REVISTA MATEMATICA de la
Universidad Complutense de Madrid
Volumen 4, números 2 y 3; 1991.

*Tritangent Planes to Toroidal Knots**

A. MONTESINOS AMILIBIA

ABSTRACT. A proof is given that, with the only exception of (3,2), all toroidal knots in $\mathbb{R}^3$ obtained in the standard way by stereographic projection of knots in S^3 have tritangent planes.

Montesinos and Nuño [2] and Morton [3] have proved that two different presentations of the trefoil knot as a toroidal knot of type (3, 2) have no tritangent planes, answering thus in the negative a conjecture of Freedman [1].

Let p, q be relative prime integers and let a, $b>0$ be such that $a^2+b^2=1$; in the following, we shall put $a=\cos A$, $b=\sin A$, with $0<A<\frac{\pi}{2}$. We have a toroidal knot in $S^3\subset\mathbb{R}^4$ given by the equations ($a\cos pt$, $a\sin pt$, $b\cos qt$, $b\sin qt$), $0\leq t\leq 2\pi$. Applying to it the stereographic projection given by $(x, y, z, w)\to\frac{1}{1-w}(x, y, z)$, we get the (p, q)—knot

$$(1)\qquad N_{p,q}(t)=\frac{1}{1-b\sin qt}(a\cos pt,\ a\sin pt,\ b\cos qt),\quad 0\leq t\leq 2\pi,$$

on the torus in $\mathbb{R}^3$ of equation

$$\left(\frac{1}{a}-\sqrt{x^2+y^2}\right)^2+z^2=\frac{b^2}{a^2}.$$

* Work partially supported by the CICYT grant no. PS87-0115-C03-01.

1991 Mathematics Subject Classification, 53A04, 57N75.
Editorial de la Universidad Complutense. Madrid, 1991.

We shall prove that all such knots, with the unique exception of the (3, 2) one, have tritangent planes.

As in [3], let $r_1 x + r_2 y + r_3 z = r$ be the equation of a plane P in $\mathfrak{R}^3$. The points of $N_{p,q}$ that belong to P correspond to the values of t that satisfy

$$r_1 a \cos pt + r_2 a \sin pt + r_3 b \cos qt + rb \sin qt = r. \tag{2}$$

Also, P is tangent to $N_{p,q}$ at $N_{p,q}(t)$ iff t satisfies (2) and its derivative with respect to t, that is

$$-pr_1 a \sin pt + pr_2 a \cos pt - qr_3 b \sin qt + qrb \cos qt = 0. \tag{3}$$

We put $z = e^{it}$ in (2) and (3) and get the equations

$$uz^p + \bar{u}\bar{z}^p + vz^q + \bar{v}\bar{z}^q - 2r = 0, \tag{4}$$

$$p(uz^p - \bar{u}\bar{z}^p) + q(vz^q - \bar{v}\bar{z}^q) = 0, \tag{5}$$

where we have put

$$u = a(r_1 - ir_2), \quad v = b(r_3 - ir). \tag{6}$$

Thus, the number of points of tangency of P with $N_{p,q}$ is the number of values of z in the unit circle of $\mathfrak{C}$ that satisfy (4) and (5).

Let $r_1 = r_2 = 0$, $r_3 = a/b$, $r = 1$. Then $u = 0$ and $v = a - ib = e^{-iA}$. With these data, P is one of the two planes that are tangent to the torus along a whole circle. Then, (5) says that vz^q must be real, and consequently, (4) is equivalent to $vz^q = r = 1$, that is $z^q = e^{iA}$. Therefore, as expected, P has q points of tangency with $N_{p,q}$, given by the roots of the last equation. So, we can henceforth assume that $q = 2$, $p \geq 5$.

Lemma. *Let $p \geq 5$ be an odd integer; then, there exist a solution, τ, of the equation*

$$p \sin t \cos \frac{p}{2} t - 2 \cos t \sin \frac{p}{2} t = 0, \tag{7}$$

such that $0 < \tau < \pi$.

Proof. Let $f(t) = p \sin t \cos \frac{p}{2} t - 2 \cos t \sin \frac{p}{2} t$. If $p \geq 7$, then

$$f\left(\frac{2\pi}{p}\right)=p\sin\frac{2\pi}{p}\cos\pi-2\cos\frac{2\pi}{p}\sin\pi=-p\sin\frac{2\pi}{p}<0$$

$$f\left(\frac{3\pi}{p}\right)=p\sin\frac{3\pi}{p}\cos\frac{3\pi}{2}-2\cos\frac{3\pi}{p}\sin\frac{3\pi}{2}=2\cos\frac{3\pi}{p}>0.$$

If $p=5$, we consider instead the interval $\left[\frac{3\pi}{5},\frac{4\pi}{5}\right]$:

$$f\left(\frac{3\pi}{5}\right)=5\sin\frac{3\pi}{5}\cos\frac{3\pi}{2}-2\cos\frac{3\pi}{5}\sin\frac{3\pi}{2}=2\cos\frac{3\pi}{5}<0$$

$$f\left(\frac{4\pi}{5}\right)=5\sin\frac{4\pi}{5}\cos 2\pi-2\cos\frac{4\pi}{5}\sin 2\pi=5\sin\frac{4\pi}{5}>0$$

In the first case, there is such a solution in $\left(\frac{2\pi}{p},\frac{3\pi}{p}\right)$; in the second, in $\left(\frac{3\pi}{5},\frac{4\pi}{5}\right)$, as claimed. ∎

In the following, τ will stand for a solution of (7) in $(0,\pi)$. We remark that from (7) it can be easily seen that $\sin 2\tau\neq 0$ and $\sin p\tau\neq 0$. Also, dividing (7) by $\cos\tau\cos\frac{p}{2}\tau$ we have

$$p\tan\tau=2\tan\frac{p}{2}\tau. \tag{8}$$

We define B by the conditions

$$\cot B=-\frac{2\sin 2\tau}{p\sin p\tau}\tan A,\quad 0<B<\pi. \tag{9}$$

We prove now that $0<\cos(A-B)/\sin B<1$. We have

$$\frac{\cos(A-B)}{\sin B}=\sin A+\cos A\cot B=\sin A\left(1-\frac{2\sin 2\tau}{p\sin p\tau}\right)$$

$$=\sin A\left(1-\frac{2\tan\tau\cos^2\tau}{p\tan\frac{p}{2}\tau\cos^2\frac{p}{2}\tau}\right)=(by\,(8))\sin A\left(1-\frac{4\cos^2\tau}{p^2\cos^2\frac{p}{2}\tau}\right)$$

$$= \sin A \left(1 - \frac{4(1+\tan^2 \frac{p}{2}\tau)}{p^2(1+\tan^2\tau)} \right) = (by\,(8)) \sin A \left(1 - \frac{4(1+\frac{p^2}{4}\tan^2\tau)}{p^2(1+\tan^2\tau)} \right)$$

$$= \sin A \left(1 - \frac{\frac{4}{p^2} + \tan^2\tau}{1+\tan^2\tau} \right).$$

Our claim follows from the fact that $\frac{4}{p^2} < 1$. We define C by

$$\sin C = \frac{\cos(A-B)}{\sin B}, \quad 0 < C < \frac{\pi}{2}. \tag{10}$$

We have now the ingredients for exhibiting a plane triply tangent to $N_{p,2}$. Let us put

$$r_1 = \cos B \cos \frac{p}{2} C, \quad r_2 = \cos B \sin \frac{p}{2} C, \quad r_3 = \sin B \cos C, \ r = \sin B \sin C$$

Then

$$uz^p = \cos A \cos B e^{ip(t - \frac{1}{2}C)}, \quad vz^2 = \sin A \sin B e^{i2(t-\frac{1}{2}C)}.$$

Hence, one half of the left hand side of (4) is

$$\cos A \cos B \cos p(t - \frac{1}{2}C) + \sin A \sin B \cos 2(t - \frac{1}{2}C) - \cos(A-B)$$

$$= \cos A \cos B (\cos p(t - \frac{1}{2}C) - 1) + \sin A \sin B(\cos 2(t - \frac{1}{2}C) - 1)$$

So, it is obvious that $t = \frac{1}{2}C$ is a solution of (4). We show that $t = \tau + \frac{1}{2}C$ so is. In this case, by substitution in the last expression we obtain

$$\cos A \cos B(\cos p\tau - 1) + \sin A \sin B (\cos 2\tau - 1)$$

$$= -2(\cos A \cos B \sin^2 \frac{p}{2}\tau + \sin A \sin B \sin^2\tau)$$

$$= -2\cos A \sin B (\cot B \sin^2 \frac{p}{2}\tau + \tan A \sin^2\tau)$$

$$= -2\sin A\sin B\left(-\frac{2\sin\tau\cos\tau}{p\sin\frac{p}{2}\tau\cos\frac{p}{2}\tau}\sin^2\frac{p}{2}\tau+\sin^2\tau\right)$$

$$= -2\sin A\sin B\frac{\sin\tau}{p\cos\frac{p}{2}\tau}\left(p\sin\tau\cos\frac{p}{2}\tau-2\cos\tau\sin\frac{p}{2}\tau\right)$$

$$=0$$

by the Lemma. Therefore, $t=\frac{1}{2}C-\tau$ is clearly another solution. Let us show that the three values satisfy (5). The left hand side of this equation is now, up to a non-zero factor,

$$p\cos A\cos B\sin p\left(t-\frac{1}{2}C\right)+2\sin A\sin B\sin 2\left(t-\frac{1}{2}C\right).$$

For $t=\frac{1}{2}C$ this zero. For $t=\tau+\frac{1}{2}C$ this is

$$p\cos A\cos B\sin p\tau+2\sin A\sin B\sin 2\tau$$

$$=p\cos A\sin B\sin p\tau\left(\cot B+\frac{2\sin 2\tau}{p\sin p\tau}\tan A\right)$$

$$=0,$$

by (9). Clearly, the same occurs for $t=\frac{1}{2}C-\tau$. Since the three values modulo 2π are different, we have proved the announced result.

Theorem. *Let $p, q\geq 2$ be relative prime integers and $(p, q)\neq(3,2)$; then, the (p, q) toroidal knot $N_{p,q}$ given by* (1) *has a tritangent plane.*

References

1. FREEDMAN, M. H.: *Planes triply tangent to curves with nonvanishing torsion*, Topology **19** (1980), 1-8.
2. MONTESINOS AMILIBIA, A. and NUÑO BALLESTEROS, J. J.: *A knot without tritangent planes*, Geometriae Dedicata **37** (1991), 141-153.
3. MORTON, H. R.: *Trefoil knots without tritangent planes*, preprint.

Departamento de Geometría y Topología.
Dr. Moliner 50,
46100 Burjasot (Valencia),
Spain
e-mail address montesin @ evalun 11 .earn

Recibido: 3 de diciembre de 1990
Revisado: 28 de febrero de 1991