Big\|_{L^\infty}d\tau,\ t\in[0,T],$$

$$\Big\|\binom{u^2}{v^2}(t)\Big\|_{L^\infty}\leq c\,e^{(c/\delta)t},\ t\in[0,T] \tag{2.2}$$

From (2.1) and (2.2) we easily obtain an estimate for $\|\binom{u}{v}(t)\|_{H^1}$ for $t\in[0,T]$ (and also for $t\in[-T,0]$) and we conclude that $\binom{u}{v}$ is a global solution, that is

$\binom{u}{v}\in C(\mathfrak{R};\ D(A))\cap C^1(\mathfrak{R};L^2)$ and verifies (2.1) for $t\in\mathfrak{R}$.

Furthermore, since $F\colon D(A)\to D(A)$ is locally Lipschitz continuous and $D(A)$ is a Hilbert space (for the H^1 norm), we get (cf. [1]) $\binom{u}{v}\in C^1(\mathfrak{R};D(A))$ if

$$\binom{u_0}{v_0}\in D(A^2)=\Big\{\binom{u}{v}\in H^2\,|\,u-v\in H^1_0(\mathfrak{R}_+),\ \frac{\partial u}{\partial r}+\frac{\partial v}{\partial r}\in H^1_0(\mathfrak{R}_+)\Big\}.$$

Hence, by (1.2), we obtain, in this case $\binom{u}{v}\in C(\mathfrak{R};D(A^2))$.

We need suplementary estimates for $\binom{u}{v}$. For this purpose we assume that $\binom{ru_0}{rv_0}\in L^2$, that is

$$\binom{u_0}{v_0}\in\big(L^2_r(\mathfrak{R}_+)\big)^2,\ \text{where } L^2_r(\mathfrak{R}_+)=L^2(\mathfrak{R}_+,\ r^2\,dr).$$

We easily deduce from (1.2), since $k\geq 0$,

$$\begin{cases}\dfrac{\partial}{\partial t}(u^2+v^2)+\dfrac{\partial}{\partial r}(u^2-v^2)+\dfrac{2}{r+\delta}(u^2-v^2)\leq 0,\\[2ex] \dfrac{\partial}{\partial t}((r+\delta)^2(u^2+v^2))+\dfrac{\partial}{\partial r}((r+\delta)^2(u^2-v^2))\leq 0.\end{cases}$$

By the integral of energy method we get, for $R>0$, $0<t<R$,

$$\int_0^{-t+R}(r+\delta)^2(u^2+v^2)(r,t)\,dr\leq\int_0^R(r+\delta)^2(u_0^2+v_0^2)\,dr\leq\int_{\mathfrak{R}_+}(r+\delta)^2(u_0^2+v_0^2)\,dr.$$

Hence $(r+\delta)(u,v)\in L^\infty(\mathfrak{R},L^2)$ and

$$\| (r+\delta) \binom{u}{v} \|_{L^\infty(\mathfrak{R};L^2)} \leq c_1 \left(\| \binom{u_0}{v_0} \|_{L^2_r} + \| \binom{u_0}{v_0} \|_{L^2} \right) \tag{2.3}$$

where c_1 does not depend on δ.

Now, let us consider the linear system in D:

$$\begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{\partial u}{\partial r} + \dfrac{1}{r}(u-v) = 0 \\ \\ \dfrac{\partial v}{\partial t} - \dfrac{\partial v}{\partial r} + \dfrac{1}{r}(u-v) = 0 \end{cases}$$

L. Tartar has pointed out to us that if we put

$$\begin{cases} w = r u_r + \dfrac{3}{2} u - \dfrac{1}{2} v \\ \\ w_1 = r v_r + \dfrac{3}{2} v - \dfrac{1}{2} u \end{cases} \text{, where } u_r = \frac{\partial u}{\partial r},$$

we obtain (formally):

$$\begin{cases} \dfrac{\partial w}{\partial t} + \dfrac{\partial w}{\partial r} = 0 \\ \\ \dfrac{\partial w_1}{\partial t} - \dfrac{\partial w_1}{\partial r} = 0 \end{cases}.$$

Let us assume $(u_0, v_0) \in D(A^2) \cap L^2_r$ and let $(u, v) \in C(\mathfrak{R}; D(A^2)) \cap C^1(\mathfrak{R}; D(A)) \cap L^\infty(\mathfrak{R}; L^2_r)$ be the solution of (1.2) for a fixed $\delta \in]0,1[$ and with initial data (u_0, v_0) (and boundary condition (1.3)). Let

$$\begin{cases} w = (r+\delta) u_r + \dfrac{3}{2} u - \dfrac{1}{2} v \\ \\ w_1 = (r+\delta) v_r + \dfrac{3}{2} v - \dfrac{1}{2} u \end{cases} \tag{2.4}$$

We deduce, from (1.2), with $\theta = k(u^2+v^2)$,

$$\begin{cases} \dfrac{\partial w}{\partial t} + \dfrac{\partial w}{\partial r} = -\theta w - (r+\delta)\theta_{,r} u \\ \\ \dfrac{\partial w_1}{\partial t} - \dfrac{\partial w_1}{\partial r} = -\theta w_1 - (r+\delta)\theta_{,r} v \end{cases} \tag{2.5}$$

and $(w, w_1) = C(\mathfrak{R}; (H^1(]0, R[))^2) \cap C^1(\mathfrak{R}; (L^2(]0, R[))^2)$, for each $R \in]0, +\infty[$.

Furthermore, we have

$$\begin{cases} \dfrac{\partial}{\partial r}[(r+\delta)(u+v)] = w + w_1 \\ \\ (u-v) + \dfrac{\partial}{\partial r}[(r+\delta)(u-v)] = w - w_1 \end{cases} \tag{2.6}$$

and

$$\begin{cases} \dfrac{\partial}{\partial r}[(r+\delta)^2 u] = (r+\delta) w + \dfrac{1}{2}(r+\delta)(u+v) \\ \\ \dfrac{\partial}{\partial r}[(r+\delta)^2 v] = (r+\delta) w_1 + \dfrac{1}{2}(r+\delta)(u+v) \end{cases} \tag{2.7}$$

By applying the Gagliardo-Nirenberg inequalities and the Sobolev theorem to $(r+\delta)(u+v) \in H^1(]0, R[)$ and $(r+\delta)^2 (u-v)^2 \in W^{1,1}]0, R[)$, respectively, we can deduce from (2.6), (1.3) and (2.3) (cf.[3], §3):

$$\|(r+\delta)(|u|+|v|)\|_{L^\infty(]0,R[)} \leq c(R)\left[1 + \|w\|^{\frac{1}{2}}_{L^2(]0,R[)} + \|w_1\|^{\frac{1}{2}}_{L^2(]0,R[)}\right] \tag{2.8}$$

where $c(R)$ does not depend on δ.

From (2.5) we deduce

$$\frac{\partial}{\partial t}(|w|^2 + |w_1|^2) + \frac{\partial}{\partial r}(|w|^2 - |w_1|^2) =$$
$$-2\theta(w^2 + w_1^2) - 2(r+\delta)\theta_r(uw + vw_1) \tag{2.9}$$

and $\theta_r = k'(u^2+v^2)+2k(u\,u_r+v\,v_r) = k'(u^2+v^2)+2\,\dfrac{k}{r+\delta}[uw+vw_1 -$

$-\dfrac{3}{2}(u^2+v^2)+uv]$.

Hence,

$$-2(r+\delta)\,\theta_r(uw+vw_1) = -2(r+\delta)\left(\frac{k'}{r^2}\right)r^2(u^2+v^2)\,(uw+vw_1) -$$

$$-4k\,(uw+vw_1)^2 - 4\,\frac{k}{r^3}\,r^3\left[-\frac{3}{2}(u^2+v^2)+uv\right](uw+vw_1).$$

Then we deduce, from (2.9) and from the properties of k,

$$\frac{\partial}{\partial t}(|w|^2+|w_1|^2)+\frac{\partial}{\partial r}(|w|^2-|w_1|^2)\leq c\,(r+\delta)^3(|u|^3+|v|^3)\,(|w|+|w_1|) \tag{2.10}$$

where c does not depend on δ.

Furthermore, we have, by (2.3) and (2.8),

$$\int_0^R (r+\delta)^3\,(|u|^3+|v|^3)\,(|w|+|w_1|)\,(r,t)\,dr \leq c\,(R)\Big[1+\|w\|^2_{L^2(]0,R[)}+ \\ +\|w_1\|^2_{L^2(]0,R[)}\Big] \tag{2.11}$$

where $c\,(R)$ does not depend on δ.

Now, if we take $T>0$ we easily obtain, by applying the integral of energy method to (2.10), and since $|w_1|^2(0,\tau)-|w|^2(0,\tau)=2\delta\left(\frac{\partial}{\partial t}|u|^2\right)(0,\tau)$ (cf. [2], §3),

$$\int_0^{-t+2T}(|w|^2+|w_1|^2)\,(r,t)\,dr \leq c\,(T)\,, \text{for all } t\in[0,T].$$

Hence, if $\Omega_T=\{(r,t)\,|\,r\in[0,-t+2T[,\,t\in]0,T[\}$,
we deduce

$$\int_{\Omega_T}(|w|^2+|w_1|^2)\,(r,t)\,dr\,dt \leq c\,(T) \tag{2.12}$$

where $c(T)$ does not depend on δ. A similar estimate holds for $T<0$. Now, for every $T>0$ and $R>0$, we deduce from (2.12), (2.7) and (2.3),

$$\begin{cases} \| (r+\delta)^2 u \|_{L^2(-T,T;H^1(]0,R[))} \leq c(R,T) \\ \| (r+\delta)^2 v \|_{L^2(-T,T;H^1(]0,R[))} \leq c(R,T) \end{cases} \tag{2.13}$$

Hence, by (2.3), (2.13) and (1.2), we obtain

$$\begin{cases} \| (r+\delta)^2 \dfrac{\partial u}{\partial t} \|_{L^2(-T,T;L^2(]0,R[))} \leq c(R,T) \\ \| (r+\delta)^2 \dfrac{\partial v}{\partial t} \|_{L^2(-T,T;L^2(]0,R[))} \leq c(R,T) \end{cases} \tag{2.14}$$

where $c(R,T)$ does not depend on δ.

Now, let

$$W(R,T) = \{ u \in L^2(-T,T;H^1_{0/}(]0,R[)) \mid \frac{\partial u}{\partial t} \in L^2(-T,T;L^2(]0,R[)) \}$$

with its natural norm, where $H^1_{0/}(]0,R[) = \{ u \in H^1(]0,R[) \mid u(0)=0 \}$. We have (cf. [4]).

$$W(R,T) \hookrightarrow C([-T,T];L^2(]0,R[)) \tag{2.15}$$

and, by Aubin's compactness theorem (cf. [4])

$$W(R,T) \hookrightarrow L^2(-T,T;L^2(]0,R[)), \text{ with compact injection} \tag{2.16}$$

Furthermore the map $u \to u(0)$ from $W(R,T)$ into $L^2(]0,R[)$ is continuous by (2.15).

Let, for each $\delta \in]0,1[$, be (u_δ, v_δ) the corresponding solution of (1.2) for the initial data (u_0, v_0) (and boundary condition (1.3)). We have, by (2.3). (2.13) and (2.14),

$$\begin{cases} \| r^2 u_\delta \|_{W(R,T)} \leq c(R,T) \\ \| r^2 v_\delta \|_{W(R,T)} \leq c(R,T) \end{cases} \tag{2.17}$$

where $c(R,T)$ does not depend on δ.

3. EXISTENCE OF A WEAK SOLUTION

Let us assume $(u_0, v_0) \in D(A^2) \cap L^2_r$. With the same notation of the last part of §2 for the couple (u_δ, v_δ), solution of the regularised problem, there exists, by (2.3), a sequence $\delta \to 0$ and $(u, v) \in L^\infty(\mathfrak{R}, L^2_r)$ such that

$$(u_\delta, v_\delta) \underset{\delta \to 0}{\to} (u, v) \text{ in } L^\infty(\mathfrak{R}, L^2_r) \text{ weak} *$$

and

$$\|(u, v)\|_{L^\infty(\mathfrak{R}, L^2_r)} \leq c(\|(u_0, v_0)\|_{L^2_r} + \|(u_0, v_0)\|_{L_2}).$$

By (2.17), (2.16) and (2.15) there exists, for each (R, T), a sub-sequence (u_δ, v_δ), $\delta \to 0$, such that

$(r^2 u_\delta, \; r^2 v_\delta) \to (r^2 u, r^2 v)$ weakly in $(W(R, T))^2$, strongly in $(L^2(-T, T; L^2(]0, R[)))^2$ and a.e. in $]0, R[\times]-T, T[$, and $(u, v)(0) = (u_0, v_0)$ a.e. in $\mathfrak{R}_+$. Furthermore, for each $T > 0$, and by a diagonalisation method, we can assume (by (2.16) and (2.17)) that

$$(u_\delta, v_\delta) \underset{\delta \to 0}{\to} (u, v) \text{ a.e. in } \mathfrak{R}_+ \times]-T, T[.$$

In particular, for fixed $T > 0$, we have, for each $R > 0$,

$$k(u_\delta^2 + v_\delta^2)\, u_\delta \underset{\delta \to 0}{\to} k(u^2 + v^2)\, u, \text{ a.e. in }]0, R[\times]-T, T[.$$

Otherwise, by (2.12) and (2.8), we have

$$\|k((u_\delta^2 + v_\delta^2)\, u_\delta\|_{L^2(]0, R[\times]-T, T[)} \leq c\|r^3((u_\delta^2 + u_\delta^2)\, u_\delta\|_{L^2(]0, R[\times]-T, T[)} \leq c(R, T),$$

where $c(R, T)$ does not depend on δ.

Hence, by lemma 1.3 in chap. 1 of [4],

$$k(u_\delta^2 + v_\delta^2)\, u_\delta \underset{\delta \to 0}{\to} k(u^2 + v^2)\, u, \text{ weakly in } L^2(]0, R[\times]-T, T[),$$

and similar conclusion for $k(u_\delta^2 + v_\delta^2)\, v_\delta$.

Now, take $\phi \in \mathscr{D}(D(R, T))$, $D(R, T) = \{(r, t) | r \in]0, R[, t \in]-T, T[\}$. we have, by (1.2),

$$\int_{-T}^{T}\int_{0}^{R}\left[\frac{\partial u_\delta}{\partial t}\,\phi+\frac{\partial u_\delta}{\partial r}\,\phi+\frac{u_\delta-v_\delta}{r+\delta}\,\phi+k\,(u_\delta^2+v_\delta^2)\,u_\delta\,\phi\right]dr\,dt=0$$

By the previous considerations we can pass to the limit, when $\delta\to 0$ (subsequence) and we obtain, since R and T are arbitrary,

$$\frac{\partial u}{\partial t}+\frac{\partial u}{\partial r}+\frac{u-v}{r}+k\,(u^2+v^2)\,u=0 \text{ in } \mathcal{D}'(\mathrm{D}).$$

The same technique applies to the second equation, and so theorem 1.1 is proved.

References

[1] T. CAZENAVE and F. WEISSLER. *The Cauchy problem for the nonlinear Schrödinger equation in H^1*. Manuscripta Math., **61** (1988), 477-494.

[2] J. P. DIAS and M. FIGUEIRA. *On the existence of weak solutions for a nonlinear time dependent Dirac equation*. Proc. Royal Soc. Edinburgh, **113** A (1989), 149-158.

[3] J. P. DIAS and M. FIGUEIRA. *Solutions faibles du problème de Cauchy pour certaines équations de Dirac non linéaires*. Portug. Math., **46** (1989), 475-484.

[4] J. L. LIONS. *Quelques méthodes de résolution des problèmes aux limites non linéaires*. Paris, Dunod, 1969.

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