REVISTA MATEMATICA de la
Universidad Complutense de Madrid
Volumen 4, números 2 y 3; 1991.

On the Homological Category of 3-Manifolds

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ABSTRACT. Let M be a closed, connected, orientable 3-manifold. Denote by $n(S^1\times S^2)$ the connected sum of n copies of $S^1\times S^2$. We prove that if the homological category of M is three then for some $n\geq 1$, $H^*(M)$ is isomorphic (as a ring) to $H^*(n(S^1\times S^2))$.

Let X be a topological space. The (complete) *homological category of X* [1] is the smallest cardinal n such that there is a family $\mathscr{F}$ of open sets of X with the following properties:

i) $\cup_{U\in\mathscr{F}} U=X$.

ii) The cardinality of $\mathscr{F}$ is n.

iii) For every $U\in\mathscr{F}$, $i>0$ and every ring of coefficients R, the inclusion induced homomorphism $H_i(U;R)\to H_i(X;R)$, in singular homology is zero.

We denote by cath X the homological category of X. Let M be a closed, connected, orientable 3-manifold. From the existence of a Heegaard splitting of M, it is clear that cath $M\leq 4$. Also, cath $M=2$ if only if M is a homology 3-sphere [1]. Therefore, it suffices to characterize 3-manifolds M with cath $M=3$. Denote by $n(S^1\times S^2)$ the connected sum of n copies of $S^1\times S^2$. We will prove in Theorem 1 that if cath $M=3$ then for some $n\geq 1$, $H^*(M)$ is isomorphic (as a ring) to $H^*(n(S^1\times S^2))$. We work in the PL-category.

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* Supported by the Heinrich Hertz Stiftung

1991 Mathematics Subject Classification: 57N10, 55M30.
Editorial de la Universidad Complutense. Madrid, 1991.

I. PRELIMINARIES

Let A be a finitely generated abelian group A. From now on, we will denote by tor A (resp. rank A) the torsion subgroup of A (resp. the number of summands Z in $A/\text{tor } A$).

The following lemma is used in the proof of Theorem 1. We feel it has independent interest; it can be used, for example, to calculate the homology of the complement of a 1-complex K in a closed orientable 3-manifold in terms of the cokernel of $H_1(K) \to H_1(M)$.

Lemma 1. *Let K be a compact proper 3-submanifold of the closed, connected and orientable 3-manifold M. Denote by b_i (resp. b'_i) the i-th Betti number of K (resp. M-K). Then*

$$H_1(M\text{-}K) \oplus Z^{b_2} \approx Ç \oplus Z^{b_1+b'_0-1}$$

where $Ç$ is the cokernel of the inclusion induced homomorphism $H_1(K) \to H_1(M)$. In particular, tor $H_1(M\text{-}K) \approx$ tor $Ç$.

Remark. The conclusion can also be stated as

$$H_1(M\text{-}K) \approx Ç \oplus Z^{b_0+b'_0-1-1/2[\chi(\partial K)]}$$

where, in case $m<0$, $Ç \oplus Z^m$ is interpreted as the unique group A such that $A \oplus Z^{-m} \approx Ç$.

Proof. We have tor $(H_1(M\text{-}K) \oplus Z^{b_2}) \approx$ tor $H_1(M\text{-}K) \approx$ tor $H^2(M, K) \approx$ $\approx$ tor $H_1(M, K) \approx$ tor $Ç \approx$ tor $(Ç \oplus Z^{b_1+b'_0-1})$. The penultimate isomorphism being a consequence of the exact sequence

$$H_1(K) \to H_1(M) \to H_1(M, K) \to H_0(K)$$

where $H_0(K)$ is free abelian.

Also $b'_1 = \text{rank } H_1(M\text{-}K) = \text{rank } H^2(M, K) = \text{rank } H_2(M, K)$, rank $H_2(M) = \text{rank } H_1(M)$, rank $H_3(M, K) = b'_0$ and the alternating sum of the ranks of the groups in the exact sequence

$$0 \to H_3(M) \to H_3(M, K) \to H_2(K) \to H_2(M) \to H_2(M, K) \to H_1(K) \to$$
$$\to H_1(M) \to Ç \to 0$$

is zero. This yields $1-b_0'+b_2+b_1'-b_1-\text{rank } Ç=0$ and, therefore, rank $(H_1(M\text{-}K)\oplus Z^{b_2})=b_1'+b_2=\text{rank } Ç+b_1+b_0'-1=$

$$=\text{rank } (Ç\oplus Z^{b_1+b_0'-1}).$$

Hence $H_1(M\text{-}K)\oplus Z^{b_2}\approx Ç\oplus Z^{b_1+b_0'-1}$. This completes the proof.

The following lemma is known ([4], p. 173, Corollarie V.8).

Lemma 2. *Let W be a compact, orientable 3-manifold with $H_1(\partial W)\approx \mathbb{Z}^{2g}$. Then, the image of the inclusion induced homomorphism $H_1(\partial W)\rightarrow H_1(W)$ has rank g.*

Let M be a closed n-manifold. We follow [3] for the definition and properties of an n-filling of M.

Lemma 3. *Let $\{T_i\}_{i=1}^3$ be a 3-filling of the closed, connected and orientable 3-manifold M. Let W be a regular neighbourhood of $F=\cup_{i=1}^3 \partial T_i$. Then, the image of $H_1(\partial W)\rightarrow H_1(W)$ contains tor $H_1(W)$.*

Proof. Let $F_k=T_i\cap T_j$, where $\{i,j,k\}=\{1,2,3\}$. Let C_i be a product neighbourhood of ∂T_i in T_i. We may assume $W=\cup_{i=1}^3 C_i$. Write $\partial_i W=C_i\cap\partial W$. We have the commutative diagram

$$\begin{array}{ccccc}
H_1(\partial T_1) & \rightarrow & H_1(F) & \rightarrow & H_1(F,\partial T_1)\\
\downarrow\approx & & & & \\
H_1(C_1) & \searrow & \Big\downarrow\approx & & \\
\uparrow\approx & & & & \\
H_1(\partial_1 W) & \rightarrow & H_1(W) & &
\end{array}$$

where the upper row is exact and the vertical arrows are isomorphism. Since $H_1(F,\partial T_1)\approx H_1(F_1,\partial F_1)\approx H^1(F_1)$ is free, the image of $H_1(\partial T_1)\rightarrow H_1(F)$ contains tor $H_1(F)$ and, therefore, the image of $H_1(\partial_1 W)\rightarrow H_1(W)$ contains tor $H_1(W)$ from which the result follows.

II. MAIN RESULT

Theorem 1. *Let M be a closed, connected, orientable 3-manifold. If cath $M=3$ then for some $n\geq 1$, $H^*(M)$ is isomorphic (as a ring) to $H^*(n(S^1\times S^2))$.*

Proof. By the arguments of [3] (see also [2]), there exists a 3-filling $\{T_i\}_{i=1}^3$ of M, where T_i is a cube with handles and $H_1(T_i) \to H_1(M)$ is trivial for $i=1,2,3$. Let W be a regular neighbourhood of $\cup_{i=1}^3 \partial T_i$ in M and let K be a closure of $M-W$. Thus K is a disjoint union of three cubes wiith handles $\{K_i\}_{i=1}^3$. Let g_i be the genus of ∂K_i and $g=g_1+g_2+g_3$. Consider the commutative diagram with exact rows.

$$\begin{array}{ccccc} H_2(M, W) & \xrightarrow{j} & H_1(W) & \to & H_1(M) \\ \uparrow \approx & & \uparrow i & & \uparrow 0 \\ H_2(K, \partial W) & \to & H_1(\partial W) & \to & H_1 K \end{array}$$

From the fact that the right vertical homomorphism is trivial we obtain $\operatorname{Im} i \subset \operatorname{Im} j$, and, since the left vertical homomorphism is onto, $\operatorname{Im} j \subset \operatorname{Im} i$. Hence $\operatorname{Im} j = \operatorname{Im} i$, which by lemmas 2 and 3, is isomorphic to $Z^g \oplus \operatorname{tor} H_1(W)$. Since $H_2(M, W) \approx H^1(K) \approx Z^g$, $\operatorname{Im} j$ can be generated by g elements and, therefore, we must have $\operatorname{tor} H_1(W)=0$. By lemma 1, it follows that $\operatorname{tor} H_1 M \approx \operatorname{tor} H_1(W)=0$ so that $H_1 M$ is free abelian. The rank n of $H_1 M$ must be positive since, otherwise, cath M would be two. Hence $H^i(M) \approx Z^n$ for $i=1,2$, that is the cohomology of M is additively the same as that of $n(S^1 \times S^2)$.

Let $\{a_1, \ldots, a_n\}$ be a basis of $H^1(M)$ and let $\{b_1, \ldots, b_n\} \subset H^2(M)$ be the dual basis; that is $a_i \smile b_j = \delta_{ij} \nu$ where $\nu \in H^3(M)$ is the fundamental class. For $r=1,2,3$, $H^1(M) \to H^1(T_r)$ is trivial so that a_i is the image of an element $a_i^{(r)} \in H^1(M, T_r)$ under $H^1(M, T_r) \to H^1(M)$. Then, for any i, j, k, $a_i \smile a_j \smile a_k$ is zero since it is the image, in $H^3(M)$, of $a_i^{(1)} \smile a_j^{(2)} \smile a_k^{(3)} \in H^3(M, T_1 \cup T_2 \cup T_3) = 0$.

Moreover, for any i, j, $a_i \smile a_j = 0$ because, if we write $a_i \smile a_j = \sum_{k=1}^n n_k b_k$, then $0 = a_k \smile a_i \smile a_j = n_k \nu$ so that $n_k = 0$ for $k=1, \ldots, n$. This proves that the cohomology ring of M is isomorphic to that of $n(S^1 \times S^2)$. This completes the proof of the theorem.

Aknowledgement. The first author wishes to thank the Ruhr-Universität Bochum for support and hopitality during the preparation of this report.

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Recibido: 3 de septiembre de 1990