

Isometries and Automorphisms of the Spaces of Spinors

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ABSTRACT. The relationships between the JB*-triple structure of a complex spin factor \mathcal{S} and the structure of the Hilbert space \mathcal{H} associated to \mathcal{S} are discussed. Every surjective linear isometry L of \mathcal{S} can be uniquely represented in the form $L(x) = \mu U(x)$ for some conjugation commuting unitary operator U on \mathcal{H} and some $\mu \in \mathbb{C}$, $|\mu| = 1$. Automorphisms of \mathcal{S} are characterized as those linear maps (continuity not assumed) that preserve minimal tripotents in \mathcal{S} and the orthogonality relations among them.

§0. INTRODUCTION

The spaces of spinors were introduced by E. Cartan in [1] to solve the problem of analytic classification of bounded symmetric domains in \mathbb{C}^n , and they also arise in the quantization of free fermionic fields [9, p. 104]. More recently, these spaces have been considered in various problems in the context of infinite dimensional holomorphy by Harris, Kaup and others. In this note, an arbitrary spinor space \mathcal{S} is considered and the relationship between the «triple structure» of \mathcal{S} and the structure of its underlying Hilbert space \mathcal{H} is discussed. In §2, we prove that any surjective linear isometry L of \mathcal{S} can be represented in the form $L = \mu U$ for some $\mu \in \mathbb{C}$, $|\mu| = 1$ and some conjugation commuting surjective linear isometry of \mathcal{H} . Since surjective linear isometries of \mathcal{S} , and conjugation commuting unitary operators on \mathcal{H} , are the same as automorphisms of the corresponding structures of \mathcal{S} and \mathcal{H} , our result can be rephrased by saying that, except for an automorphism, the spaces of spinors are uniquely determined by their underlying Hilbert space. This is a result that anyone could expect, though the authors have found no precise reference for the statement and proof. In §3, the sets $\text{Min}(\mathcal{S})$ and

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Extr(\mathcal{S}) of minimal and maximal tripotents of \mathcal{S} are discussed. In §4 we prove that any linear mapping (continuity is not assumed) that preserves minimal tripotents and the orthogonality relations among them is an automorphism of \mathcal{S} , and that any holomorphic automorphism of the unit ball of \mathcal{S} is uniquely determined by its values at the set $\text{Min}(\mathcal{S}) \cup \{0\}$.

§1. NOTATION AND PRELIMINARY RESULTS

Let H be a complex Hilbert space with $\dim(H) > 1$. We recall ([2] p. 16, [4] p. 358) that a Cartan factor of type IV, also called a spinor or a spin factor, is a norm closed selfadjoint complex subspace \mathcal{S} of $\mathcal{L}(H)$ such that $\{a^2 \mid a \in \mathcal{S}\} \subset \mathbb{C}1_H$, where $\mathcal{L}(H)$ is the C^* -algebra of bounded linear operators on H and 1_H is the identity operator. For a and b in \mathcal{S} , there is a unique complex number $(a|b)$ such that

$$(1.1) \quad ab^* + b^*a = 2(a|b)1_H$$

and the mapping $(a, b) \rightarrow (a|b)$ defines an inner product on \mathcal{S} whose associated norm, denoted by $\|\cdot\|$, is equivalent to the usual operator norm $\|\cdot\|_\infty$ induced by $\mathcal{L}(H)$ on \mathcal{S} . Let us define

$$\chi =: \{a \in \mathcal{S} \mid a = a^*\}.$$

The norms $\|\cdot\|$ and $\|\cdot\|_\infty$ coincide on χ , and we have the topological direct sum decomposition

$$\mathcal{S} = \chi \oplus i\chi$$

In particular, \mathcal{S} is a complex Hilbert space with conjugation $a \rightarrow \bar{a} =: a^*$, $a \in \mathcal{S}$, the hilbertian norm and the operator norm being related by

$$\|a\|_\infty^2 = \|a\|^2 + [\|a\|^4 - |(a|a^*)|^2]^{1/2}$$

On the other hand, \mathcal{S} is a J^* -algebra of operators, i.e., \mathcal{S} is a norm closed complex subspace of $\mathcal{L}(H)$ such that the triple product

$$(1.2) \quad \{ab^*c\} =: \frac{1}{2}(ab^*c + cb^*a)$$

is in \mathcal{S} whenever a, b , and c are in \mathcal{S} . The J^* -algebra structure and the Hilbert space structure are linked by the formula ([4] p. 358)

$$(1.3) \quad \{aa^*a\} = 2(a|a)a - (a|a^*)a^* \quad (a \in \mathcal{S})$$

An alternative introduction of the spaces of spinors is the following: Let K be a complex Hilbert space with conjugation $\bar{\cdot}$ and inner product $(\cdot|\cdot)$. Define a triple product by

$$\{x y^* z\} = (x|y)z - (z|\bar{x})\bar{y} + (z|y)x.$$

Then K with the conjugation $\bar{\cdot}$, the triple product $\{\dots\}$ and the norm $\|\cdot\|_\infty$ given by

$$\|x\|_\infty^2 = \|x\|^2 + [\|x\|^4 - |(x|\bar{x})|^2]^{1/2}$$

is a Cartan factor of type IV. The norms $\|\cdot\|$ and $\|\cdot\|_\infty$ are referred to as the Hilbert and the Lie norm on K , their unit balls being denoted by B and B_∞ .

Some other basic facts on J^* -algebras are needed in the sequel. Let $L: \mathcal{A} \rightarrow \mathcal{B}$ be a vector space isomorphism (continuity not assumed) between the J^* -algebras \mathcal{A} and \mathcal{B} . Then L commutes with the triple product, i.e.,

$$L\{ab^*c\} = \{L(a)L(b)^*L(c)\} \quad (a, b, c \in \mathcal{A}),$$

if and only if L is an isometry. In that case, L is said to be a J^* -isomorphism. An element $a \in \mathcal{S}$ is said to be a tripotent if $a \neq 0$ and $\{a a^* a\} = a$, and in that case $\|a\|_\infty = 1$. The tripotent a is said to be minimal if, to each $x \in \mathcal{S}$, there exists $\lambda_x \in \mathbb{C}$ such that

$$\{a x^* a\} = \lambda_x a.$$

J^* -isomorphisms preserve tripotents and minimal tripotents. The set $\text{Min}(\mathcal{S})$ of minimal tripotents of \mathcal{S} is given by ([3] p. 179)

$$\text{Min}(\mathcal{S}) = \{a \in \mathcal{S} \mid a^2 = 0\}.$$

§2. ISOMETRIES OF THE SPACES OF SPINORS

In this section, \mathcal{S} and χ stand for a fixed space of spinors and its selfadjoint part.

Lemma. *Let $L: \mathcal{S} \rightarrow \mathcal{S}$ be any surjective linear $\|\cdot\|_\infty$ -isometry of \mathcal{S} . Then there exists a complex number $\lambda \in \mathbb{C}$, $|\lambda| = 1$, such that*

$$(2.1) \quad L(a)^* = \lambda L(a) \quad (a \in \chi).$$

As a consequence, $\|La\| = \|a\|_\infty$ holds for all a in χ .

Proof: Let $a \in \chi$, $a \neq 0$, be given. By (1.3)

$$\{a a^* a\} = \|a\|^2 a.$$

Thus applying L

$$L\{a a^* a\} = \|a\|^2 L(a)$$

whence again by (1.3)

$$(2.2) \quad L\{a a^* a\} = \{L(a)L(a)^*L(a)\} = 2\|L(a)\|^2 L(a) - (L(a) | L(a)^*) L(a)^*$$

If $(L(a) | L(a)^*) = 0$ then by (1.1) and (1.3)

$$2L(a)^2 = 2(L(a) | L(a)^*) 1_H = 0$$

i.e., $L(a)$ is a minimal tripotent; but then a is also a minimal tripotent and so $a^2 = 0$ which, together with $a \in \chi$, implies $a = 0$, a contradiction. From (2.2)

$$(2.3) \quad L(a)^* = \lambda L(a) \quad (a \in \chi)$$

where $|\lambda| = 1$ and

$$\lambda = \frac{2\|L(a)\|^2 - \|a\|^2}{(L(a) | L(a)^*)} \quad (a \in \chi).$$

We claim that λ does not depend on $a \in \chi$. Indeed, let $b \in \chi$ be given so that a, b are linearly independent (this is possible since by assumption $\dim \mathcal{S} > 1$). By (2.3) there are unitary numbers $\lambda, \mu, \nu \in \mathbb{C}$, such that

$$L(a) = \bar{\lambda} L(a)^*, \quad L(b) = \bar{\mu} L(b)^*, \quad L(a+b) = \bar{\nu} [L(a+b)]^*$$

whence by the linearity of L and the independence of $L(a)$ and $L(b)$, we get $\lambda = \mu = \nu$. Using (2.3) and the expression of λ ,

$\|L(a)\|^2 = (L(a) | L(a)) = (L(a) | \bar{\lambda} L(a)^*) = \lambda (L(a) | L(a)^*) = 2\|L(a)\|^2 - \|a\|^2$
whence, by the coincidence of $\|\cdot\|_\infty$ and $\|\cdot\|$ on χ , we get

$$\|L(a)\|^2 = \|a\|^2 = \|a\|_\infty^2$$

Theorem. *Let \mathcal{S} be any spin factor, and let $L: \mathcal{S} \rightarrow \mathcal{S}$ be any surjective vector space isomorphism. Then the following statements are equivalent:*

1. L is an $\|\cdot\|_\infty$ -isometry of \mathcal{S} .

2. There is a unitary operator U on the Hilbert space \mathcal{S} with $U(a^*) = U(a)^*$ for all a in \mathcal{S} , and there is a number $\mu \in \mathbb{C}$ with $|\mu| = 1$ such that

$$L(a) = \mu U(a) \quad (a \in \mathcal{S}).$$

In particular, any $\|\cdot\|_\infty$ -isometry of \mathcal{S} is an isometry for the underlying Hilbert space.

Proof: “2 \Rightarrow 1” is immediate, so we show “1 \Rightarrow 2”. By the previous lemma

$$L(a)^* = \lambda L(a) \quad (a \in \chi)$$

for some $\lambda \in \mathbb{C}$, $|\lambda| = 1$. Let $\mu \in \mathbb{C}$ be such that $\mu^2 = \lambda$, and define $U: \mathcal{S} \rightarrow \mathcal{S}$ by $U =: \mu L$. Then, for $a \in \chi$ we have

$$U(a) = \mu L(a) = \bar{\mu} L(a)^* = [(\mu L)(a)]^* = U(a)^*$$

i.e., $U(\chi) \subset \chi$, and so

$$U(b^*) = [U(b)]^* \quad (b \in \mathcal{S}).$$

Since $L = \bar{\mu}U$, in order to prove the theorem it suffices to show that U is an isometry for the hilbertian norm $\|\cdot\|$ on \mathcal{S} . But this is a consequence of the sesquilinearity of the scalar product and the fact that U is an $\|\cdot\|$ -isometry on the selfadjoint part χ of \mathcal{S} .

Note. The authors thank Prof. Rodríguez Palacios for simplifying their original proof of this theorem.

§3. EXTREME POINTS OF THE UNIT LIE BALL OF \mathcal{S}

A tripotent e of \mathcal{S} is said to be regular if the operator $\xi \rightarrow \{e e^* \xi\}$, $\xi \in \mathcal{A}$, is invertible in $\mathcal{L}(\mathcal{A})$, and this occurs ([7] p. 190) if and only if e is a (real or complex) extreme point of the unit Lie ball of \mathcal{S} , whose set is denoted by $\text{Extr}(\mathcal{S})$. We have ([9] p. 37)

Theorem. *If \mathcal{S} is any spin factor, then the following equalities hold:*

$$\begin{aligned} \{e \in \mathcal{S} \mid e \text{ is a unitary operator}\} &= \{e \in \mathcal{S} \mid e \text{ is a normal operator and } \|e\|_\infty = 1\} \\ &= \{e \in \mathcal{S} \mid e \text{ is a normal operator and } \|e\| = 1\} = \{e \in \mathcal{S} \mid e \text{ is a real extreme point of } B\} \\ &= \{e \in \mathcal{S} \mid \|e\|_\infty = 1 \text{ and } e^* = \lambda e, \lambda \in \mathbb{C} \mid \lambda| = 1\}. \end{aligned}$$

Proof: Let us denote by S_k , $1 \leq k \leq 5$, the sets above. The inclusion $S_1 \subset S_2$ is clear, and $S_2 \subset S_3$ follows by (1.1). We now prove $S_3 \subset S_4$. Let $e \in S_3$; from (1.1), the assumption $\|e\| = 1$ and the normality of e we get

$$(3.1) \quad ee^* = 1_H = e^*e$$

whence by composing with e , $\{ee^*e\} = e$ which shows that e is a tripotent. From (3.1) $\{ee^*x\}1 = \frac{1}{2}(ee^*x + xe^*e) = x$ for $x \in \mathcal{S}$, which shows the regularity of e , hence $e \in \text{Extr}(\mathcal{S})$

We now prove " $S_4 \subset S_5$ ". Let $e \in S_4$, hence in particular

$$e = \{ee^*e\} = ee^*e$$

If here we compose with e and use the fact that $e^2 = \alpha 1_H$ for some $\alpha \in \mathbb{C}$ (recall the definition of \mathcal{S}), we obtain $\alpha(ee^* - 1_H) = 0$. But $\alpha \neq 0$ as otherwise $e^2 = 0 = e^{*2}$ and so

$$\{ee^*e^*\} = \frac{1}{2}(ee^{*2} + e^{*2}e) = 0$$

which contradicts the regularity of e . Thus $ee^* = 1_H$ and multiplying by e on the left, $\alpha e^* = e$, where $|\alpha| = 1$ since $e \in \text{Extr}(\mathcal{S})$ entails $\|e\|_\infty = 1$. The inclusion $S_5 \subset S_1$ is trivial.

§4. BOUNDARY BEHAVIOUR OF AUTOMORPHISMS

In this section, \mathcal{S} denotes an arbitrary JB^* -triple of finite rank ([8], p. 5.4), $\|\cdot\|_\infty$ denotes its unique JB^* -norm, B_∞ is the unit ball of \mathcal{S} , and $\mathcal{G} = \text{Aut}(B_\infty)$ is the group of all holomorphic automorphisms of B_∞ . It is known ([5], prop. 3.2) that each $g \in \mathcal{G}$ extends to a holomorphic mapping on a neighbourhood of \bar{B}_∞ , and that g maps the boundary ∂B_∞ of B_∞ onto itself. It is also known that ([8], p. 3.10) that each $x \in \mathcal{S}$, $x \neq 0$, admits a spectral representation of the form

$$(4.1) \quad x = \sum_{k=1}^n \lambda_k e_k$$

for some pairwise orthogonal minimal tripotents e_k and some uniquely determined scalars λ_k , $1 \leq k \leq n$ with

$$(4.2) \quad \lambda_1 \geq \lambda_2 \geq \dots > 0, \quad \|x\|_\infty = \max \{\lambda_k \mid 1 \leq k \leq n\}$$

Theorem. *Let \mathcal{S} be any finite rank JB*-triple, and let $L: \mathcal{S} \rightarrow \mathcal{S}$ be any linear mapping (continuity not assumed). Then the following statements are equivalent:*

1. L is a J^* -automorphism of \mathcal{S} .
2. L maps $\text{Min}(\mathcal{S})$ onto itself and preserves the orthogonality relations on $\text{Min}(\mathcal{S})$,

$$(4.3) \quad L[\text{Min}(\mathcal{S})] = \text{Min}(\mathcal{S}) \quad [a, b \in \text{Min}(\mathcal{S}), a \perp b] \Rightarrow L(a) \perp L(b).$$

Proof: We show that “2 \Rightarrow 1” as the converse is trivial. Let $x \in \mathcal{S}$ and let (4.1) be its spectral decomposition. Then

$$L(x) = \sum_{k=1}^n \lambda_k L(e_k)$$

where by (4.3), $L(e_k)$, $1 \leq k \leq n$, are pairwise orthogonal minimal tripotents and so, by the properties of the spectral representation and (4.2),

$$\|L(x)\|_\infty = \|x\|_\infty.$$

Besides, L is surjective. Indeed, if

$$y = \sum_1^m \mu_j e_j \in \mathcal{S},$$

by (4.3) there are tripotents f_j , $1 \leq j \leq m$ (orthogonality is not needed now), such that $L(f_j) = e_j$, hence $x = \sum_1^m \mu_j f_j$ satisfies $L(x) = y$. Thus L is a J^* -automorphism.

Corollary. *Any J^* -automorphism L of a finite rank JB*-triple \mathcal{S} is uniquely determined by its values at the set $\text{Min}(\mathcal{S})$.*

Corollary. *A holomorphic automorphism of the unit ball B_∞ of a finite rank JB*-triple is uniquely determined by its values at the set $\{0\} \cup \text{Min}(\mathcal{S})$.*

Proof: Let f and g in $\text{Aut}(B_\infty)$ be such that $f(0) = g(0) = a$ and $f(e) = g(e)$ for all $e \in \text{Min}(\mathcal{S})$. Take any $h \in \text{Aut}(B_\infty)$ such that $h(a) = 0$. Then $L =: (hg)^{-1} \circ (hf)$ fixes the origin, hence by Cartan’s uniqueness theorem, L is linear. Since L fixes any minimal tripotent of \mathcal{S} , we have $L = Id_{\mathcal{S}}$ and $f = g$.

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