REVISTA MATEMATICA de la
Universidad Complutense de Madrid
Volumen 5, números 2 y 3; 1992.

On Slice Knots in the Complex Projective Plane

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ABSTRACT. We investigate the knots in the boundary of the punctured complex projective plane. Our result gives an affirmative answer to a question raised by Suzuki. As an application, we answer to a question by Mathieu.

1. INTRODUCTION

Throughout this paper, we work in the smooth category, all manifolds are oriented and all the homology groups are with integral coefficients.

Let M be a closed 4-manifold, B^4 an embedded 4-ball in M, and K a knot in $\partial(M-\text{Int}\, B^4)$. If K bounds a properly embedded 2-disk in $M-\text{Int}\, B^4$ then we call the knot K a *slice knot in M*. Let *Slice*(M) be the set of slice knots in M. It is well-known that *Slice*(S^4) is proper subset of the set of knots (Fox and Milnor [3]) and *Slice*(S^4) is a subset of *Slice*(M). In [17], Suzuki proved that *Slice*$(S^2\times S^2)$ is equal to the set of knots, and asked the following question.

Question 1. *Is there a 4-manifold M such that Slice*(S^4) *is a proper subset of Slice*(M) *and Slice*(M) *is a proper subset of the set of knots?*

In [20], the author has proved that *Slice*(CP^2) does not contain a $(-2,15)$-torus knot. This assertion gives an affirmative answer to Question 1 since *Slice*(S^4) is a proper subset of *Slice*(CP^2) (Kervaire and Milnor [6]). In [20], the author could not find a knot that belongs to neither *Slice*(CP^2) nor *Slice*$(\overline{CP^2})$. In Section 2, we show that there exist the knots that belongs to neither *Slice*(CP^2) nor *Slice*$(\overline{CP^2})$.

1991 Mathematics Subject Classification: 57M25; 57M13, 57R95.
Editorial Complutense. Madrid, 1992.

Let K be a knot in $\partial(n_1CP^2\#n_2\overline{CP^2}-\text{Int}\,B^4)$. The knot K is an *evenly slice knot in* $n_1CP^2\#n_2\overline{CP^2}$ if K bounds a properly embedded 2-disk in $n_1CP^2\#n_2\overline{CP^2}-\text{Int}\,B^4$ that represents an element $z(\varepsilon_1\gamma_1+\dots+\varepsilon_{n_1}\gamma_{n_1}+\bar{\varepsilon}_1\bar{\gamma}_1+\dots+\bar{\varepsilon}_{n_2}\bar{\gamma}_{n_2})$ in $H_2(n_1CP^2\#n_2\overline{CP^2}-\text{Int}\,B^4,\partial)$, where $\gamma_1,\dots,\gamma_{n_1},\bar{\gamma}_1,\dots,\bar{\gamma}_{n_2}$ are standard generators of $H_2(n_1CP^2\#n_2\overline{CP^2}-\text{Int}\,B^4,\partial)$, $\varepsilon_i=\pm1$, $\bar{\varepsilon}_j=\pm1$ and z is an integer. Let *e-Slice*$(n_1CP^2\#n_2\overline{CP^2})$ be the set of evenly slice knots in $n_1CP^2\#n_2\overline{CP^2}$. (Note that *e-Slice*$(CP^2)=$*Slice*$(CP^2)$ and *e-Slice*$(\overline{CP^2})=$*Slice*$(\overline{CP^2})$.) In Section 3, we deal with in the case $n_1=n_2=1$ or $n_1=0$.

Let K_0 be a knot and D^2 a 2-disk intersecting transversely K_0 with the linking number $lk(\partial D^2, K_0)=l$. Let p be a positive integer and $\varepsilon=\pm1$. By performing $\frac{\varepsilon}{p}$–Dehn surgery along ∂D^2, we have a new knot. The new knot is said to be the knot obtained from K_0 by an *$(\varepsilon p, l)$-twisting*. Let $\mathscr{K}_p$ be the set of knots obtained from a trivial knot by an $(\varepsilon p, l)$-twisting for some integer l and $\varepsilon=\pm1$. Section 4 is devoted to two applications. Our first application is to find infinitely many knots that give a negative answer to the following question given by Mathieu [12].

Question 2. *For any knot K, is there a positive integer p such that $K\in\mathscr{K}_p$?*

Our second one is to find infinitely many counterexamples to the following conjecture made by Akbulut and Kirby.

Conjecture. *If K is a knot with Arf invariant zero, then K is obtained from a slice knot by a $(\pm1,\pm1)$-twisting.* (Problem 1.46 (B) of [9].)

It is shown that a $(2,7)$-torus knot cannot be obtained from a ribbon knot by a $(1,1)$-twisting by using Donaldson's outstanding theorem [1, Theorem 1] (see [10]). Since then Donaldson improved this result to drop "simply connectedness assumption" [2, Theorem 1], a $(2,7)$-torus knot cannot be obtained from a slice knot by a $(1,1)$-twisting. Here we give infinitely many counterexamples in different knot cobordism classes.

Similar results for Question 2 were obtained independently by Katura Miyazaki [13].

1. PRELIMINARIES

In this section we introduce some useful lemmas to us. In particular, Lemmas 1.8 and 1.11 are key lemmas in this paper.

Let α, β be the standard generators of $H_2(S^2\times S^2)$ with $\alpha^2=\beta^2=0$, $\alpha\cdot\beta=1$ and let γ or γ_i (resp. $\bar{\gamma}$ or $\bar{\gamma}_i$) be the standard generator of $H_2(CP^2)$ (resp. $H_2(\overline{CP^2})$) with $\gamma^2=\gamma_i^2=1$ (resp. $\bar{\gamma}^2=\bar{\gamma}_i^2=-1$). From now on a homology class in $H_2(M-\text{Int}\,B^4, \partial)$ is identified with its image by the homomorphism

$$H_2(M-\text{Int}\,B^4, \partial)\overset{\cong}{\leftarrow} H_2(M-\text{Int}\,B^4)\rightarrow H_2(M).$$

Let l and m be nonnegative integers and $\varepsilon=\pm1$. An $(\varepsilon l, m)$*-torus link* is the link that wraps around the standardly embedded solid torus in S^3 in the longitudinal direction l times and in the meridional direction m times, where the intersection number of the meridian and longitude is ε. When l and m are relatively prime, it is a knot and called an $(\varepsilon l, m)$*-torus knot.* An $(\varepsilon l, m)$-torus knot is denoted by $T(\varepsilon l, m)$.

Let L be a μ-component link in S^3. Let $f_i: I\times I\rightarrow S^3$, $i=1,\dots,m-1$ $(m\leq\mu)$ be mutually disjoint embeddings such that

(i) $f_i(I\times I)\cap L=f_i(I\times\partial I)$ for each i $(i=1,\dots,m-1)$ and

(ii) the link $L'=Cl(L\cup\bigcup f_i(\partial I\times I)-\bigcup f_i(I\times\partial I))$ has the orientation compatible with that of $L-\bigcup f_i(I\times\partial I)$ and $\bigcup f_i(\partial I\times I)$.

The link L' is said to be the link obtained from L by *m-fusion* if the number of the components of L' is $\mu-m$. In particular if the number of the components of L' is one, then L' is said to be the knot obtained from L by *complete fusion.* We call the images $f_1(I\times I),\dots, f_m(I\times I)$ the *strips connecting* L. Let $\mathscr{T}_{\varepsilon x}$ $(\varepsilon=\pm1, x\geq0)$ be the set of knots obtained from a $(2\varepsilon, 4x)$-torus link by 1-fusion. Note that a knot K belongs to $\mathscr{T}_x$ if and only if the reflected inverse $-K^!$ belongs to $\mathscr{T}_{-x}$.

1.1. Lemma. *For any knot $K\in\mathscr{T}_{\varepsilon x}$, there exists an embedded 2-disk Δ in $S^2\times S^2-\text{Int}\,B^4$ such that Δ represents an element $2\alpha+2\varepsilon x\beta$ in $H_2(S^2\times S^2-\text{Int.}\,B^4, \partial)$ and $\partial\Delta\subset\partial(S^2\times S^2-\text{Int.}\,B^4)$ is $-K^!$.*

Proof. We first deal with the case that $K\in\mathscr{T}_x$. It is easily seen that there exist mutually disjoint $2x+2$ properly embedded 2-disks $\Delta_1,\dots,\Delta_{2x+2}$ in $S^2\times S^2-\text{Int}\,B^4$ such that $\bigcup\Delta_i$ represents an element $2\alpha+2x\beta$ and $\partial(\bigcup\Delta_i)\subset\partial(S^2\times S^2-\text{Int}\,B^4)$ is a Figure 1. Since a $(-2,4x)$-torus link is obtained from $\partial(\bigcup\Delta_i)$ by $2x$-fusion, there exist $2x+1$ strips $b_1,\dots,b_{2x+1}$ connecting the link $\partial(\bigcup\Delta_i)$ such that $\Delta=\Delta_1\cup\dots\cup\Delta_{2x+2}\cup b_1\cup\dots\cup b_{2x+1}$ is an embedded 2-disk in $S^2\times S^2-\text{Int}\,B^4$ and $\partial\Delta\subset(S^2\times S^2-\text{Int}\,B^4)$ is $-K^!$.

The above argument remains valid in case $K\in\mathscr{T}_{-x}$ $\square$

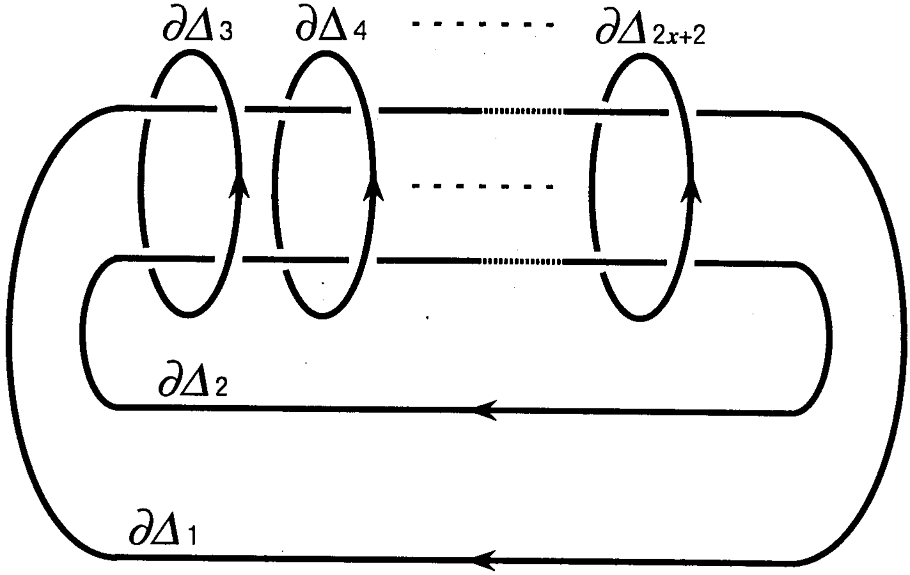

Figure 1

1.2. Lemma. *For any knot $K \in \mathscr{T}_{\varepsilon x}$, there exists an embedded 2-disk Δ in $CP^2 \# \overline{CP^2} - \text{Int}\, B^4$ such that Δ represents an element $(2x+\varepsilon)\gamma + (2x-\varepsilon)\bar{\gamma}$ in $H_2(CP^2 \# \overline{CP^2} - \text{Int.}\, B^4, \partial)$ and $\partial\Delta \subset \partial(CP^2 \# \overline{CP^2} - \text{Int}\, B^4)$ is $-K^!$.*

Proof. We first deal with the case that $K \in \mathscr{T}_x$. Let $O_1 \cup O_{-1}$ be a 2-component trivial link in ∂B^4 such that O_j is framed by j ($j = \pm 1$). By considering the "Kirby's calculus"[8] as Figure 2, we note that there exist mutually disjoint $2x+1$ properly embedded 2-disks $\Delta_1, \ldots, \Delta_{2x+1}$ in $CP^2 \# \overline{CP^2} - \text{Int}\, B^4$ such that $\cup \Delta_i$ represents an element $(2x+1)\gamma + (2x-1)\bar{\gamma}$ in $H_2(CP^2 \# \overline{CP^2} - \text{Int}\, B^4, \partial)$ and $\partial(\cup\Delta_i) \subset \partial(CP^2 \# \overline{CP^2} - \text{Int}\, B^4)$ is as Figure 3. Since a $(-2,4x)$-torus link is obtained from $\partial(\cup\Delta_i)$ by $(2x-1)$-fusion, there exist $2x$ strips $b_1, \ldots, b_{2x}$ *connecting the link* $\partial(\cup\Delta_i)$ such that $\Delta = \Delta_1 \cup \ldots \cup \Delta_{2x+1} \cup b_1 \cup \ldots \cup b_{2x}$ is an embedded 2-disk in $CP^2 \# \overline{CP^2} - \text{Int}\, B^4$ and $\partial\Delta \subset \partial(CP^2 \# \overline{CP^2} - \text{Int}\, B^4)$ is $-K^!$.

By considering the Kirby's calculus as in Figure 4, the above argument remains valid in case $K \in \mathscr{T}_{-x}$ □

1.3. Lemma. (Rohlin [16]) *Let M be a connected, simply connected, closed 4-manifold. If $\xi \in H_2(M)$ is represented by an embedded 2-sphere in M, then*

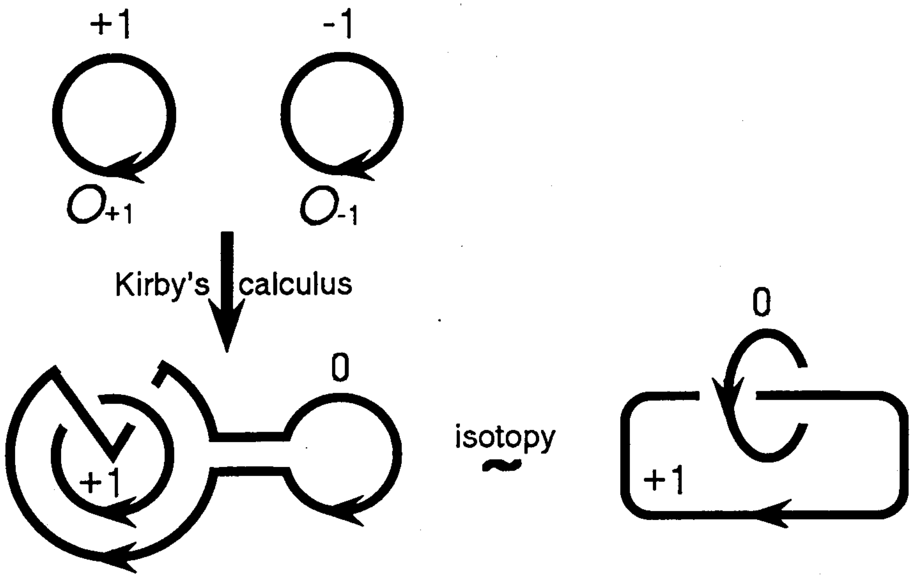

Figure 2

(a) $\left|\frac{\xi^2}{2} - \sigma(M)\right| \leq$ rank $H_2(M)$ *if* ξ *is divisible by* 2,

(b) $\left|\frac{\xi^2(q^2-1)}{2q^2} - \sigma(M)\right| \leq$ rank $H_2(M)$ *if* ξ *is divisible by an odd prime integer* q, *where* $\sigma(M)$ *is the signature of* M.

1.4. Lemma. (Weintraub [18], Yamamoto [19]) *Let* K *be a knot. If the unknotting number of* K *is less than or equal to* u *then there exists embedded 2-disk* Δ *in* $u(CP^2\#\overline{CP^2}) - \text{Int } B^4$ *such that* Δ *represents the zero element in* $H_2(u(CP^2\#\overline{CP^2}) - \text{Int } B^4, \partial)$ *and* $\partial\Delta \subset \partial(u(CP^2\#\overline{CP^2}) - \text{Int } B^4)$ *is* $-K^!$.

1.5. Lemma. (Lawson [11]) *Let* $\xi \in H_2(CP^2\#2\overline{CP^2})$ *be a characteristic element. The element* ξ *is represented by a 2-sphere in* $CP^2\#2\overline{CP^2}$ *if and only if* $\xi^2 = -1$.

1.6. Lemma. (Lawson [11]) *Let* $\xi \in H_2(CP^2\#n\overline{CP^2})$ $(n \geq 3)$ *be a characteristic element. If* ξ *is represented by a 2-sphere in* $CP^2\#n\overline{CP^2}$ *then* $\xi^2 \leq -2$.

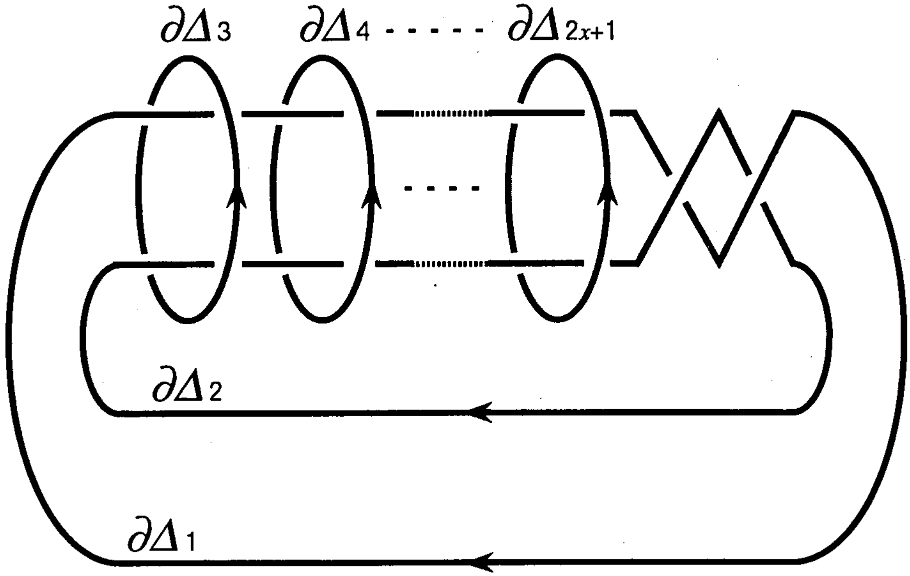

Figure 3

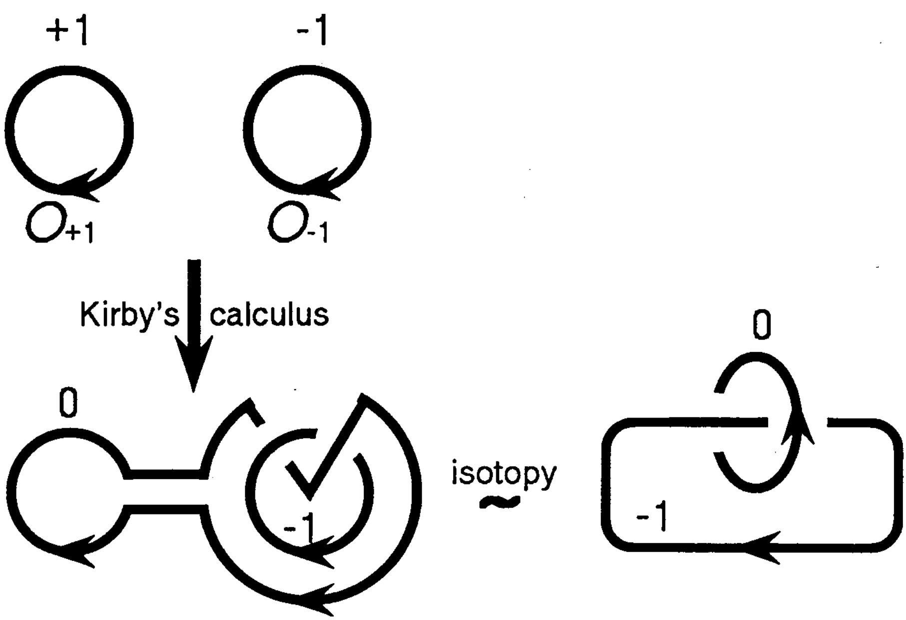

Figure 4

1.7. Lemma. (Kikuchi [7]) *Let $\xi \in H_2(CP^2 \# 3\overline{CP^2})$ be a characteristic element. The element ξ is represented by a 2-sphere in $CP^2 \# 3\overline{CP^2}$ if and only if $\xi^2 = -2$.*

1.8. Lemma. *Let p be a positive integer and x a nonnegative integer. Let $K \in \mathscr{T}_x$ be a knot such that the unknotting number of K is less than or equal to u. If $K \in$ e-Slice$(p\overline{CP^2})$ then there exists an integer z such that z satisfies a condition*

(a) $\dfrac{8x-4}{p} \leq z^2 \leq \dfrac{4u}{p} + 4$ *and z is even, or*

(b) $\begin{cases} z^2 = 8x+1 \text{ if } p=1, \\ z^2 = 4x+1 \text{ if } p=2, \\ \dfrac{8x+2}{p} \leq z^2 \leq \dfrac{9}{2}\left(\dfrac{u}{p}+1\right) \text{ and } z \text{ is odd if } p \geq 3. \end{cases}$

Proof. Suppose that $K \in \mathscr{T}_x \cap$ *e-Slice* $(p\overline{CP^2})$ and the unknotting number of K is less than or equal to u. Since $K \in \mathscr{T}_x \cap$ *e-Slice* $(p\overline{CP^2})$, there exists an integer z such that

(1) $2\alpha + 2x\beta + z(\bar{\varepsilon}_1 \bar{\gamma}_1 + \ldots + \bar{\varepsilon}_p \bar{\gamma}_p) \in H_2(S^2 \times S^2 \# p\overline{CP^2})$ is represented by a 2-sphere in $S^2 \times S^2 \# p\overline{CP^2}$ and

(2) $(2x+1)\gamma + (2x-1)\bar{\gamma} + z(\bar{\varepsilon}_1 \bar{\gamma}_1 + \ldots + \bar{\varepsilon}_p \bar{\gamma}_p) \in H_2(CP^2 \# (p+1)\overline{CP^2})$ is represented by a 2-sphere in $CP^2 \# (p+1)\overline{CP^2}$,

by Lemmas 1.1, 1.2 and the definition of evenly slice knots. Since the unknotting number of K is less than or equal to u, by Lemma 1.4,

(3) $z(\bar{\varepsilon}_1 \bar{\gamma}_1 + \ldots + \bar{\varepsilon}_p \bar{\gamma}_p)$ is represented by a 2-sphere in $p\overline{CP^2} \# u(CP^2 \# \overline{CP^2})$.

In case that z is even. By Lemma 1.3, (1) and (3),

$$\left| \frac{8x - pz^2}{2} + p \right| \leq p + 2,$$

$$\left| \frac{-pz^2}{2} + p \right| \leq p + 2u.$$

It follows that

$$\frac{8x-4}{p} \leq z^2 \leq \frac{4u}{p} + 4.$$

In case that z is odd and $|z| \geq 3$. By Lemma 1.3 and (3), there exists an odd prime integer q such that

$$\left| \frac{-pz^2(q^2-1)}{2q^2} + p \right| \leq p + 2u.$$

This implies

(1-1) $$z^2 \leq \frac{9}{2}\left(\frac{u}{p} + 1\right).$$

We note that

(1-2) $$1 < \frac{9}{2}\left(\frac{u}{p} + 1\right).$$

The inequations (1-1) and (1-2) imply that any odd integer z satisfies

(1-3) $$1 \leq z^2 \leq \frac{9}{2}\left(\frac{u}{p} + 1\right).$$

Moreover if z is odd then $(2x+1)\gamma + (2x-1)\bar{\gamma} + z(\bar{\varepsilon}\bar{\gamma}_1 + \dots + \bar{\varepsilon}_p\bar{\gamma}_p)$ is a characteristic element in $H_2(CP^2 \# (p+1)\overline{CP^2})$. By Lemmas 1.5, 1.6, 1.7 and (2),

(1-4) $$8x - z^2 = -1 \text{ if } p = 1,$$

(1-5) $$8x - 2z^2 = -2 \text{ if } p = 2,$$

(1-6) $$8x - pz^2 \leq -2 \text{ if } p \geq 3.$$

By (1-3), (1-4), (1-5) and (1-6), we have

$$z^2 = 8x + 1 \text{ if } p = 1,$$

$$z^2 = 4x + 1 \text{ if } p = 2,$$

$$\frac{8x+2}{p} \leq z^2 \leq \frac{9}{2}\left(\frac{u}{p}+1\right) \text{ if } p \geq 3.$$

This completes the proof. □

Suppose that knots K_+ and K_- have representatives in S^3 that are identical outside a 3-ball within which they are as in Figure 5. Then we say that K_- is obtained from K_+ by *changing a positive crossing* and that K_+ is obtained from K_- by *changing a negative crossing.* We define the *positive unknotting number* (resp. *negative unknotting number*) of a knot K, to be the minimum, over all sequences transforming K to be a trivial knot, of the number of positive (resp. negative) crossings which are changed. If K cannot be a trivial knot by changing only positive (resp. negative) crossings, then we define the positive unknotting number (resp. negative unknotting number) of K is *infinite.*

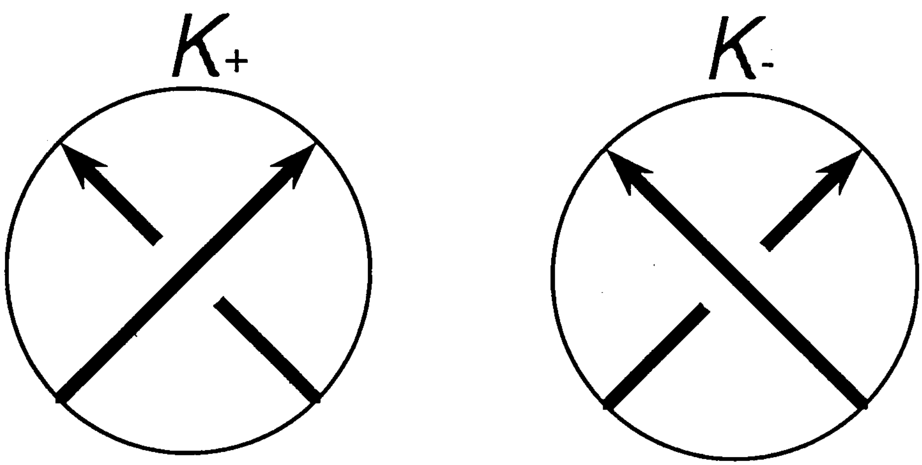

Figure 5

1.9. Lemma. (Weintraub [18]) *Let K be a knot. If the positive unknotting number (resp. negative unknotting number) of K is less than or equal to u, then there exists an embedded 2-disk Δ in $u\overline{CP^2}$ —* Int B^4 *(resp. uCP^2 — Int B^4) such that Δ represents the zero element in $H_2(u\overline{CP^2}$ —* Int B^4, ∂) *(resp.* $H_2(uCP^2$ — Int $B^4, \partial)$) *and $\partial\Delta \subset \partial(u\overline{CP^2}$ — Int B^4) (resp. $\partial\Delta \subset \partial(uCP^2$ —*Int B^4)) *is — $K^!$.*

1.10. Lemma. (Kervaire and Milnor [6]) *Let M be a connected, simply connected, closed 4-manifold. Let $\xi \in H_2(M)$ be a characteristic element. If ξ is represented by an embedded 2-sphere in M, then $\xi^2 \equiv \sigma(M)$* mod 16.

1.11. Lemma. *Let p be a positive integer and x a nonnegative integer. Let $K \in \mathcal{T}_{-x}$ be a knot such that the negative unknotting number of K is finite. If $K \in$ e-Slice$(p\overline{CP^2})$ then there exists an integer z such that z satisfies a condition*

(a) $z^2 \leq 4 + \dfrac{4-8x}{p}$ *and z is even, or*

(b) $\begin{cases} z^2 = 1 \text{ only if } x = 0 \text{ and } p = 1, 2, \\ z^2 = 1 \text{ only if } x \equiv 0 \bmod 2 \text{ and } p \geq 3. \end{cases}$

Proof. Suppose $K \in \mathcal{T}_{-x} \cap$ *e-Slice*$(p\overline{CP^2})$ and the negative unknotting number of K is u. Since $K \in \mathcal{T}_{-x} \cap$ *e-Slice*$(p\overline{CP^2})$, there exists an integer z such that

(4) $2\alpha - 2x\beta + z(\bar{\varepsilon}_1\bar{\gamma}_1 + \ldots + \bar{\varepsilon}_p\bar{\gamma}_p) \in H_2(S^2 \times S^2 \# p\overline{CP^2})$ is represented by a 2-sphere in $S^2 \times S^2 \# p\overline{CP^2}$ and

(5) $(2x-1)\gamma + (2x+1)\bar{\gamma} + z(\bar{\varepsilon}_1\bar{\gamma}_1 + \ldots + \bar{\varepsilon}_p\bar{\gamma}_p) \in H_2(CP^2 \# (p+1)\overline{CP^2})$ is represented by a 2-sphere in $CP^2 \# (p+1)\overline{CP^2}$,

by Lemmas 1.1, 1.2 and the definition of evenly slice knots. Since the negative unknotting number of K is u, by Lemma 1.9,

(6) $z(\bar{\varepsilon}_1\bar{\gamma}_1 + \ldots + \bar{\varepsilon}_p\bar{\gamma}_p)$ is represented by a 2-sphere in $p\overline{CP^2} \# uCP^2$.

In case that z is even. By Lemma 1.3 and (4),

$$\left| \frac{-8x - pz^2}{2} + p \right| \leq p + 2.$$

This implies

$$z^2 \leq 4 + \frac{4-8x}{p}.$$

In case that z is odd. If $|z| \geq 3$, then by Lemma 1.3 and (6), there exists an odd prime integer q such that

$$\left| \frac{-pz^2(q^2-1)}{2q^2} + p - u \right| \leq p + u.$$

It follows that

$$z^2 \leq \frac{9}{2}.$$

This is a contradiction. Thus $|z| = 1$. Moreover, by Lemmas 1.5, 1.7, 1.10 and (5), we have

$$-8x - pz^2 = -p \text{ if } p = 1, 2,$$

$$-8x - pz^2 \equiv -p \bmod 16.$$

Since $|z| = 1$,

$$-8x = 0 \text{ if } p = 1, 2,$$

$$-8x \equiv 0 \bmod 16.$$

This implies

$$x = 0 \text{ if } p = 1, 2,$$

$$x \equiv 0 \bmod 2.$$

This completes the proof. □

2. SLICE KNOTS IN CP^2 or $\overline{CP^2}$

In this section we shall prove the following two theorems.

2.1. Theorem. *Let x be a positive integer.*

(a) *If $Slice(\overline{CP^2})$ contains $T(2, 4x-1)$, then $2x-1$, $2x$ or $8x+1$ is a square number.*

(b) *If $Slice(\overline{CP^2})$ contains $T(2, 4x+1)$, then $2x$, $2x+1$ or $8x+1$ is a square number.*

2.2. Theorem. *Let t be a nonnegative integer. The set $Slice(\overline{CP^2})$ does not contain $T(-2, 2t+1)$ if and only if $t \geq 2$.*

2.3. Remark. Since $Slice(CP^2)$ contains a knot K if and only if $Slice(\overline{CP^2})$ contains $-K^!$, $Slice(CP^2)$ contains $T(l, m)$ if and only if

Slice $(\overline{CP^2})$ contains $T(-l, m)$. It follows that Theorems 2.1 and 2.2 imply that there exist infinitely many integer $x_i (i=1, 2, \ldots)$ such that $T(2, 2x_i+1)$ belongs to neither *Slice* (CP^2) nor *Slice* $(\overline{CP^2})$ for any x_i.

2.4. Lemma. *For any* $T(2\varepsilon, 4x+1)$ $(\varepsilon=\pm 1, x\geq 0)$, *there exists an embedded* 2-*disk* Δ *in* $CP^2 \# \overline{CP^2} - \mathrm{Int}\, B^4$ *such that* Δ *represents an element* $(2x+1+\varepsilon)\gamma + (2x+1-\varepsilon)\bar{\gamma}$ *in* $H_2(CP^2 \# \overline{CP^2} - \mathrm{Int}\, B^4, \partial)$ *and* $\partial\Delta \subset \partial(CP^2 \# \overline{CP^2} - \mathrm{Int}\, B^4)$ *is* $T(-2\varepsilon, 4x+1)$.

Proof. By considering the Kirby's calculus as in Figure 2, we note that there exist mutually disjoint $2x+2$ properly embedded 2-disk $\Delta_1, \ldots, \Delta_{2x+2}$ in $CP^2 \# \overline{CP^2} - \mathrm{Int}\, B^4$ such that $\cup \Delta_i$ represents an element $(2x+2)\gamma + 2x\bar{\gamma}$ in $H_2(CP^2 \# \overline{CP^2} - \mathrm{Int}\, B^4, \partial)$ and $\partial(\cup \Delta_i) \subset \partial(CP^2 \# \overline{CP^2} - \mathrm{Int}\, B^4)$ is as Figure 6. Since a $(-2, 4x+2)$-torus link is obtained from $\partial(\cup \Delta_i)$ by $2x$-fusion, there exist $2x+1$ strips $b_1, \ldots, b_{2x+1}$ connecting the link $\partial(\cup \Delta_i)$ such that $\Delta = \Delta_1 \cup \ldots \cup \Delta_{2x+2} \cup b_1 \cup \ldots \cup b_{2x+1}$ is an embedded 2-disk in $CP^2 \# \overline{CP^2} - \mathrm{Int}\, B^4$ and $\partial\Delta \subset \partial(CP^2 \# \overline{CP^2} - \mathrm{Int}\, B^4)$ is $T(-2, 4x+1)$.

By considering the Kirby's calculus as in Figure 4, the above argument remains valid for $T(-2, 4x+1)$. □

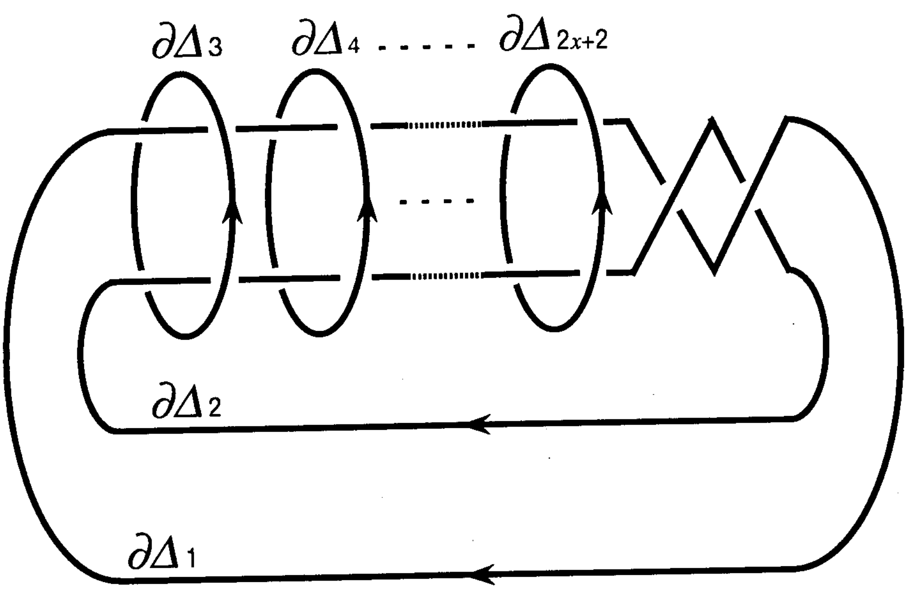

Figure 6

Proof of Theorem 2.1. Suppose $T(2,4x-1)\in Slice(\overline{CP^2})$. Since the unknotting number of $T(2,4x-1)$ is $2x-1$, $T(2,4x-1)\in\mathcal{T}_x$ and $e\text{-}Slice(\overline{CP^2})=Slice(\overline{CP^2})$, by Lemma 1.8, there exists an integer z such that z satisfies a condition

(2-7) $$8x-4\leq z^2\leq 8x \text{ and } z \text{ is even, or}$$

(2-8) $$z^2=8x+1.$$

We set $z=2k$ in (2-7), then we have

$$2x-1\leq k^2\leq 2x.$$

It follows that

(2-9) $$k^2=2x-1,\ 2x.$$

By (2-8) and (2-9), we obtain Theorem 2.1 (a).

Suppose $T(2,4x+1)\in Slice(\overline{CP^2})$. Since the unknotting number of $T(2,4x+1)$ is $2x$ and $T(2,4x+1)\in\mathcal{T}_x$, by LEmma 1.8, there exists an integer z such that z satisfies a condition

(2-10) $$8x-4\leq z^2\leq 8x+4 \text{ and } z \text{ is even, or}$$

(2-11) $$z^2=8x+1.$$

The fact that $T(2,4x+1)$ belongs to $Slice(\overline{CP^2})$ and Lemma 2.4 imply that $(2x+2)\gamma+2x\bar{\gamma}+z\bar{\gamma}_1\in H_2(CP^2\#2\overline{CP^2})$ is represented by a 2-sphere in $CP^2\#2\overline{CP^2}$. If z is even, then by Lemma 1.3, we have

$$\left|\frac{8x+4-z^2}{2}+1\right|\leq 3.$$

This implies

(2-12) $$8x\leq z^2\leq 8x+12.$$

By (2-10) and (2-12), we have

(2-13) $$8x\leq z^2\leq 8x+4 \text{ and } z \text{ is even.}$$

We set $z=2k$ in (2-13) then

$$2x\leq k^2\leq 2x+1.$$

It follows that

(2-14) $$k^2=2x,\ 2x+1.$$

By (2-11) and (2-14), we obtain Theorem 2.1 (b). □

2.5. Proposition. *If $t\geq 3$ then Slice$(\overline{CP^2})$ does not contain $T(-2, 2t+1)$.*

Proof. Note that $\mathcal{T}_{-x}$ contains both $T(-2, 4x-1)$ and $T(-2, 4x+1)$ and that the negative unknotting number of $T(-2,4x-1)$ and that the negative unknotting number of $T(-2,4x+1)$ are finite. If *Slice*$(\overline{CP^2})$ contains $T(-2,4x-1)$ or $T(-2,4+1)$, then by Lemma 1.11, there exists an integer z such that z satisfies a condition

(2-15) $$z^2=8-8x \text{ and } z \text{ is even, or}$$

(2-16) $$z^2=1 \text{ and } x=0.$$

The conditions (2-15) and (2-16) imply

$$x=0,1.$$

This completes the proof. □

2.5.1. Remark. By the proofs of Lemma 1.11 and Proposition 2.5, we note that if *Slice*$(\overline{CP^2})$ contains $T(-2,5)$ then there exists a properly embedded 2-disk Δ in $\overline{CP^2}-\text{Int}\,B^4$ such that Δ represents the zero element in $H_2(\overline{CP^2}-\text{Int}\,B^4, \partial)$ and $\partial\Delta\subset\partial(\overline{CP^2}-\text{Int}\,B^4)$ is $T(-2,5)$.

2.6. Proposition. *The set Slice$(\overline{CP^2})$ does not contain $T(-2,5)$.*

Proof. Suppose *Slice*$(\overline{CP^2})$ contains $T(-2,5)$. Remark 2.5.1 and Lemma 2.4 imply that $2\gamma+4\bar{\gamma}\in H_2(CP^2\#\overline{CP^2})$ is represented by a 2-sphere in $CP^2\#2\overline{CP^2}$. By Lemma 1.3, we have

$$\left|\frac{4-16}{2}+1\right|\leq 3.$$

This is a contradiction. □

Proof of Theorem 2.2. By Propositions 2.5 and 2.6, if $t \geq 2$ then $Slice(\overline{CP^2})$ does not contain $T(-2, 2t+1)$. If $t=0$ or 1 then $Slice(\overline{CP^2})$ contains $T(-2, 2t+1)$, see Proposition 3.7. □

3. EVENLY SLICE KNOTS IN $\mathbf{n_1 CP^2 \# n_2 \overline{CP^2}}$

In [15], Norman proved that $Slice(CP^2 \# \overline{CP^2})$ is equal to the set of knots, but the following theorem implies that there exist infinitely many knots that do not belong to $e\text{-}Slice(CP^2 \# \overline{CP^2})$, i.e., $e\text{-}Slice(CP^2 \# \overline{CP^2})$ is a proper subset of $Slice(CP^2 \# \overline{CP^2})$.

3.1. Theorem. *Let t be a nonnegative integer and $\varepsilon = \pm 1$. The set $e\text{-}Slice(CP^2 \# \overline{CP^2})$ contains $T(2\varepsilon, 2t+1)$ if and only if $t=0$ or* 1.

3.2. Lemma. (Hirai [4]) *Let $\xi \in H_2(2(CP^2 \# \overline{CP^2}))$ be a characteristic element. The element ξ represented by a 2-sphere in $2(CP^2 \# (\overline{CP^2})$ if and only if $\xi^2 = 0$.*

3.3. Proposition. *For $\varepsilon = \pm 1$, if $t \geq 3$ then $e\text{-}Slice(CP^2 \# \overline{CP^2})$ does not contain $T(2\varepsilon, 2t+1)$.*

Proof. Let x be a nonnegative integer. If either $T(2\varepsilon, 4x-1)$ or $T(2\varepsilon, 4x+1)$ belongs to $e\text{-}Slice(CP^2 \# \overline{CP^2})$ then there exists an integer z such that

(7) $2\alpha + 2\varepsilon x \beta + z(\varepsilon_1 \gamma_1 + \bar{\varepsilon}_1 \bar{\gamma}_1) \in H_2(S^2 \times S^2 \# CP^2 \# \overline{CP^2})$ is represented by a 2-sphere in $S^2 \times S^2 \# CP^2 \# \overline{CP^2}$ and

(8) $(2x+\varepsilon)\gamma + (2x-\varepsilon)\bar{\gamma} + z(\varepsilon_1 \gamma_1 + \bar{\varepsilon}_1 \bar{\gamma}_1) \in H_2(2(CP^2 \# \overline{CP^2}))$ is represented by a 2-sphere in $2(CP^2 \# \overline{CP^2})$,

by Lemmas 1.1, 1.2 and the definition of evenly slice knots. If z is even, then by Lemma 1.3 and (7),

$$\left| \frac{8\varepsilon x}{2} \right| \leq 4.$$

This implies

$$x = 0, 1.$$

If z is odd, then by Lemma 3.2 and (8),

$$8\varepsilon x=0.$$

It follows that if $x\geq 2$, then neither $T(2\varepsilon, 4x-1)$ nor $T(2\varepsilon, 4x+1)$ belongs to $e\text{-}Slice(CP^2\#\overline{CP^2})$. This completes the proof. □

3.4. Proposition. *The set* $e\text{-}Slice(CP^2\#\overline{CP^2})$ *does not contain* $T(2\varepsilon, 5)$ *for* $\varepsilon=\pm 1$.

Proof. Suppose $e\text{-}Slice(CP^2\#\overline{CP^2})$ contains $T(2\varepsilon, 5)$. Proof of Proposition 3.3 and Lemma 2.4 implies that there exists an even integer z such that $(3+\varepsilon)\gamma+(3-\varepsilon)\bar{\gamma}+z(\varepsilon_1\gamma_1+\bar{\varepsilon}_1\bar{\gamma}_1)\in H_2(2(CP^2\#\overline{CP^2}))$ is represented by a 2-sphere in $2(CP^2\#\overline{CP^2})$. By Lemma 1.3, we have

$$\left|\frac{12\varepsilon}{2}\right|\leq 4.$$

This is a contradiction. □

Proof of Theorem 3.1. By Propositions 3.3 and 3.4, if $t\geq 2$ then $e\text{-}Slice(CP^2\#\overline{CP^2})$ does not contain $T(2\varepsilon, 2t+1)$. If $t=0$ or 1 then $e\text{-}Slice(CP^2\#\overline{CP^2})$ contains $T(2\varepsilon, 2t+1)$, see Proposition 3.7. □

The same arguments as proof of Theorem 2.1 and Proposition 2.5 lead to the following Theorem 3.5 and Proposition 3.6, respectively.

3.5. Theorem. *Let x be a positive integer.*

(a) *If* $e\text{-}Slice(2\overline{CP^2})$ *contains* $T(2,4x-1)$ *then x or $4x+1$ is a square number.*

(b) *If* $e\text{-}Slice(2\overline{CP^2})$ *contains* $T(2,4x+1)$ *then x, $x+1$ or $4x+1$ is a square number.*

3.6. Proposition. *If $t\geq 3$ then* $e\text{-}Slice(2\overline{CP^2})$ *does not contain* $T(-2, 2t+1)$.

3.7. Proposition. *Let K be a knot. If the positive unknotting number or the negative unknotting number of K is less than or equal to p, then both* $e\text{-}Slice(pCP^2)$ *and* $e\text{-}Slice(p\overline{CP^2})$ *contain K.*

Proof. Suppose K is a knot and the positive or negative unknotting number of K is less than or equal to p. Let L_ε be the Hopf link in $\partial(CP^2 - \mathrm{Int}\, B^4)$ with linking number $\varepsilon(\varepsilon = \pm 1)$. It is easily seen that L_ε bounds a properly embedded 2-disk in $CP^2 - \mathrm{Int}\, B^4$ that represents an element $(1-\varepsilon)\gamma$ in $H_2(CP^2 - \mathrm{Int}\, B^4, \partial)$. Since the positive or negative unknotting number of K is less than or equal to p, K is obtained from the p copies of L_ε by complete fusion. It follows that K bounds a properly embedded 2-disk in $pCP^2 - \mathrm{Int}\, B^4$ that represents an element $(1-\varepsilon)(\varepsilon_1\gamma_1 + \ldots + \varepsilon_p\gamma_p)$ in $H_2(pCP^2 - \mathrm{Int}\, B^4, \partial)$. This implies that K belongs to *e-Slice* (pCP^2).

The above argument remains valid to show that K belongs to *e-Slice* $(p\overline{CP^2})$. This completes the proof. □

By Propositions 3.6 and 3.7, we have the following theorem.

3.8. Theorem. *Let t be a nonnegative integer. The set e-Slice$(2\,\overline{CP^2})$ does not contain $T(-2, 2t+1)$ if and only if $t \geq 3$.*

3.9. Theorem. *For any integer $p \geq 3$, e-Slice$(p\overline{CP^2})$ contains neither $T(2, 8p+3)$ nor $T(-2, 8p+3)$.*

Proof. Suppose that *e-Slice*$(p\overline{CP^2})$ contains $T(2, 8p+3)$. Since $T(2, 8p+3)$ belongs to $\mathcal{T}_{2p+1}$ and the unknotting number of $T(2, 8p+3)$ is $4p+1$, by Lemma 1.8, there exists an integer z such that z satisfies a condition

$$\text{(3-17)} \qquad \frac{16p+4}{p} \leq z^2 \leq \frac{16p+4}{p} + 4 \text{ and } z \text{ is even, or}$$

$$\text{(3-18)} \qquad \frac{16p+10}{p} \leq z^2 \leq \frac{9}{2}\left(\frac{4p+1}{p} + 1\right) \text{and } z \text{ is odd.}$$

Since $p \geq 3$, (3-17) and (3-18) imply

$$16 < z^2 < 25 \text{ and } z \text{ is even,}$$

$$16 < z^2 < 25 \text{ and } z \text{ is odd.}$$

This is a contradiction.

Suppose that *e-Slice*$(p\overline{CP^2})$ contains $T(-2, 8p+3)$. Since $T(-2, 8p+3)$ belongs to $\mathcal{T}_{-2p-1}$ and the negative unknotting number of $T(-2, 8p+3)$ is finite, by Lemma 1.11, there exists an integer z such that z satisfies the following condition

$$z^2 \leq 4 + \frac{-16p-4}{p} < 0.$$

This is a contradiction. □

3.10. Claim. Let K be a knot. Neither *e-Slice* (pCP^2) nor *e-Slice* $(p\overline{CP^2})$ contains K if and only if *e-Slice* $(p\overline{CP^2})$ contains neither K nor $-K^!$.

3.11. Remark. By Theorem 3.9 and Claim 3.10, we have that $T(2, 8p+3)$ belongs to neither *e-Slice* (pCP^2) nor *e-Slice* $(p\overline{CP^2})$ for any $p \geq 3$.

4. APPLICATIONS

4.1. Proposition. *If $K \in \mathscr{K}_p$ then K belongs to either e-Slice (pCP^2) or e-Slice $(p\overline{CP^2})$.*

Proof. If $K \in \mathscr{K}_p$ then there exists a 2-disk D^2 and a trivial knot K_0 in S^3 such that K is obtained from K_0 by $\frac{\varepsilon}{p}$—Dehn surgery along ∂D^2. We take the parallel copies $D_1^2, \ldots, D_p^2$ of D^2 as in Figure 7. It is easily seen that K is obtained from K^0 by Dehn surgery along $\partial(\cup D_i^2)$ in which the surgery coefficients are all ε. Suppose that K_0 and $\cup D_i^2$ are in the boundary of a 4-ball B_0^4, then K_0 bounds a properly embedded 2-disk Δ in B_0^4. Let $\{h_i^2\}$ $(1 \leq i \leq p)$ be 2-handles on B_0^4 whose attaching sphere are $\{\partial D_i^2\}$ and all framings are ε. We note that $K_0 \subset \partial(B_0^4 \cup \cup h_i^2)$ is K, K bounds the 2-disk Δ in $B_0^4 \cup \cup h_i^2$ and $B_0^4 \cup \cup h_i^2$ is deffeomorphic to either punctured pCP^2 or punctured $p\overline{CP^2}$. Let the punctured pCP^2 and punctured $p\overline{CP^2}$ be denoted by $pCP^2 - \mathrm{Int}\, B^4$ and $p\overline{CP^2} - \mathrm{Int}\, B^4$, respectively. Suppose the linking number $lk(K_0, \partial D^2) = z$ then $lk(K_0, \partial D_i^2)$ $(1 \leq i \leq p)$ are the same number as z. It is not hard to see that Δ represents either an element $z(\varepsilon_1 \gamma_1 + \ldots + \varepsilon_p \gamma_p)$ in $H_2(pCP^2 - \mathrm{Int}\, B^4, \partial)$ or an element $z(\bar{\varepsilon}_1 \bar{\gamma}_1 + \ldots \bar{\varepsilon}_p \bar{\gamma}_p)$ in $H_2(p\overline{CP^2} - \mathrm{Int}\, B^4, \partial)$. This implies that K belongs to either *e-Slice* (pCP^2) or *e-Slice* $(p\overline{CP^2})$. □

By Remark 3.11, Proposition 4.1 and the definition of evenly slice knots, we have the following theorem.

4.2. Theorem. *For any integer $p \geq 3$, $\mathscr{K}_p$ does not contain any knot that is cobordant to $T(2, 8p+3)$.*

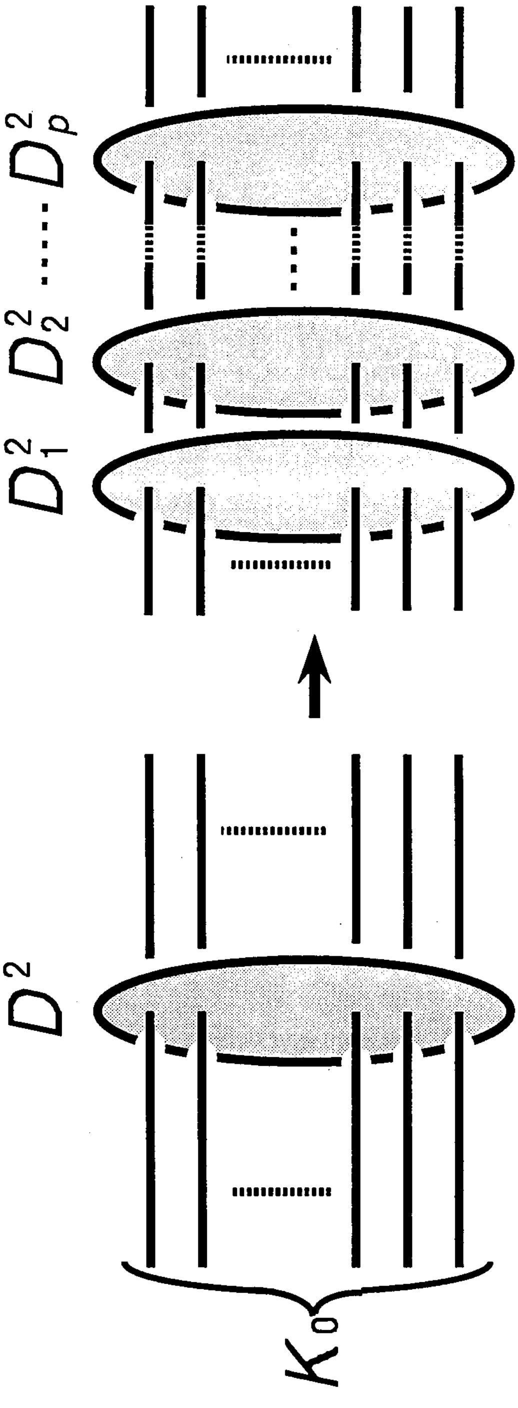

Figure 7

By Lemmas 1.8 and 1.11, we have the following proposition.

4.3. Proposition. *For any p $(1 \leq p \leq 5)$, e-Slice$(p\overline{CP^2})$ contains neither $T(2,75)$ nor $T(-2,75)$.*

By Claim 3.10, Propositions 4.1, 4.3 and the definition of evenly slice knots, we have the following proposition.

4.4. Proposition. *For any p $(1 \leq p \leq 5)$, $\mathscr{K}_p$ does not contain any knot that is cobordant to $T(2,75)$.*

4.5. Lemma. (Motegi [14]) *If $p \geq 6$ then $\mathscr{K}_p$ does not contain any composite knot.*

Let K be a nontrivial slice knot. Proposition 4.4 and Lemma 4.5 imply that $\mathscr{K}_p$ does not contain $T(2,75)\# K$ for any $p \geq 1$. Hence we have the following theorem that gives a negative answer to Question 2.

4.6. Theorem. *There exist infinitely many knots that do not belong to any $\mathscr{K}_p$ $(p \geq 1)$.*

Let K be a knot in $\partial(CP^2 \# \overline{CP^2} - \text{Int}\, B^4)$. If K is obtained from a slice knot by a $(\pm 1, \pm 1)$-twisting, then by proof of Proposition 4.1, K bounds a properly embedded 2-disk in $CP^2 \# \overline{CP^2} - \text{Int}\, B^4$ that represents an element $\pm\gamma_1$ or $\pm\bar{\gamma}_1$ in $H_2(CP^2 \# \overline{CP^2} - \text{Int}\, B^4, \partial)$. It follows that K bounds a properly embedded 2-disk in $CP^2 \# \overline{CP^2} - \text{Int}\, B^4$ that represent an element $\pm\gamma_1 + \bar{\gamma}_1$ or $\gamma_1 \pm \bar{\gamma}_1$ in $H_2(CP^2 \# \overline{CP^2} - \text{Int}\, B^4, \partial)$. We have the following proposition.

4.7. Proposition. *If K is obtained from a slice knot by $(\pm 1, \pm 1)$-twisting, then K belongs to e-Slice$(CP^2 \# \overline{CP^2})$.*

Since a $(\pm 1, \pm 1)$-twisting does not change the Arf invariant of a knot, thus $T(2\varepsilon, 3)$ cannot be obtained from a slice knot by a $(\pm 1, \pm 1)$-twisting. By Theorem 3.1, Proposition 4.7 and the definition of evenly slice knots, we have the following theorem.

4.8. Theorem. *Let t be a nonnegative integer and $\varepsilon = \pm 1$. A knot cobordant to $T(2\varepsilon, 2t+1)$ is obtained from a slice knot by a $(\pm 1, \pm 1)$-twisting if and only if $t = 0$.*

If $2t+1\equiv\pm1 \bmod 8$, then the Arf invariant of $T(2\varepsilon, 2t+1)$ is zero (for example, see p266 in [5]). Thus Theorem 4.8 gives infinitely many counterexamples to Conjecture.

Acknowledgement. The author would like to thank Professor Shin'ichi Suzuki for his encouragement.

References

[1] S. K. DONALDSON: *An application of gauge theory to four-dimensional topology,* J. Diff. Geometry, 18 (1983), 279-315.

[2] S. K. DONALDSON: *The orientation of Yang-Mills moduli spaces and 4-manifold topology,* J. Diff. Geometry, 26 (1987), 397-428.

[3] R. H. FOX and J. W. MILNOR: *Singularities of 2-spheres in 4-space and cobordism of knots,* Osaka J, Math., 3 (1966), 257-267.

[4] Y. HIRAI: *Representing homology classes of connected sums of 2-sphere bundles over 2-spheres,* Kobe J. Math., 6 (1989), 233-240.

[5] L. H. KUAFFMAN: *On knots,* Ann. of Math. Studies, 115, Princeton Univ. Press, Princeton, N. J., (1987).

[6] M. A. KERVAIRE and J. W. MILNOR: *On 2-sphere in 4-manifolds,* Proc. Nat. Acad. Sci. U.S.A., 47 (1961), 1651-1657.

[7] K. KIKUCHI: *Representing positive homology classes of $CP^2\#2\overline{CP^2}$ and $CP^2\#3\overline{CP^2}$,* to appear in Proc. Amer. Math. Soc.

[8] R. KIRBY: *A calculus for framed links in S^3,* Invent. Math. 45 (1978), 35-56.

[9] R. KIRBY: *Problems in low-dimensional manifold theory,* Proc. Symp. Pure Math. 32, American Mathematical Society, Providence, Rhode Island (1978), 273-312.

[10] R. KIRBY: *4-manifold problems,* Contemp. Math., 35 (1984), 513-528.

[11] T. LAWSON: *Representing homology classes of almost definite 4-manifolds,* Michigan Math. J., 34 (1987), 85-91.

[12] Y. MATHIEU: Sur des noeuds qui ne sont pas déteminés par leur complément et problémes de cirugie dans les variétés de dimension 3, These, L'Université de Provence, 1990.

[13] K. MIYAZAKI: private communication.

[14] K. MOTEGI: *Primeness of twisted knots,* prepint.

[15] R. A. NORMAN: *Dehn's Lemma for certain 4-manifolds,* Invent. Math., 7 (1969), 143-147.

[16] V. A. ROHLIN: *Two-dimensional submanifolds of four-dimensional manifolds,* Functional Anal. Appl., 5 (1974), 39-48.

[17] S. SUZUKI: *Local knots of 2-spheres in 4-manifolds,* Proc. Japan Acad., 45 (1969), 34-38.

[18] S. H. WEINTRAUB: *Inefficiently embedded surfaces in 4-manifolds,* Algebraic Topology Aarhus 1978 (J. L. Dupont and I. H. Madsen, eds.), Lecture Notes in Math., 763, Springer-Verlag, Berlin and New York, 1979.

[19] M. YAMAMOTO: *Lower bounds for the unknotting numbers of certain torus knots,* Proc. Amer. Math. Soc., 86 (1982), 519-524.

[20] A. YASUHARA: (2, 15)-*torus knot is not slice in* CP^2, Proc. Japan Acad. ser. A. Math. Sci., 67 (1991), 353-355.

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Recibido: 18 de marzo de 1992.